# Optimization and Chemical/ Bioprocess Optimization

# 11 [3]

*Do not worry about your difficulties in mathematics. I can assure you mine are still greater.*

Albert Einstein

*Pleasure in the job puts perfection in the work.*

Aristotle

*An investment in knowledge always pays the best interest.*

Benjamin Franklin

## 11.1 Chapter Purpose

This chapter is about understanding an important and fundamental topic for all engineers: optimization. The importance of this chapter lies in its two objectives: to familiarize and excite you with the applications of optimization you will encounter in your career in chemical and bioprocess engineering and provide you with some elementary tools (using graphics and spreadsheets) to solve interesting and challenging optimization problems. We will introduce you to this captivating topic through examples, and you will be presented with some classic problems in chemical engineering, environmental engineering, food engineering, biochemical engineering, biotechnology, and others, i.e., classic chemical engineering problems including most of the branches of bioprocess engineering.

Paraphrasing Aristotle, we would like to engage and fascinate you with the topic of optimization and enable you to reach your highest potential. Oh, and remember what Einstein told us about our difficulties in mathematics.

## 11.2 What Is Optimization?

Recall from Chap. 4 that understanding the logic behind the construction of a honeycomb can help us improve and optimize the design and construction of various pieces of equipment and structures. Honeycomb construction can help us understand the concept of optimization while at the same time, as mentioned in Chap. 4, helping us to apply this logic and improve the design of equipment and processes. But the question persists: what is optimization? Considering the honeycomb construction we can infer that this storage device is optimum because it is structurally adequate and able to hold a

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specific amount of honey utilizing a minimum amount of wax. As depicted in Chap. 4 the amount of wax utilized is vital because honeybees need approximately 4–6 [g] of honey to produce 1 [g] of wax. Taking a dictionary definition we can say that optimization is to achieve the best outcome. In the honeycomb construction the "best" means to construct a reliable storage device utilizing the minimum amount of wax.

As we will see in this chapter, including in the solved and proposed problems (Sects. 11.7 and 11.8), optimization has a wide array of applications that range far beyond the field of chemical and bioprocess engineering. A rereading of Chap. 4 would reveal that chapter was almost entirely, though indirectly, devoted to optimization. The beauty of Chap. 4 is that it invites us to discover the wonders of nature but also subtly teaches us, through examples, the concept of optimization. In addition, optimization is a daily occurrence in all our lives. For example, when you were in high school, you could have taken several routes to get to school, but you probably chose the route that was the best for you, the optimum, optimum in the sense that you probably based your decision on the distance traveled (minimizing time), but perhaps you considered other variables as well, such as personal safety (maximizing safety). All the time and every time (more than we think) we, perhaps unwittingly, make decisions making systematic use of the concept of optimization (maximization and minimization). A little reflection would show us that we apply the concept of optimization all the time, or at least maximize or minimize the variables that are of interest to us. Examples of optimization abound in our everyday lives, but a few will suffice: (a) choosing a restaurant, (b) picking a vacation spot, (c) choosing a girlfriend/boyfriend, (d) choosing a college or university, and (e) buying a car. Try to come up with five more examples of optimization like those in items (a)–(e). All these examples have in common at least two main features: (a) the optimum is not absolute (universal) in the sense that, for example, the best restaurant for me is not necessarily the best restaurant for you, and (b) the optimum is not decided by maximizing or minimizing just one variable; there are several variables and some of them might be in conflict. For example, what is the optimum route to go to school? If you choose a route so as to minimize the time to get there, it might put you in harm's way; on the other hand, if your aim is to maximize safety, then it might take too much time. So what is the best route? One way to handle the situation is, for example, to have as your primary consideration minimizing the time it takes to get there but given a certain minimum level of safety. So your problem first had two variables, but you have reduced it to a problem with one variable and one constraint. But now how do you decide on the minimum level of safety? Is there a way to quantify the level of safety depending on the route? As we will see in our engineering problems, most of the time, the final decision will be based on our engineering criterion, which represents our cumulative experience.

## 11.3 Do We Have the Required Knowledge to Deal with Problems of Optimization and Process Optimization?

Unfortunately not. In fact, as you will discover and learn as an engineer, most practical and real optimization problems require an advanced level of mathematics. But, as stated at the beginning, our goal is to familiarize and engage you with this absorbing, fascinating, and important topic in chemical and bioprocess engineering. Fortunately, with very basic mathematical tools (high-school level) not only will you be fully familiarized with the concept and potential applications of optimization, but you will also be able to solve some interesting process engineering problems.

Paraphrasing Einstein, do not be worried about your knowledge of mathematics: just with your current preparation in mathematics and some understanding of spreadsheets, we will be able to guide you and tackle exciting problems while at the same time fulfilling one of our main goals, which is to familiarize and captivate you with chemical and bioprocess engineering. Because not every freshman engineering student has received solid training in spreadsheets, in Sects. 11.5 and 11.6 we will give you, through warm-up examples, the required tools to solve the optimization problems presented in this book.

Of course, you do not have the knowledge required to solve these interesting problems, but you already have the capability to at least approach them. With the help of spreadsheets you will be able to attempt and solve problems that are stimulating and that will give you a broad awareness of chemical and bioprocess engineering. In addition, with confidence we can state that the most important part of dealing with engineering problems is to have the capability to correctly formulate them, and this capability was developed and heavily stressed in previous chapters.

## 11.4 Understanding Optimization in Chemical and Bioprocess Engineering

Most, if not all, engineering problems are subject to optimization. We cannot mention all the possibilities of optimization that you will encounter in chemical and bioprocess engineering, but in the sections on solved and proposed problems (Sects. 11.7 and 11.8), you will be exposed to a wide variety of optimization situations. In this section, we present several real optimization situations in different applications in chemical engineering, food processing, biochemical and bioprocess engineering, and environmental engineering.

**Multieffect evaporation**

Multieffect evaporators are a sequence of heat exchangers connected in series where a liquid is boiled to produce vapor and a concentrated liquid solution. A classic example in the chemical industry is the determination of the optimal number of effects in an evaporation system, were the optimum is found when there is an economic balance between energy saving and added investment, this is, a minimization of the total cost. Normally multieffect evaporators, in various applications, have three to six stages, although in some specific applications they have much more than that. As an example, in the tomato concentration industry multieffect evaporators are, most of the time, manufactured with five effects. To understand the optimization problem, we should mention that evaporators can be constructed from one, two, three, or sometimes over ten effects. The tradeoff is that adding an effect to the equipment will lead to energy savings during the operation (making it more efficient), but on the other hand, the equipment cost will be higher. Then, to determine the optimum number of effects, it is necessary to carry out an economic analysis to establish the number of effects that minimizes the total cost, taking into account additional effects means greater energy savings (decreased variable costs) higher equipment cost (increased fixed costs).

**Number of fermenters**

Industrial fermentation plants consist of three main sections: preparation, fermentation, and product recovery. The preparation section usually contains operations such us medium preparation and sterilization and inoculum propagation. The fermentation section is the heart of the plant where the transformation of raw material into products takes place. The product recovery section encompasses the downstream operations needed to obtain the product of interest with the required purity (Reisman 1988). In the design of fed-batch and batch fermentation plants, one faces the problem of figuring out the adequate combination of the number and size of fermenters to be used to meet the desired production schedule. In principle, the problem has an infinite number of solutions because for any given fermenter size, a number of units of that size will do the work. Nevertheless, not all solutions are equal from an economic standpoint (Simpson et al. 2005).

**Anaerobic digestion**

Pig waste slurry is normally revalued by obtaining methane gas (biogas). At present, the production of biogas is a well-known technique, but it can be improved, economically speaking, by the addition of organic waste from the food industry. However, this may lead to problems of organic overload and inhibition, among others. Prediction and optimization of these input parameters, and the corresponding operating conditions (e.g., pH, temperature, microorganism concentration), will contribute to determining the optimum operation and performance of the digestion plant.

**Optimum number of retorts in canned-food plants**

Batch processing with a battery of individual retorts (autoclaves) is a common mode of operation in many food-canning plants (canneries). Although high-speed processing with continuous rotary or hydrostatic retort systems can be found in very large canning factories (where they are cost-justified by high-volume throughput), such systems are not economically feasible in the majority of small to medium-sized canneries (Norback and Rattunde 1991). In such smaller canneries, retort operations are carried out as batch processes in a cook room where the battery of retorts is located. Although the unloading and reloading operations for each retort are labor intensive, a well-designed and managed cook room can operate with surprising efficiency if it has the optimum number of retorts and the optimum schedule of retort operation. Using just one retort entails a long downtime; adding more retorts can make the operation more efficient but will raise investment costs. As in the previous example (fermenters) there exist literally an infinite number of alternatives, but only one is optimum in economic terms (Simpson et al. 2002).

**Optimum temperature to dry food products**

Dehydration is a common process used to reduce the water activity of foods, thereby preventing the proliferation of deteriorative microorganisms and extending the product shelf life. Usually the drier uses hot air (70–90 °C) to remove water (lowering water activity) from the food product, and the dehydration process takes from 30 [min] to hours, and the food product will be dehydrated but also exhibit quality losses. So the question is: what is the best temperature at which to dehydrate food products? The higher the temperature, the shorter the process (maximizing production output); on the other hand, the product will be much more affected in terms of quality (vitamin loss, color change, and shrinkage). Clearly, there is a tradeoff, and there exists an optimum strategy to carry out the dehydration process.

**Manufacturing and processing plant location**

Real-world engineering problems, including chemical and bioprocess engineering problems, are generally characterized by the presence of many conflicting and nonmeasurable objectives or particular objective functions. These types of optimization problem are called multiobjective (multipurpose, multicriterion) problems. Taking into account the nature of real-life problems, it is better to look at these problems as multiobjective optimization problems. One interesting multiobjective optimization problem is to find the best location for a manufacturing or processing plant. Why? Because there are several variables that affect the best location, for example, distance from the market, quality, availability and cost of raw materials, availability of skilled and unskilled labor, availability and cost of water and energy, and environmental regulations.

**Optimizing industrial water utilization and management**

Nowadays chemical, food, and bioprocessing companies are major consumers of water, and the water sources that are available have great relevance. Many studies indicate that water resources are often used inefficiently. In addition, at this time the demand for potable water, environmental regulations, increasing costs of supply and wastewater treatments, and other factors require that industries find efficient solutions, i.e., optimizing the use and management of water for processing.

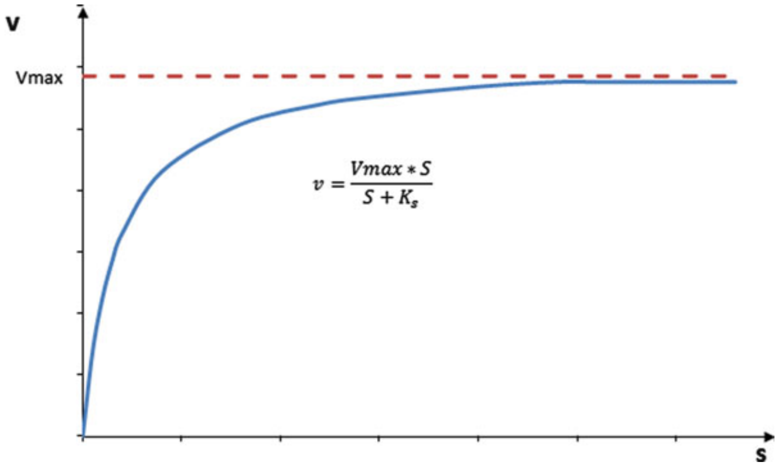



**Fig. 11.1** Michaelis-Menten model

**Determining the optimal number of experiments**
Experimentation is normally expensive, and sometimes very tedious, and difficult. As an example, advances in nanotechnology make it easier to examine nature at a deep level because today we have advanced microscopes and tools that allow us to better observe nature and that have improved our understanding of the micro and nano worlds, but such equipment is very expensive and the experiments rather difficult. When planning new experiments it is important to determine, first, what experiments to do. Several mathematical techniques can help us, in specific cases, to determine what and how many experiments represent the optimal set. One of these techniques is the D-optimal design that normally significantly reduces the number of experiments and delivers good results. For example, if you have a mathematical model such as the one depicted in Fig. 11.1 (Michaelis-Menten model) and you want to fit the curve with experimental data, then to determine the parameters of the model ($V_{\max}$ and $K_s$), the question is how many experiments are required and for what values of the $S$-axis. D-optimal design helps to efficiently fix the set of experiments.

## 11.5 Maximum, Minimum, and a Warm-Up Example

As shown in Fig. 11.2, $x = a$ and $x = b$ represent a relative maximum and minimum for $f(x)$, respectively. We use the term *relative* because if we extend the amplitude of $x$ values, we can probably find other maximum and minimum values. Why do we call $x = a$ the maximum? Because in the vicinity of $x = a, f(a)$ is the largest value for $f(x)$. In the same way, $x = b$ represents the minimum because $f(b)$ is the minimum value of $f(x)$ in the vicinity of $x = b$.

As indirectly mentioned in Sect. 11.3, you are probably not yet familiar with the concept of derivatives, which is essential in handling maximum and minimum problems. Although we will present and solve interesting and applicable maximum and minimum engineering problems, it is not our intention to familiarize you with the concept of derivatives. Soon in your career you will be introduced by experts to this important and key mathematical concept for all engineers. You are most probably familiar with the use of spreadsheets. In this section we will show you how to solve maximum and minimum problems using spreadsheets.

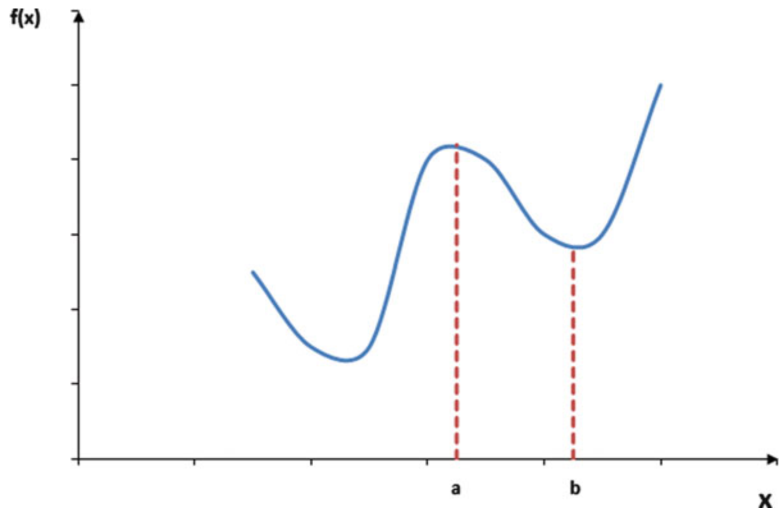



**Fig. 11.2** Maximum and minimum in a $f(x)$ vs. $x$ graph

### 11.5.1 Warm-Up Example: Minimization

**Airplane and Cessna [6]**. A commercial airliner is arriving in Columbus, Ohio, from the south at a speed of 800.0 [km/h]. When the airliner is 600.0 [km] from the Columbus airport, a Cessna aircraft leaves Columbus for the East Coast at a speed of 250.0 [km/h]. When will the minimum distance occur between the airliner and the Cessna? **Assumption**: The two aircraft are moving at an angle of 90° to each other.

**Solution**

**Step I**

Graphical representation and mathematical formulation

At the beginning ($t = 0$) the distance that separates the airliner and the Cessna is 600.0 [km] (Fig. 11.3a). Then $t$ hours later the location of each plane will be as depicted in Fig. 11.3b.

As shown in Fig. 11.3b, $D$ represents the distance between both planes at any instant $t$ ($t \geq 0$). According to the Pythagorean theorem, we can write the following equation:

$$D^2 = (600 - 800t)^2 + (250t)^2.$$

Then

$$D = \sqrt{(600 - 800t)^2 + (250t)^2}. \tag{11.1}$$

One way to test the validity of (11.1) is replacing $t = 0$ and confirming that $D = 600.0$ [km].

**Step II**

Determine the right procedure to solve the problem. At your level, to obtain the value of $t$, to minimize the value of $D$ in (11.1), we will use spreadsheets where we have two interesting alternatives to find the value of $t$ to minimize $D$ in (11.1).

1. Approximate graphical solution. Plot $D$ ($y$-axis) against $t$ ($x$-axis) and, first, visually determine if the function has a minimum or a maximum, whichever is applicable. If the function has a minimum or

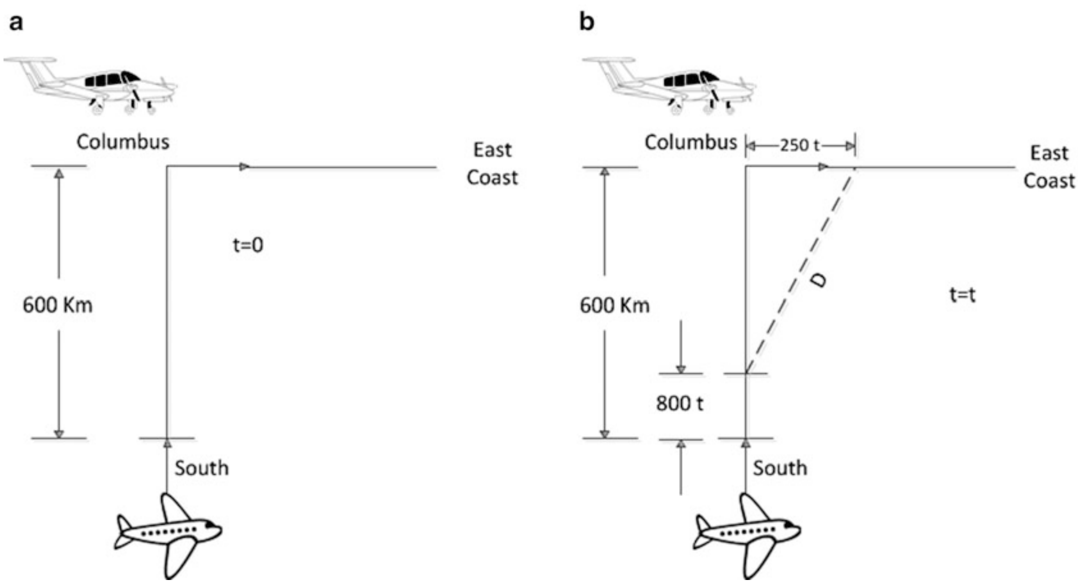



**Fig. 11.3** (**a**) Initial position of the Cessna and the airplane (**b**) Position of the Cessna and the airplane after *t* hours

a maximum, construct a plot with sufficient data to approximately, from the graph, determine the maximum or minimum; in this specific case determine the value of *t* that minimizes *D*. We would like to emphasize that this procedure is just a rough approximate, not the true solution. It is not a rigorous procedure but will help us to have an approximate idea of the real solution.

2. Solver tool. Use the Solver tool (Excel spreadsheet) to find the maximum or the minimum of the function, in this case the value of *t* that minimizes *D*.

Although the Solver approach is more accurate than the graphical visualization, we recommend using the Solver tool but also, first, doing a graph to have a clear visualization of the problem and see if the function has a minimum or a maximum.

Now we will show you how to approach this warm-up example problem with both procedures as follows:

(a) Graphical solution.
To find the value of *t* that minimizes the distance *D*, we first construct a table using Excel.

Table 11.1 shows that the time ($t$) that minimizes the distance ($D$) between the planes is $t = 0.7$ [h]. Figure 11.4 shows the Excel graph of $D$ against $t$ ($0 < t < 1$) and with time increments of 0.1 [h].

According to Fig. 11.4, the minimum distance between the two planes is at $t = 0.7$ [h]. This solution is approximate because it depends on the time increment chosen and so should be refined. To get a more accurate solution, instead of dividing the values of $t$ by 0.1 [h], we will divide by 0.01 [h] to get the results presented in Table 11.2.

According to Table 11.2, the time $t$ that minimizes the distance between the planes is $t \sim 0.68$ [h] and the distance between the planes is $D \sim 179$ [km]. A more accurate solution can be obtained using the Solver tool of Excel as follows.

(b) Solver tool.
First, we assign $t$ a cell (e.g., D9) with an initial value of 0. Then, in a different cell (e.g., G9), we write (11.1) (taking into account that our variable $t$ is now represented by cell D9). Therefore, the Excel

**Table 11.1**

| Time, *t* [h] | Distance, *D* [km] |
|---|---|
| 0.0 | 600.0 |
| 0.1 | 520.6 |
| 0.2 | 442.8 |
| 0.3 | 367.7 |
| 0.4 | 297.3 |
| 0.5 | 235.8 |
| 0.6 | 192.1 |
| **0.7** | **179.5** |
| 0.8 | 204.0 |
| 0.9 | 255.0 |
| 1.0 | 320.2 |

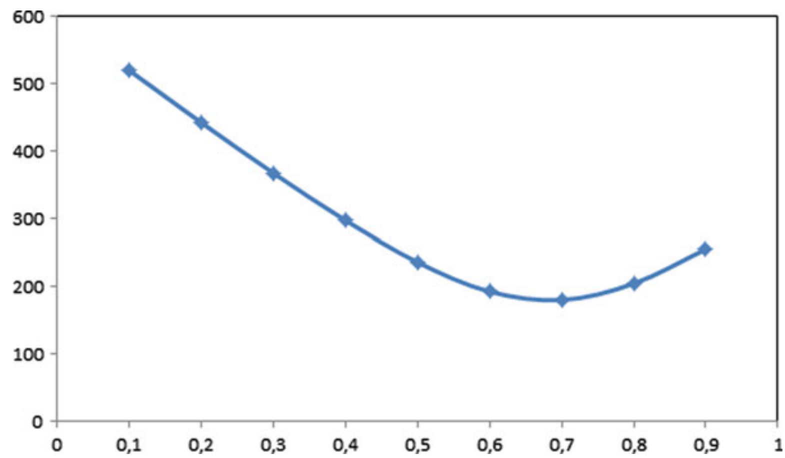



**Fig. 11.4** Distance of the Cessna and the airplane against time

**Table 11.2**

| Time, *t* [h] | Distance, *D* [km] |
|---|---|
| 0.60 | 192.1 |
| 0.61 | 189.2 |
| 0.62 | 186.7 |
| 0.63 | 184.5 |
| 0.64 | 182.6 |
| 0.65 | 181.1 |
| 0.66 | 180.0 |
| 0.67 | 179.3 |
| **0.68** | **179.0** |
| 0.69 | 179.1 |
| 0.70 | 179.5 |

spreadsheet should look like this (cells D8 and G8 are just labels to indicate that cells D9 and G9 are time $t$ and (11.1), respectively) (Fig. 11.5):

Cell G9 is equal to 600 because $t = 0$ (cell D9).

Second, click on Data on the toolbar. Solver is on the right-hand side (if Solver is not displayed, you should ask your tutor or professor for a detailed explanation on how to install Solver in an Excel spreadsheet).

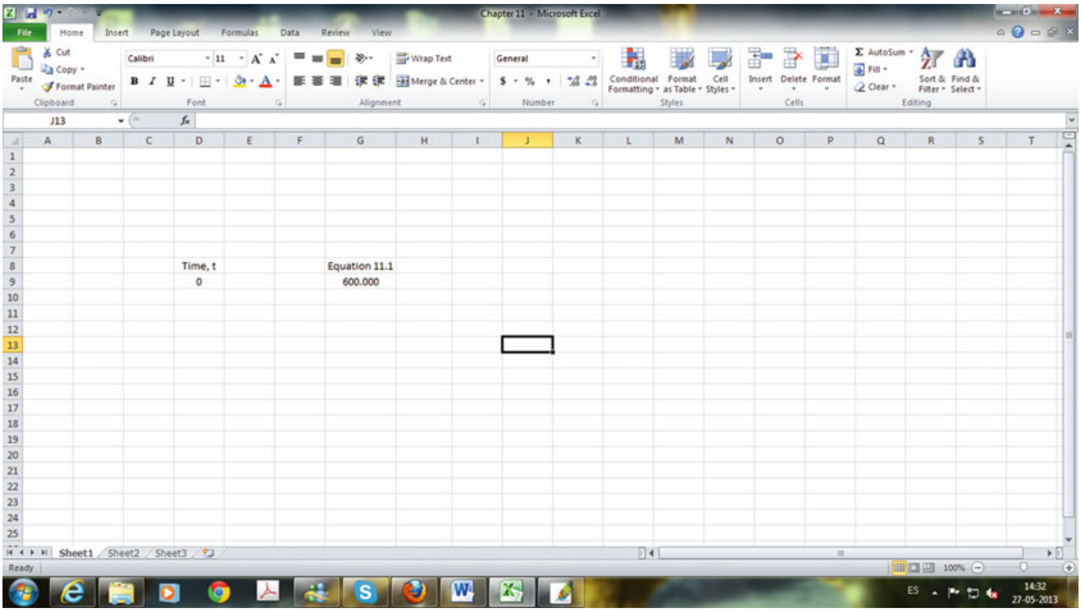



**Fig. 11.5** Microsoft Excel screen to calculate distance against time (Eq. 11.1)

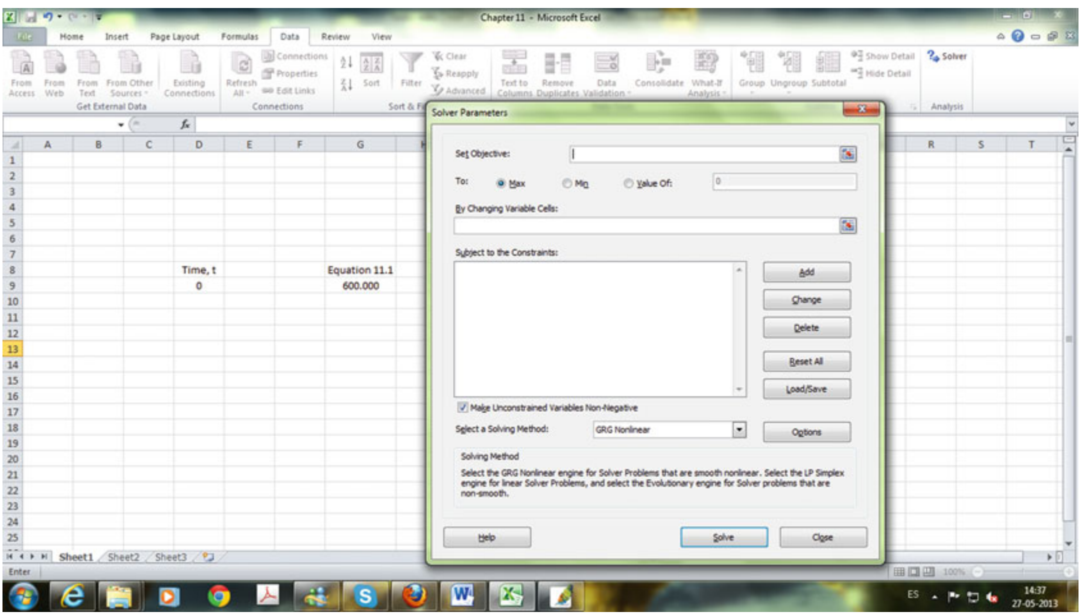



**Fig. 11.6** Solver window in Microsoft Excel spread sheet

Now clicking on Solver you will see the following screen (Fig. 11.6):

Set cell G9 as the objective function (Set Objective), then choose Min (minimize), and select cell D9 in By Changing Variable Cells. In addition, click on the box to make D9 (unconstrained variables) nonnegative (remember $t \geq 0$) and the screen will now look like this (Fig. 11.7):

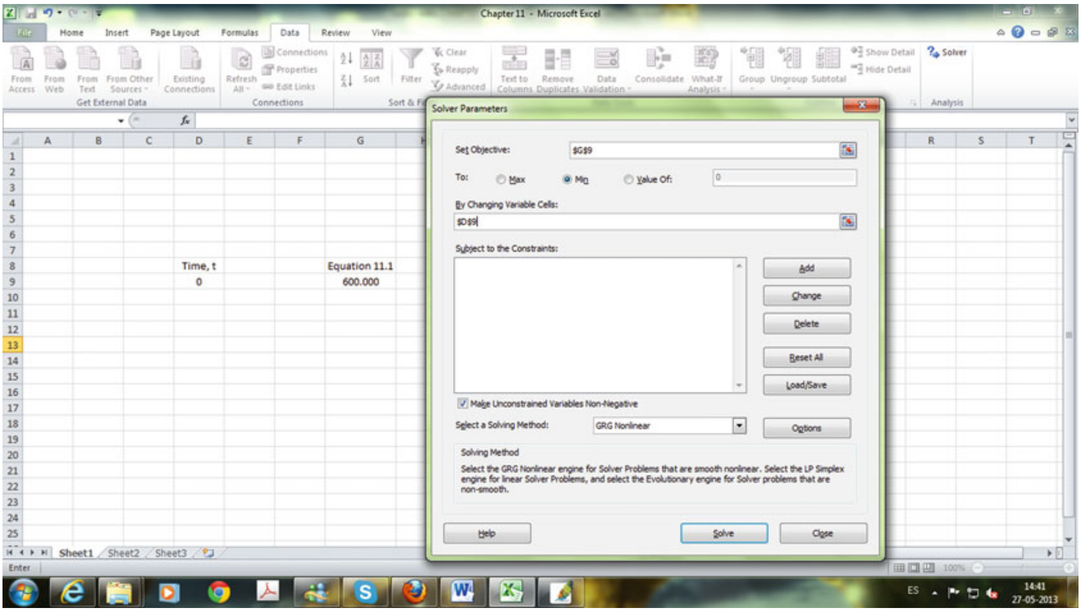



**Fig. 11.7** Solver window in Microsoft Excel spread sheet showing the objective function (cell G9) and the changing variable (cell D9)

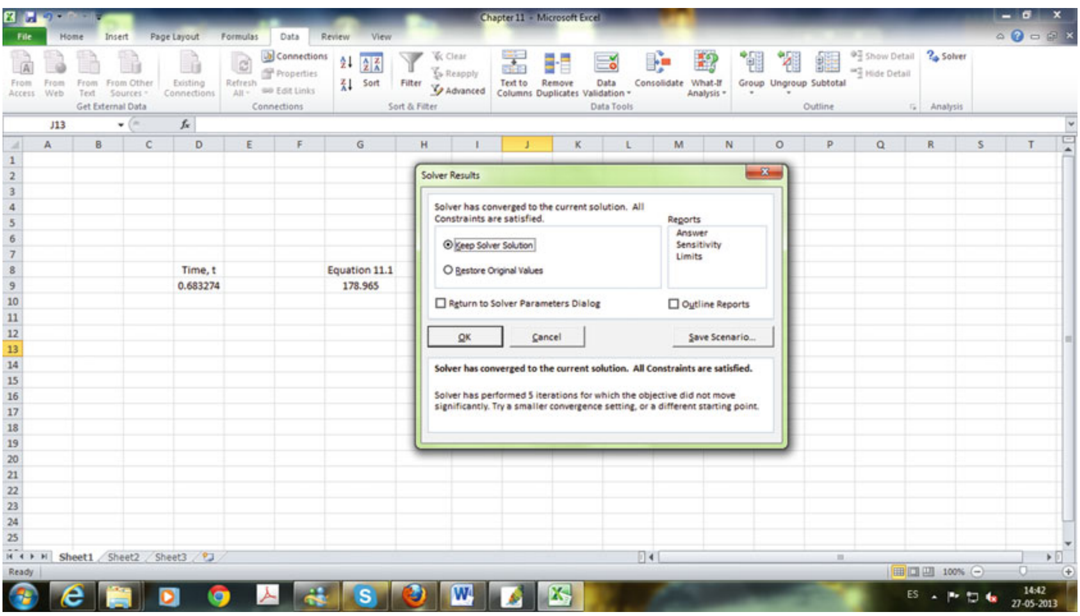



**Fig. 11.8** Final screen depicting the solution found by Solver tool

Next, click on Solve to bring up the following screen (Fig. 11.8):

Indicating that Solver has found a solution, $t = 0.683274$ [h] and $D = 178.965$ [km]. Note that in an Excel sheet, you can define the number of decimal places. Also, although the Solver solution is more accurate than the graphical solution, the values are fairly close.

## 11.6 Origins of Operations Research and Process Optimization

### 11.6.1 Historical Background

History teaches us that until the mid-nineteenth century, the production of goods was mainly in workshops that operated in the traditional way, i.e., with low production volumes, starting the manufacture of a new product when the previous was finished (one to one), and without further specialization of labor, meaning a craftsman normally made the product in full. This form of craftsmanship in some cases led to high-quality products, of which those existing today are considered works of art.

From the time indicated, the Industrial Revolution brought about the birth of the first industries, the division of labor, interest in optimizing workers' time [the first operations research (OR) studies were devoted to this area], optimal placement of machines, specialization of individuals and industries, and the creation of various functional areas within companies, which sometimes had conflicting objectives. Then came the need to coordinate and define activity levels and better allocate and use resources (concept of efficiency) to achieve these levels of activity, with optimal results for the company (from a global point of view) and avoiding internal conflicts.

In the early twentieth century in the steel and coal industries, Frederic Taylor, a mechanical engineer and the father of scientific management, conducted the first studies OR in the USA on work organization, process improvement, optimization of workers' time, and machine location (layout). Henry Fayol, a mining engineer from France, made a contribution analogous to Taylor's but oriented toward administration, separating the functional areas of a company and stating the 14 general principles of management. At the same time, A. Erlang, working for the Danish telephone company in Copenhagen, studied timeouts of private branch exchange users, giving rise to the issue of waiting lines (generally applicable to any service) and making an important contribution to telephone engineering.

The Gantt chart was invented by Henry Gantt in the early part of the last century. Gantt charts coordinate various activities (with a common ultimate goal) of various people. Such charts are the predecessor of the most powerful methods of the Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT).

OR underwent tremendous development and use during World War II; it was applied to solve problems of logistics equipment and staff, as well as to plan military operations themselves. Consider the following examples:

- Choosing the type and quantity of materiel to manufacture with limited resources.
- Assigning materiel to different battlefronts and deciding how to transport materiel.
- Allocating human resources to the various fronts.
- Determining how to deploy patrol antisubmarine aircraft and ships.

After the war, these experiences were transferred to the commercial sector by people who were demobilized from the military and as a result of articles published in various journals. This happened mainly in England and the USA. The USA was in the position of global economic leader, given that European countries (particularly Germany) and Asia (especially Japan) saw their economies destroyed by the end of the war. The USA had the opportunity to export its products and services to almost everyone and, in turn, imported raw materials from the rest of the world, becoming an "economic engine," globally speaking.

US companies experienced strong growth in operational levels, and OR was used for various purposes, including determining the optimal levels of activity, planning, coordination, control, and allocation of resources.

During the postwar period, several tools were developed to assist in decision making to address various types of recurring industrial problems in the area of operations. The most applicable tools to today's world are linear programming and the CPM method. One of the first textbooks on OR appeared in 1957, *Introduction to Operations Research*, by Churchman, Ackoff, and Arnoff.

### 11.6.2 Operations Research and the Scientific Method

OR uses the scientific method, through experimentation or simulation, utilizing a general sequential procedure or methodology (which may be iterative) to determine the structure of a situation and find the cause and effect relationships between variables. The typical stages of a real OR application are as follows:

(a) Detect and identify the problem or situation on which you have to make a decision. (If there are alternatives, for example, in the area of resource distribution, which is the optimum?)
(b) Define the problem and its organizational environment, temporal, physical, and decisional context; raising of study objectives.
(c) Collect data and relevant parameters, relationships between variables of interest.
(d) Formulate a mathematical model to evaluate the alternatives available; this implies determining the factors, parameters, variables, and elements to be considered in a process of abstraction from reality.*
(e) Use an algorithm to find the best alternative.*
(f) Test the model: validate it using various parameters to determine the ability of the model to predict the behavior of the real system under various circumstances. Perform sensitivity analysis on different parameter values.
(g) Decide on the best alternative. Consider the key elements for a successful implementation, for example, the need for training, changes in organizational structure, and technological changes.
(h) Prepare a solution for implementation. Provide documentation and training.
(i) Implement the solution, establishing specific actions.
(j) Evaluate the results.

At this stage in your career, we will show you, in some detail, two of the ten stages (marked with *) of OR application: the formulation of a mathematical model and an application algorithm. In earlier chapters, you were continuously tested to formulate mathematical expressions to solve a wide variety of problems, so you are fully prepared to learn the right procedure to formulate and, with the help of an appropriate tool, solve these interesting and engaging problems.

### 11.6.3 Linear Programming

Linear programming (LP) is a mathematical model used initially in solving problems in the area of defense in order to optimize the effect of weapons systems in the event of war. It was later used in the private sector. In the latter, it has been used to help make the best decisions for distributing scarce resources among several potential products, i.e., situation planning and production scheduling to determine product types and quantities manufactured, so as to obtain optimal results for the company. In 1947, Dantzig developed a simplex algorithm that facilitates solving powerful LP models.

In the early 1950s, many US companies experienced strong growth in operational levels, and LP was used to determine activity levels and optimal resource allocation.

LP (understood as being synonymous with linear planning) is currently one of the most widely used tools in OR. Furthermore, LP has expanded its scope of application to optimize a variety of situations, including investment portfolio problems, transportation and distribution of products, staff scheduling, inventory management, facility location (plants, warehouses), and combinations of these situations.

A decision-making situation using LP comprises three phases:

(a) Formulate the problem as a LP model.
(b) Resolve the LP model.
(c) Interpret the solution and perform a sensitivity analysis on it.

At this point, we would like to reemphasize that our goal is to familiarize you with phase (a), problem formulation. In addition, we will show you how to solve these problems utilizing the Solver tool of Microsoft Excel.

### 11.6.4 Modeling and Solving LP and IP Problems

The steps for formulating a LP model can be summarized as follows.

**Step I**

**Define decision variables and codification**. Correct definition and codification are very important in order to have a clear understanding of all variables involved in the problem. Variables should correspond to real-world variables, which make up the problem in a mathematical way; in LP models we attempt to determine the optimal values of these decision variables.

**Step II**

**Objective function**. Formulate the objective function (OF), which is a mathematical expression that must be expressed as a linear function of the decision variables. This expression should be optimized (maximize or minimize), subject to the constraints explained in the following step (e.g., limited resources).

**Step III**

**Constraints.** Formulate the constraints that limit or delimit the problem. These can derive from the availability of, for example, machinery, labor, raw materials, demand, and financial resources.

**Step IV**

**Mathematical solution**. In this aspect, we will not go into detail but just mention that the most common procedure for solving LP problems is the simplex method. In more advanced courses students will have the opportunity to learn this method and others that are used to solve integer programming (IP) problems, nonlinear problems, and others.

In this case, to solve these problems (LP and IP), we will detail the use of the Microsoft Excel Solver tool, which allows us to solve both LP and IP problems.

The fact that there are decision variables implies that there are several courses of action or alternatives from which to choose. For example, if a company has the option of producing three different products, the management team can utilize a LP model to decide which products to produce and at what level.

An important aspect in LP is the requirement that all mathematical expressions must be linear. Obviously, in practice, not all factors are linear, so nonlinear expressions are sometimes necessary to model specific situations, for example, in the case of economies of scale. Fortunately, it has been shown that many factors (work hours, use of machinery, benefits) are reasonably linear or can be approximated by expressions of this type. However, if there is found to be a factor that definitely cannot be expressed linearly, then the problem cannot be solved by LP. This is certainly is the case with nonlinear programming (NLP), which is beyond the scope of this textbook.

LP models can be used to model and solve various situations. The models must meet the following conditions:

(a) LP models must be able to express the situation by means of a mathematical model.
(b) All factors or terms considered in the objective function and constraints must be linear, i.e., they must be formed by the addition of one or more terms where each term must be the product of a coefficient and a decision variable, with an exponent having a value of one for the decision variable. If this condition is not met, then we are faced with a NLP problem.
(c) There should be one or more constraints on the decision variables of the problem, which determines if there will be different courses of action from which to choose, as established by the range of values that the decision variables can take.
(d) The decision variables can take continuous values (decimal). If this condition is not met, then we are faced with a problem of IP. IP is a special case of LP that is of great interest, and some examples and problems related to it will be presented in this textbook.

**Integer programming (IP)**: corresponds to a special case of LP in which the decision variables can take only integer values. When values are also limited to 0 and 1, we talk of binary integer programming. Moreover, there are also mixed models in which some variables are integers and others are continuous.

### 11.6.5 Warm-Up Example

Below is an example where it is necessary to decide the level of weekly production of products 1, 2, 3, and 4 in order to maximize the benefits to the company.

A company can produce four products and wants to maximize its profits. The benefit of each of the products is 5, 4, 3, and 4 respectively in arbitrary units. In addition, the company has a staff of 200 skilled workers and 150 unskilled workers laboring 40 h/week. The time required to produce one unit of each product is presented in the following table. Formulate and solve the LP problem in order to determine the optimum production levels of each of the four products.

| | Product 1 | Product 2 | Product 3 | Product 4 | Available hours per week |
|---|---|---|---|---|---|
| Required hours per skilled worker per unit of product | 5 | 3 | 1 | 2 | 8,000 (200 × 40) |
| Required hours per unskilled worker per unit | 5 | 7 | 4 | 8 | 6,000 (150 × 40) |
| Benefits per product | 5 | 4 | 3 | 4 | |

**Solution**

As was mentioned in previous chapters, it is highly recommended to approach problems, in this case LP problems, with the right attitude and, more importantly, with a methodology. As discussed in Sect. 11.6.4, a right solution procedure includes four steps, as follows: **Step I**. Variable definition and codification. **Step II**. Formulation of objective function. **Step III**. Formulation of all constraints. **Step IV**. Implementation and solution with the Solver tool (Microsoft Excel).

**Step I**

**Variable definition and codification**

$P_{1SW}$: Units of product 1 manufactured by skilled workers

$P_{1UW}$: Units of product 1 manufactured by unskilled workers

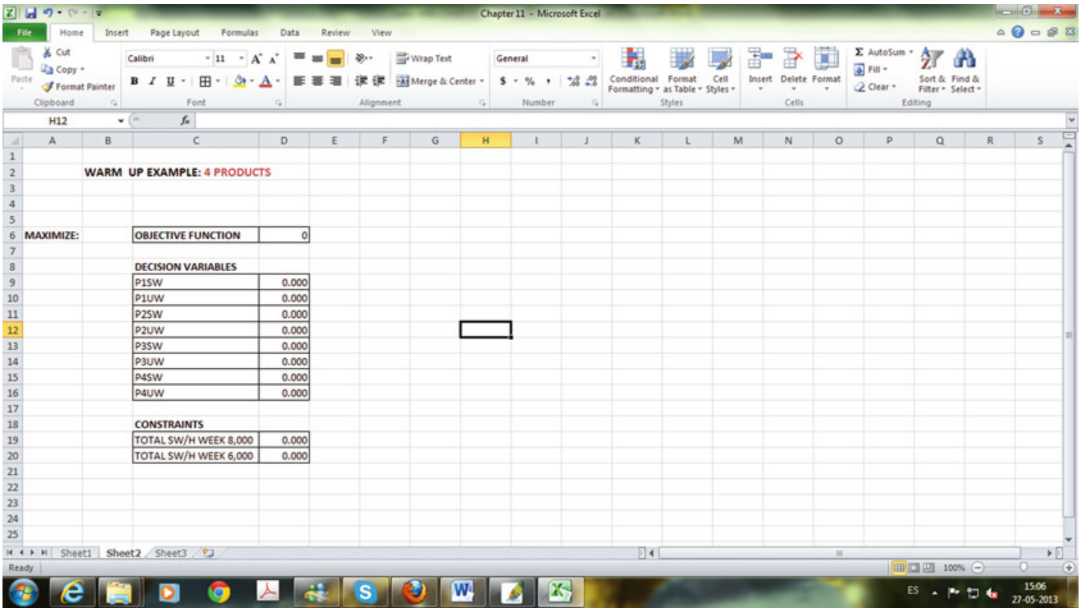



**Fig. 11.9** Microsoft Excel screen with the assigned cells for the objective function, decision variables and constraints

$P_{2\text{SW}}$: Units of product 2 manufactured by skilled workers
$P_{2\text{UW}}$: Units of product 2 manufactured by unskilled workers
$P_{3\text{SW}}$: Units of product 3 manufactured by skilled workers
$P_{3\text{UW}}$: Units of product 3manufactured by unskilled workers
$P_{4\text{SW}}$: Units of product 4 manufactured by skilled workers
$P_{4\text{UW}}$: Units of product 4 manufactured by unskilled workers

Summarizing we have eight decision variables.

**Step II**
**Formulation of objective function**
The total benefits will be as follows:

$$\textbf{Benefits} = 5 \times P_{1\text{SW}} + 5 \times P_{1\text{UW}} + 4 \times P_{2\text{SW}} + 4 \times P_{2\text{UW}} + 3 \times P_{3\text{SW}} + 3 \times P_{3\text{UW}} + 4 \times P_{4\text{SW}} + 4 \times P_{4\text{UW}}.$$

**Step III**
**Constraints**
We have two constraints, 8,000 h/week of skilled workers and 6,000 h/week of unskilled workers. Therefore:

1. SKILLED WORKERS: $5 \times P_{1\text{SW}} + 3 \times P_{2\text{SW}} + 1 \times P_{3\text{SW}} + 2 \times P_{4\text{SW}} \leq 8{,}000$,
2. UNSKILLED WORKERS: $5 \times P_{1\text{UW}} + 7 \times P_{2\text{UW}} + 4 \times P_{3\text{UW}} + 8 \times P_{4\text{UW}} \leq 6{,}000$.

In addition, all decision variables should be equal to or greater than 0 ($P_{1\text{SW}} \geq 0, P_{1\text{UW}} \geq 0, \ldots, P_{4\text{UW}} \geq 0$).

**Step IV**
As shown in this example, we will use Solver from Microsoft Excel.
1. The following image shows the first step for the warm-up example (Fig. 11.9).

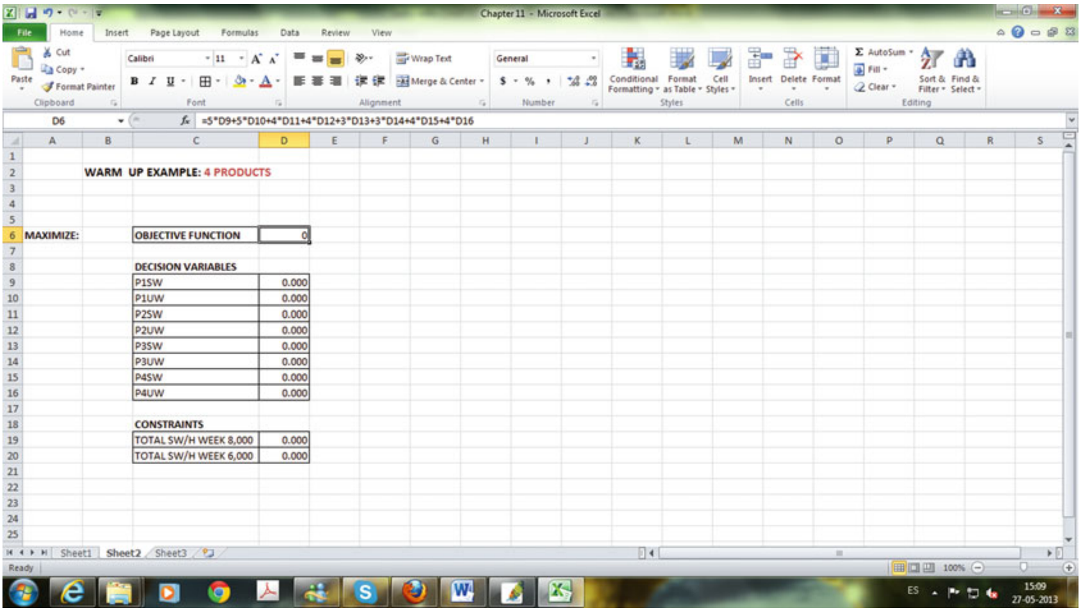



**Fig. 11.10** Microsoft Excel screen with the objective function written in cell D6

(a) We have called this **WARM-UP EXAMPLE: 4 PRODUCTS** and, in addition, we have assigned cells for the objective function, decision variables and constraints. In the left as a reminder we included a cell indicating that it is a maximization problem.

(b) In cell D6 we need to write the objective function and then, according to the assigned cells for $P_{1SW}$, $P_{1UW}$, …, $P_{4UW}$. Clicking on D6, the screen should look like this (Fig. 11.10): where $f(x) = 5 \times \text{D9} + 5 \times \text{D10} + 4 \times \text{D11} + 4 \times \text{D12} + 3 \times \text{D13} + 3 \times \text{D14} + 4 \times \text{D15} + 4 \times \text{D16}$.

This is equivalent to **Benefits** $= 5 \times P_{1SW} + 5 \times P_{1UW} + 4 \times P_{2SW} + 4 \times P_{2UW} + 3 \times P_{3SW} + 3 \times P_{3UW} + 4 \times P_{4SW} + 4 \times P_{4UW}$.

D9 represents the value assigned to $P_{1SW}$ and is multiplied by 5 because the benefit of each unit of product 1 is 5, D10 corresponds to the value assigned to $P_{1UW}$ and is also multiplied by 5 because the benefit of each unit of product 1 is 5, and so on.

(c) Writing the constraints: as shown in the previous screen, constraints are written in cells D19 and D20. Clicking on D19 we see (Fig. 11.11)

Where $f(x) = 5 \times \text{D9} + 3 \times \text{D11} + \text{D13} + 2 \times \text{D15}$.

This corresponds to the available hours of SKILLED WORKERS: $5 \times P_{1SW} + 3 \times P_{2SW} + 1 \times P_{3SW} + 2 \times P_{4SW} \leq 8{,}000$.

As you see, we have written the equation but we have not included the constraint that limits us to 8,000 h. This will be considered in the Solver tool box that will be shown later.

In the same way, clicking on D20 we see (Fig. 11.12)

Where $f(x) = 5 \times \text{D10} + 7 \times \text{D12} + 4 \times \text{D14} + 8 \times \text{D16}$.

This corresponds to the available hours of UNSKILLED WORKERS: $5 \times P_{1UW} + 7 \times P_{2UW} + 4 \times P_{3UW} + 8 \times P_{4UW} \leq 6{,}000$.

As you see, again we have written the equation but we have not included the constraint that limits us to 6,000 h. This will be considered in the Solver tool box that will be shown later.

Now that we have written the mathematical formulation of the problem in a Microsoft Excel spreadsheet, we can explain how to use the Solver tool box to obtain the optimum solution, in this case a maximum.

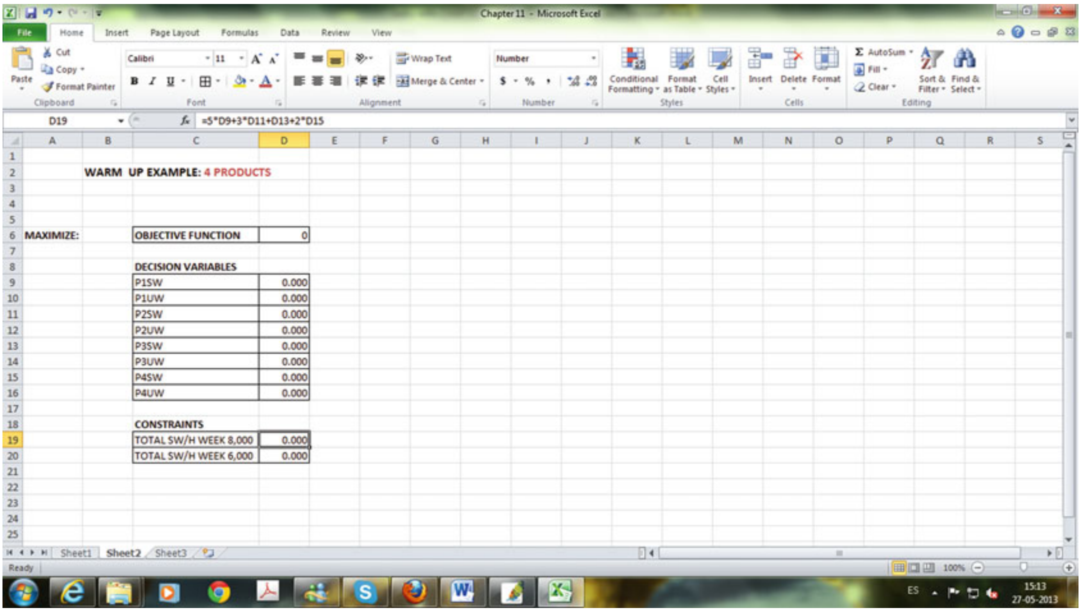



**Fig. 11.11** Microsoft Excel screen with the first constraint written in cell D19

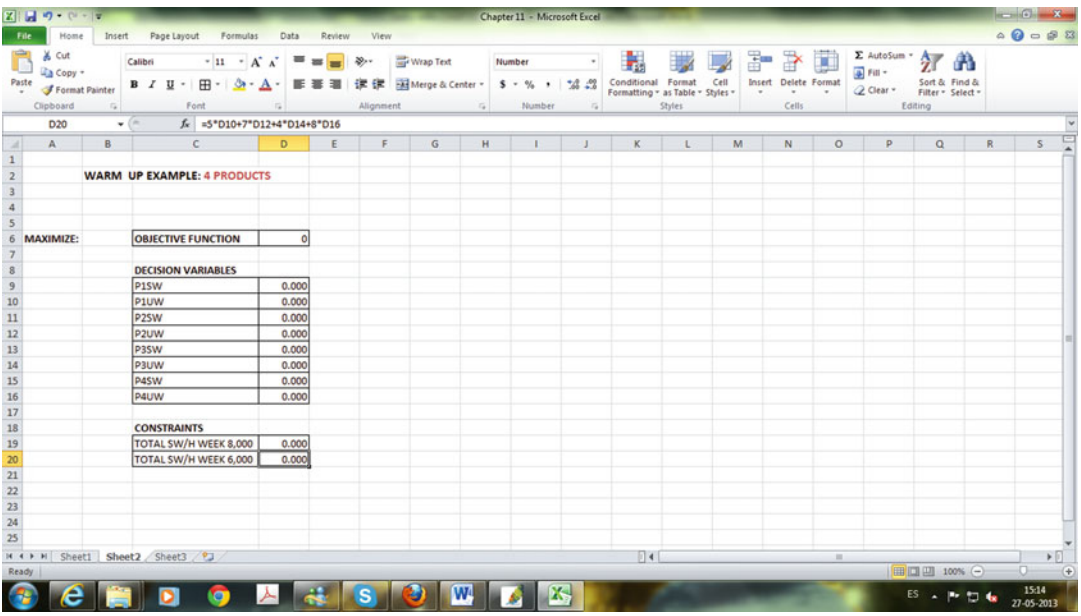



**Fig. 11.12** Microsoft Excel screen with the second constraint written in cell D20

In Excel spreadsheet, click Data in the tool bar. On the right-hand side will appear Solver, as shown in the following image (Fig. 11.13):

If Solver does not appear, you need to install (see appendix 1 of this chapter).

Now, clicking in Solver, you will get (Fig. 11.14)

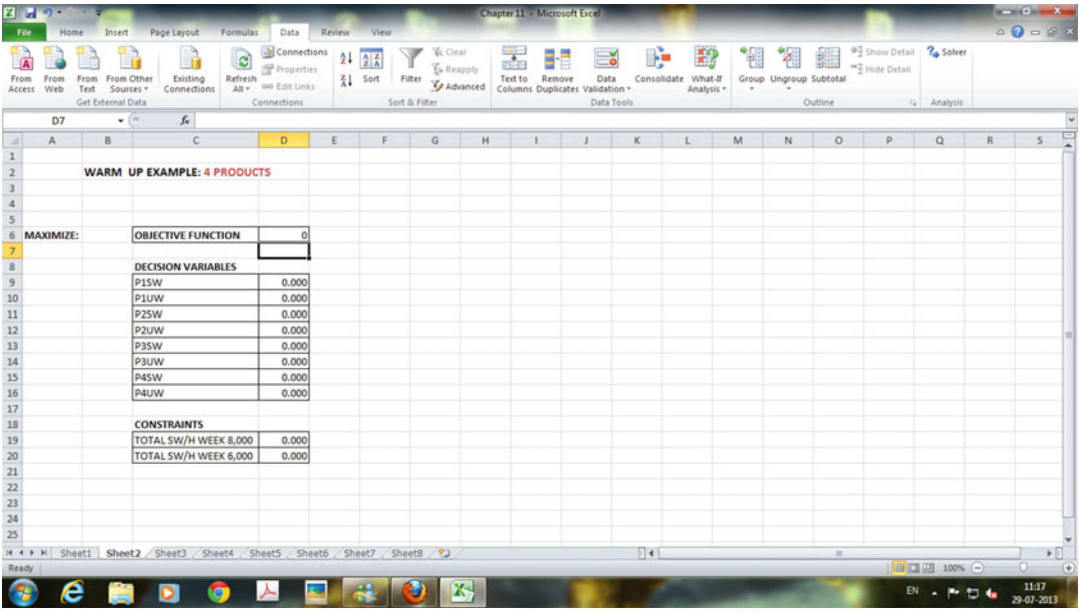


**Fig. 11.13** Microsoft Excel screen showing Solver in the tool bar

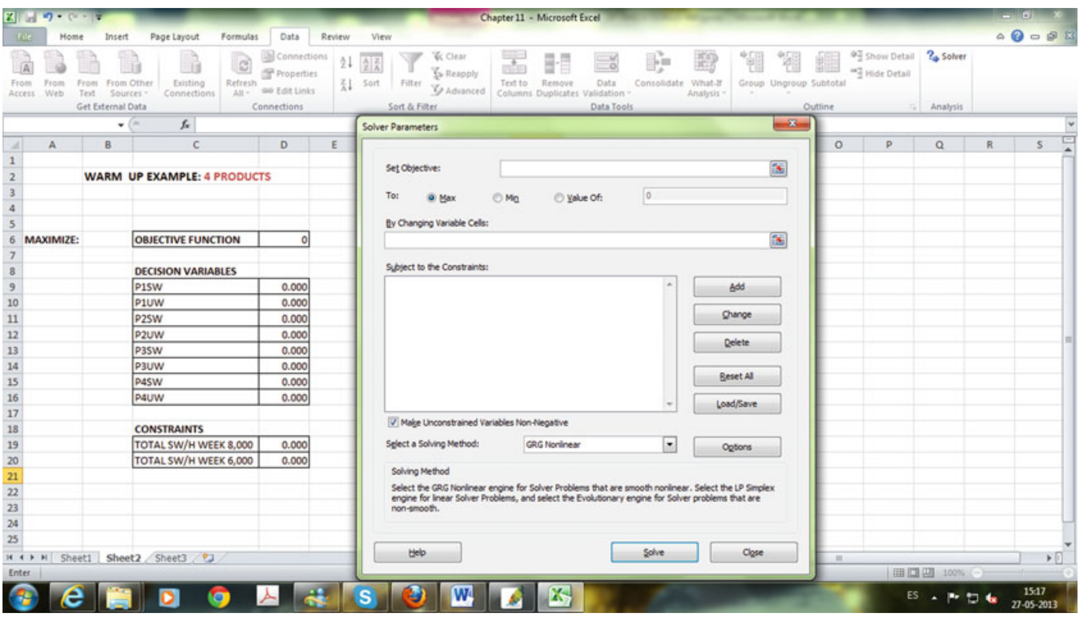


**Fig. 11.14** Microsoft Excel screen showing Solver window

Let us see now what the Solver window will look like after the completion of all data (Fig. 11.15):

As you can appreciate, cell D6 has been set as the objective function, cells D9–D16 have been set as the decision variables, and we have included both constraints in cells D19 and D20. How do we get to this screen?

Initially, the Solver screen will look like this (Fig. 11.16):

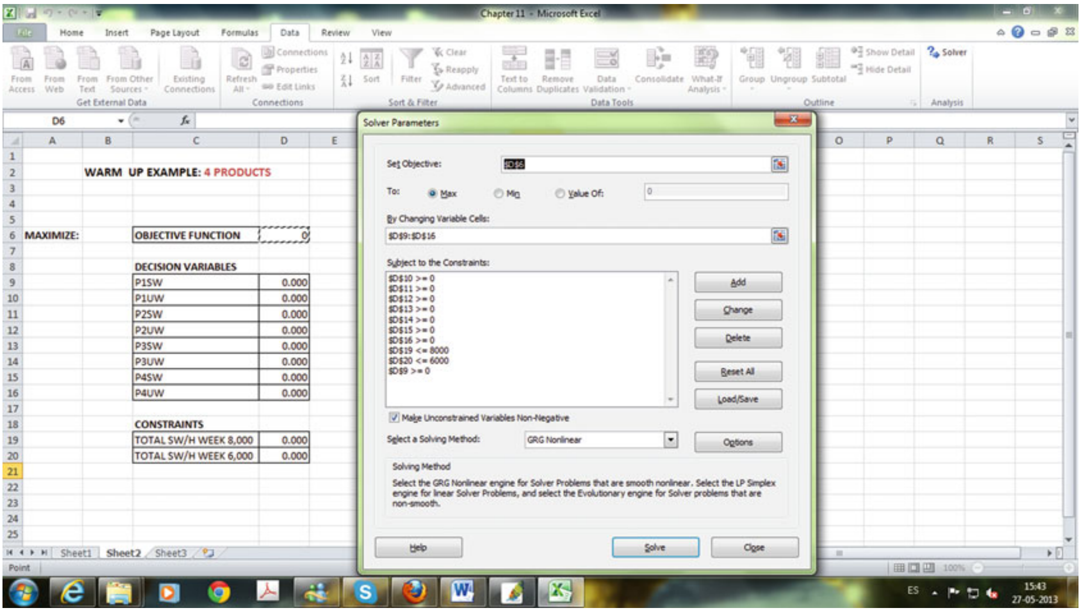


**Fig. 11.15** Solver window showing cell D6 as the objective function, Cells D9 through D16 as the decision variables and constraints in Cells D19 and D20

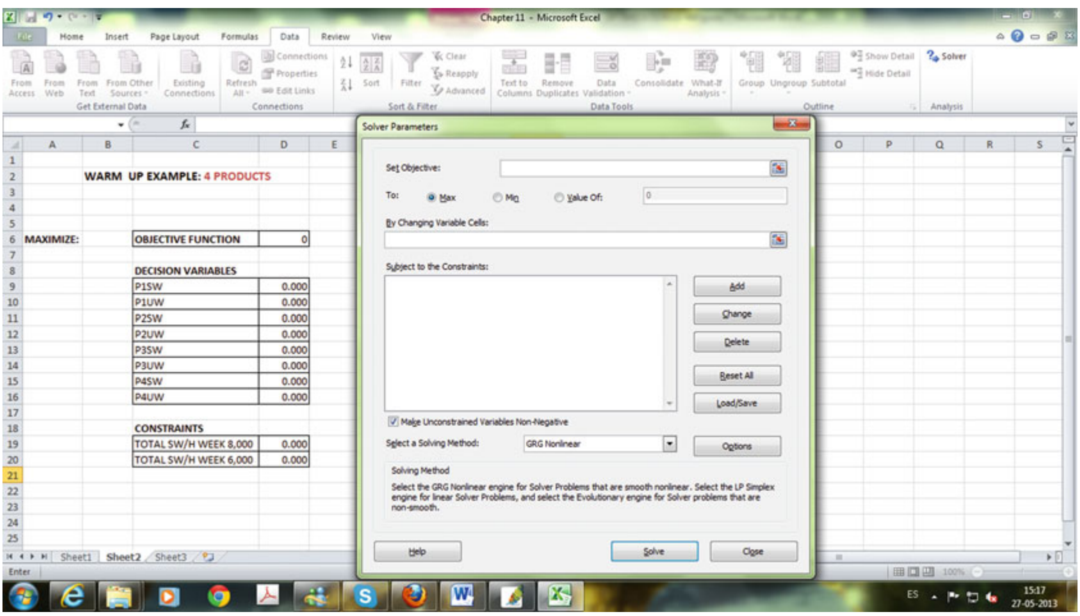


**Fig. 11.16** Starting with the Solver window

The first step is to click on D6 to include the objective function in the Solver screen and, in addition, select **Max** (because we are maximizing). Then the screen will look like this (Fig. 11.17):

The second step is to include the decision variables (D9–D16). Therefore, you need to write D9: D16 below the statement that says **By Changing Variable Cells**, and the screen will look like this (Fig. 11.18):

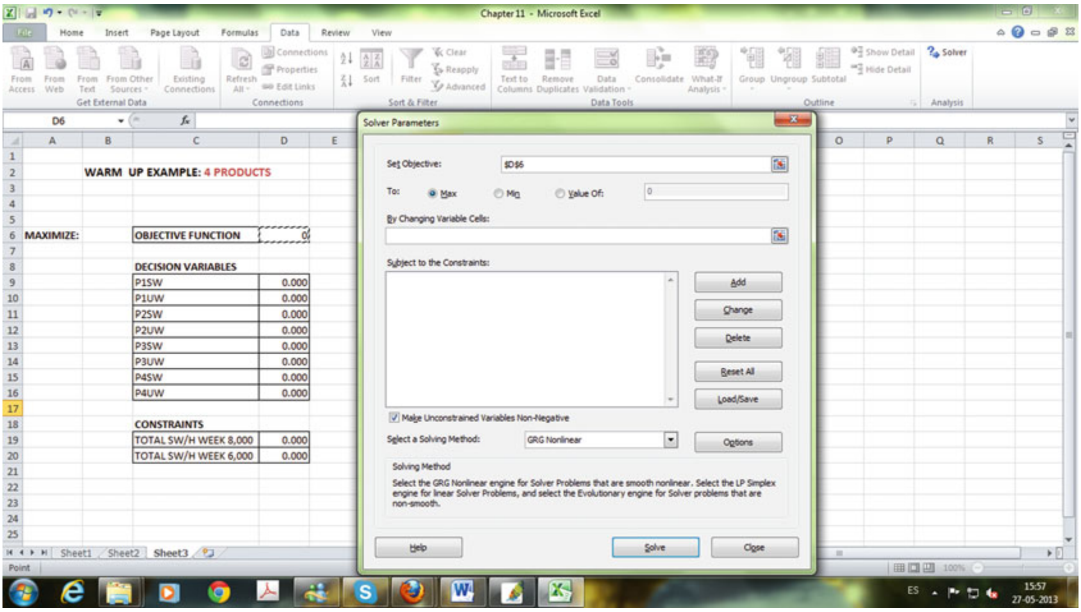



**Fig. 11.17** Solver window with the addition of the objective function (D6)

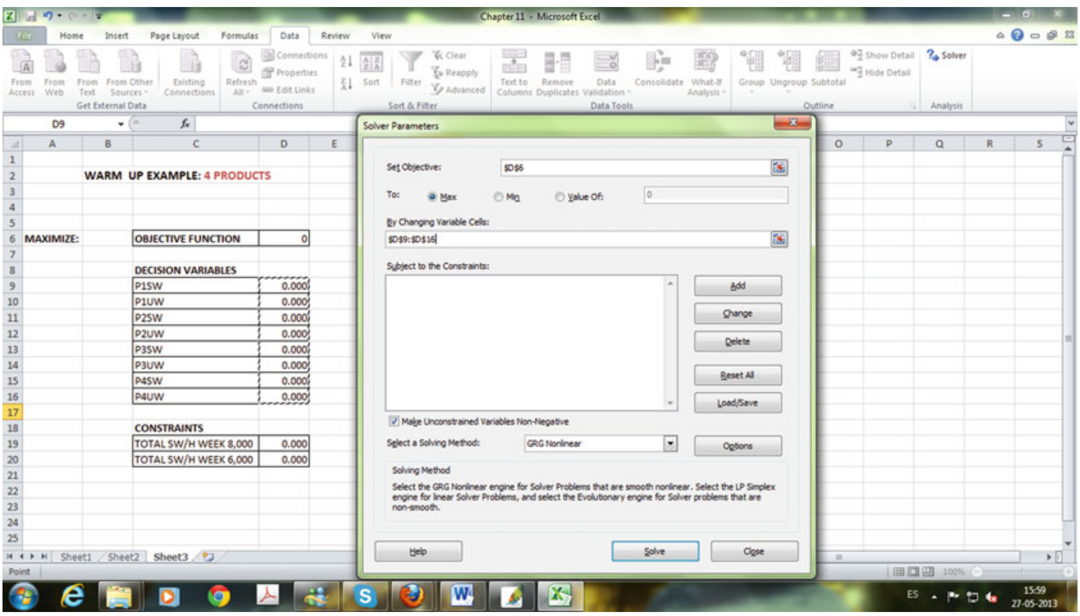



**Fig. 11.18** Solver window with the addition of the changing variables (Cells D9 through D16)

The third step in completing the Solver window is to include all the constraints. First, we have the constraints that limit the total hours for skilled workers (8,000) and the total hours for unskilled workers (6,000). Then clicking Add in the Solver screen we get (Fig. 11.19)

Therefore, to include the first constraint, we need to click on D19 and select $\leq$ and 8,000 under the Constraint, and the screen should look like this (Fig. 11.20):

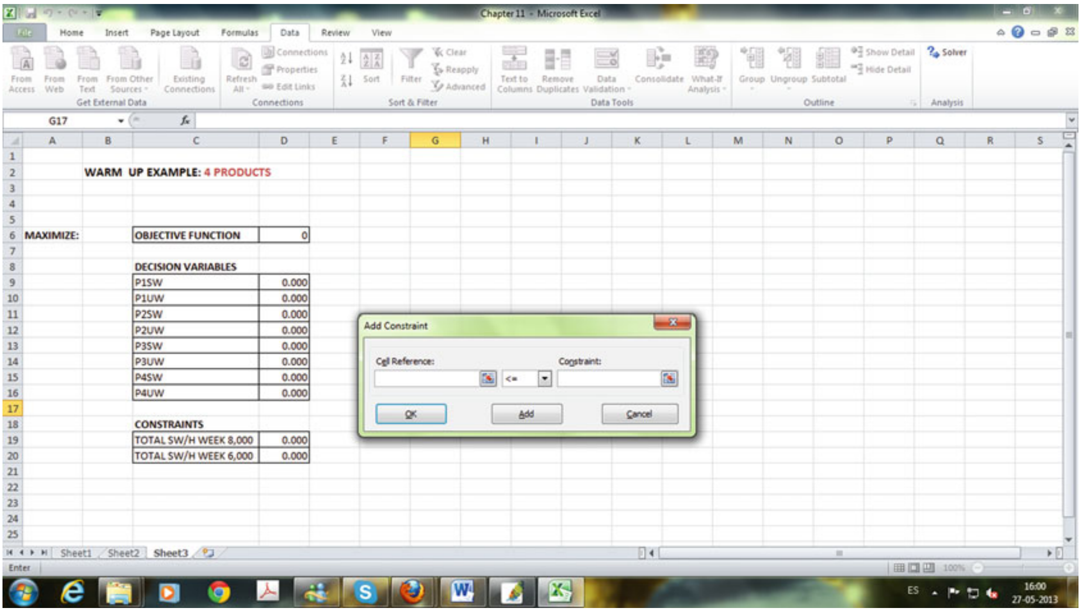



**Fig. 11.19** Solver window to for the addition of all the constraints

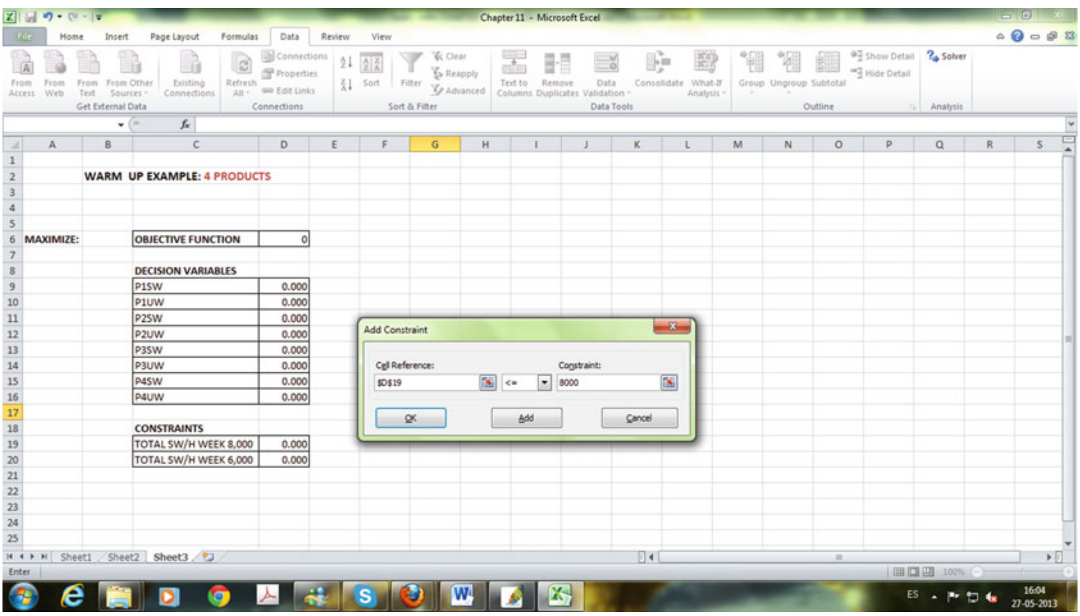



**Fig. 11.20** Solver window showing the constraints that limit the total hours for skilled workers (8,000)

Finally, click on OK to get (Fig. 11.21)

In the same way you need to add all the constraints, including D20 $\leq$ 6,000 and the cells for $P_{1\mathrm{SW}}$, $P_{1\mathrm{UW}}$, …, $P_{4\mathrm{UW}}$ (D9–D16 should be $\geq$0). Finally, the Solver screen should look like the following image (Fig. 11.22):

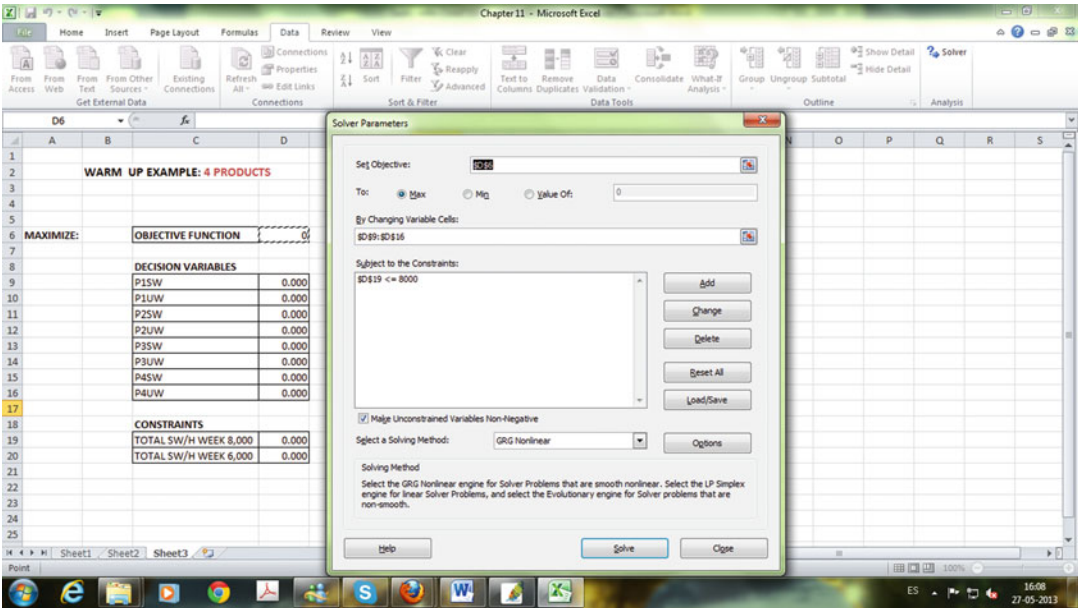



**Fig. 11.21** Solver window with the addition of the first constraint

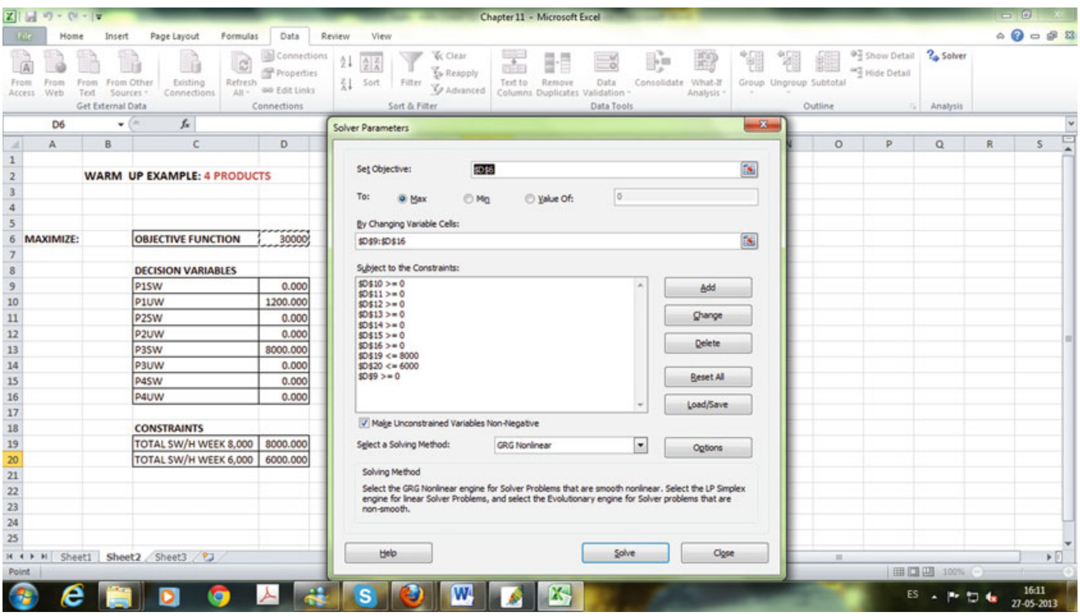



**Fig. 11.22** Solver window with the addition of all the constraints, including that D20 ≤ 6,000 and the cells for $P_{1SW}$, $P_{1UW}$, . . ., $P_{4UW}$ (D9 through D16 should be ≥ 0)

Now that we have included the objective function, selected Max (for maximization), included all the decision variables (D9–D16), and included all the constraints, we can solve the problem by clicking Solve to get the following screen (Fig. 11.23):

Indicating that Solver has found a solution that satisfies all the constraints and the optimality conditions. Therefore, we can click Ok to get (Fig. 11.24)

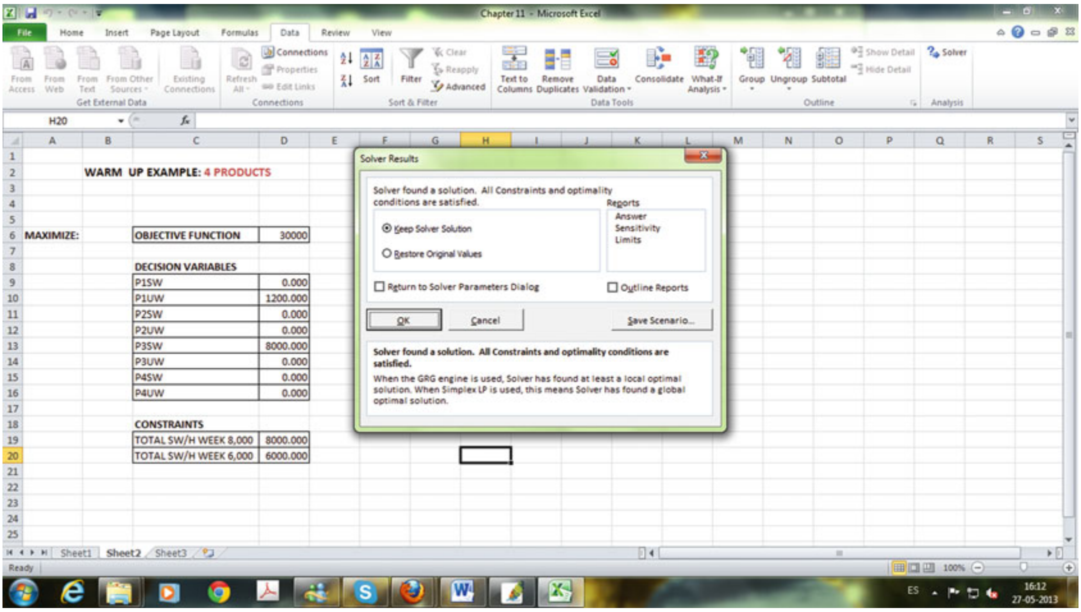



**Fig. 11.23** Solver window showing that a solution has been found

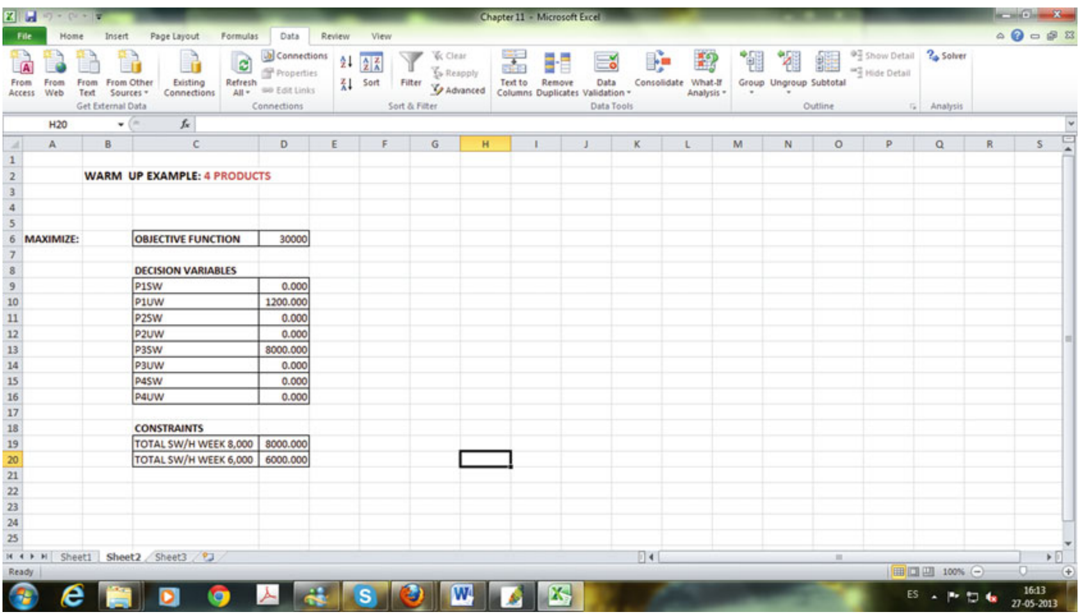



**Fig. 11.24** Final screen depicting the solution found by Solver tool

According to this screen the optimum solution is

**Objective function** = 30,000 and

$P_{1\text{SW}} = 0$

$P_{1\text{UW}} = 1{,}200$ units of product 1 manufactured by unskilled workers

$P_{2\text{SW}} = 0$

$P_{2UW} = 0$
$P_{3SW} = 8{,}000$ units of product 3 manufactured by skilled workers
$P_{3UW} = 0$
$P_{4SW} = 0$
$P_{4UW} = 0$

Finally, we can check that both restrictions are satisfied and all the working hours are used; thus, we have no slack in labor resources. In the same way we can verify that with the number of units to be manufactured, the value of the objective function is 30,000.

## 11.7 Solved Problems

In this section we present and develop three maximum and minimum problems and two OR problems. In both cases the emphasis will be on problem formulation because in the warm-up examples we already detailed the solution procedure with the help of the Excel spreadsheet.

### 11.7.1 Maximum, Minimum, and Applications

**1. Maximum or minimum? [3]**. (a) Determine whether the following function [$f(x)$] presents a maximum or a minimum. (b) What is the value of $x$ that maximizes or minimizes the $f(x)$ function?

$$f(x) = 2x^2 - 5x + 3.$$

**Solution**
**(a) Graphical solution**
As suggested in the warm-up example, we will first plot a graph to find the shape of the curve and a tentative solution. Then we will use the Solver tool to confirm the graphic solution or get a more accurate solution.

Plotting $f(x)$ for x values from 0 to 2 (Fig. 11.25).

According to the graphical solution, $f(x)$ presents a **minimum** value for $x = 1.25$.

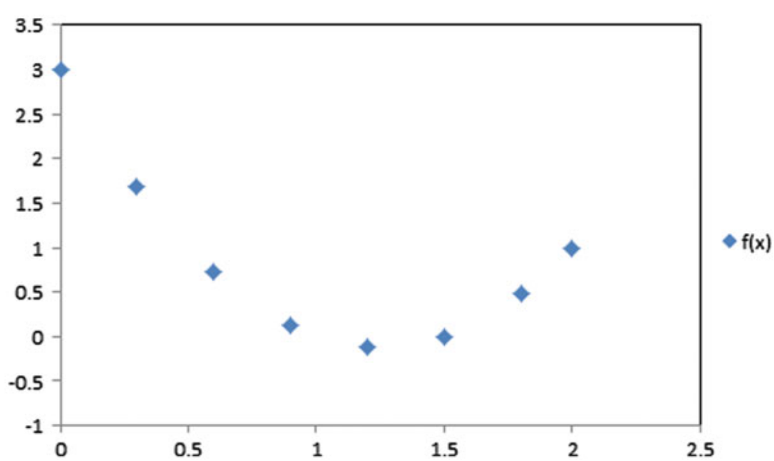



**Fig. 11.25** Plotted values for the function $f(x) = 2x^2 - 5x + 3$

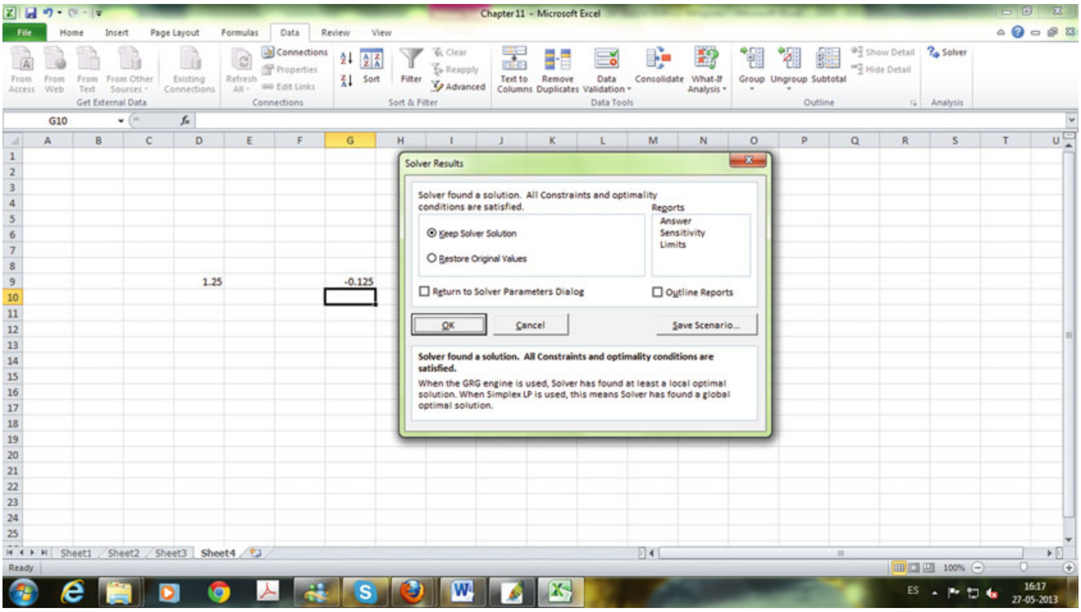



**Fig. 11.26** Solver window for the minimum value of $x$ in function $f(x) = 2x^2 - 5x + 3$ ($x = 1.25$ and $f(x) = -0.25$)

**(b) Solver**

As explained in the warm-up example, we first choose a cell for $x$ (D9) and assign it a value of 0. Then we write the function in cell G9. Now click on Data and then Solver. In the Solver tool box we set the objective function (G9), choose minimum, Min, then in By Changing Variable Cells, we select D9 and finally click on Solve to get (Fig. 11.26)

Showing the same result as was obtained with the graphical solution, x = 1.25 and $f(x) = -0.25$.

**2. Book edition [5]**. An important publisher is planning to edit a new and innovative book that combines chemical and bioprocess engineering. After a careful cost and marketing study the production manager reports to the general manager the following information:

$$Q = 3{,}000 - 20P, \tag{11.2}$$

$$C = 4{,}000/Q + 18. \tag{11.3}$$

$Q$: Quantity of books manufactured
$P$: Selling price per each book ($)
$C$: Cost per book ($)

Then the general manager decides to produce 1,500 books at a price of $75 and a cost of $20.67, giving the company profits of $81,500. What do you think? Is the general manager taking the optimum decision?

**Solution**

**Step I**

Graphical representation and mathematical formulation

In this case it is not necessary to do a graphical representation of the problem statement, but we need a mathematical formulation. The objective should be to maximize company profits, where we can define the profits ($U$) as follows: *Profits* = *input* − *output*. Therefore, we can write $U = PQ - CQ$.

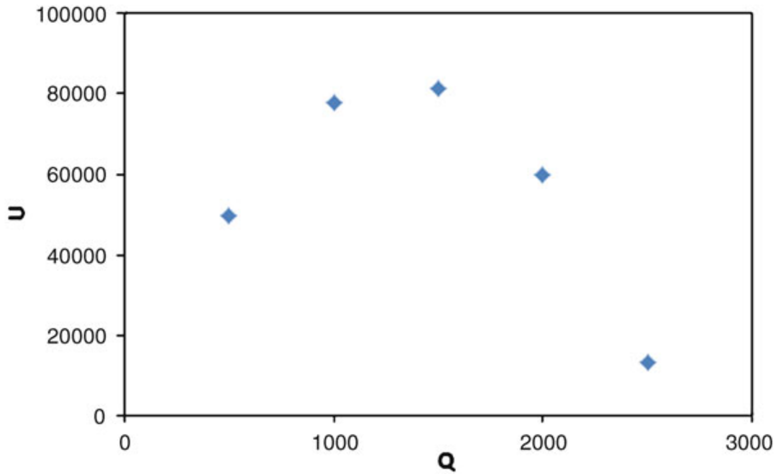



**Fig. 11.27** Plotted values for $U$ (profits) against $Q$ (quantity of books manufactured)

Then from (11.2) and (11.3) and rearranging we get

$$U = \left(\frac{(3,000 - Q)}{20}\right)Q - \left(\frac{4,000}{Q} + 18\right)Q,$$

$$U = 132Q - \frac{Q^2}{20} - 4,000. \tag{11.4}$$

**Step II**

(a) Graphical solution

A plot of $U$ (profits) against $Q$ (quantity of books made) is depicted in the following graph (Fig. 11.27):

According to the graph the optimum value is at $Q = 1{,}500$. Refining the graph we get (Fig. 11.28)

Now we can see that the optimum is approximately $Q = 1{,}300$ and $U = 83{,}100$, suggesting that the general manager's decision was not optimal. Before coming to a final opinion and solution for the problem we will solve it with the Solver tool.

(b) **Solver**

We assign $Q$ to cell F9 (with a starting value of 1,000) and write the objective function ($U$) in cell H9. Then in the Solver tool box we set H9 as the objective function, Max for maximization, and F9 as a changing variable cell. Then the screen of the spreadsheet will look like this (Fig. 11.29):

Clicking in Solve we get (Fig. 11.30)

Where the optimum value is $Q = 1{,}320$ and $U = 83{,}120$, which is better than our approximately graphical solution and confirming that the general manager's decision was not optimal.

3. **Box design [5]**. A fine chemical (4 [L]) should be shipped in a special box. The material to manufacture these special parallelepiped boxes comes in a sheet of the following dimensions: length 50 [cm], width 30 [cm] (Fig. 11.31). (a) What value of $x$ maximizes the volume of the box? (b) What is the volume of the box? Use both graphical and Solver procedures to solve the problem.

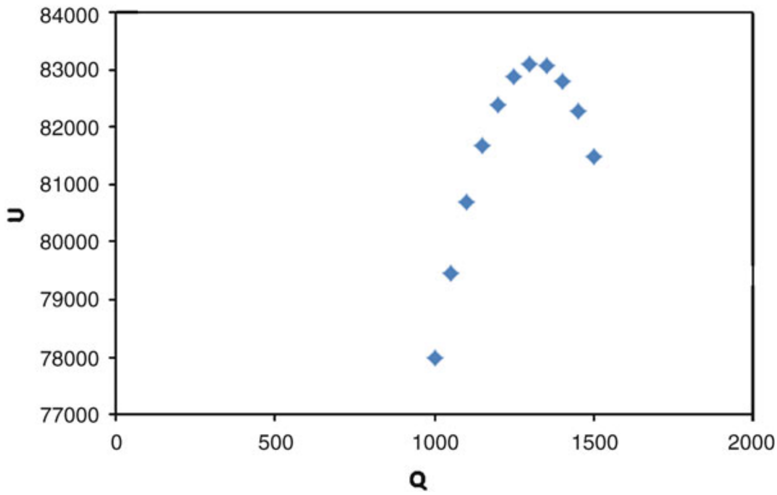



**Fig. 11.28** Refined plotted values for $U$ (profits) against $Q$ (quantity of books manufactured)

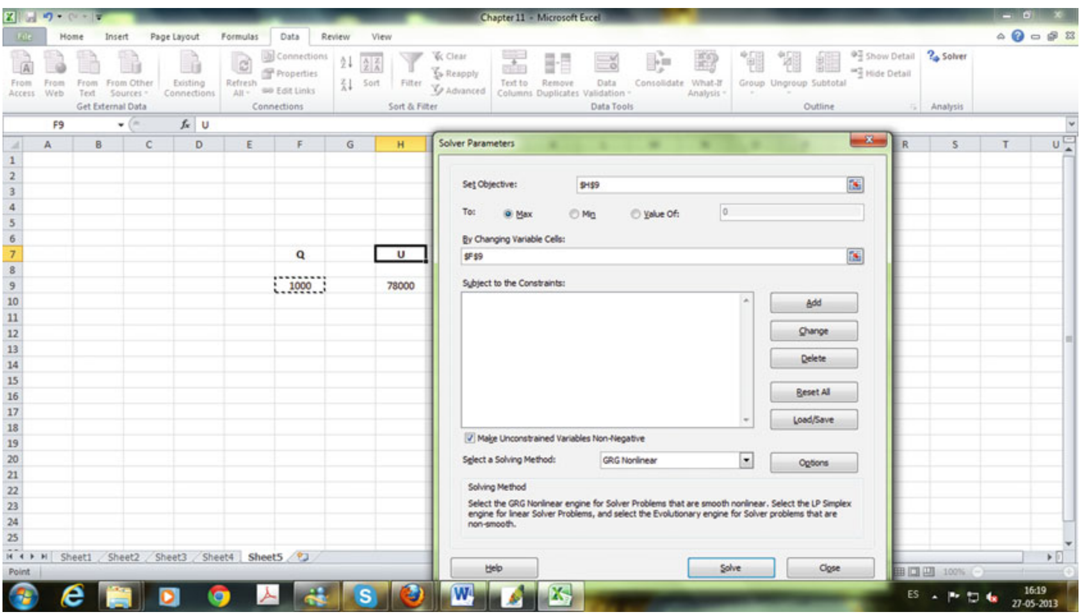



**Fig. 11.29** Solver window where cell F9 has been assigned for $Q$ (with a starting value of 1,000) and the objective function ($U$) in cell H9

**Solution**

**Step I**

Graphical representation and mathematical formulation

According to Fig. 11.32, the volume of the box as a function of $x$ is equal to

$$V = (50 - 2x)(30 - 2x)x \quad \text{or} \quad V = 4x^3 - 160x^2 + 1,500x.$$

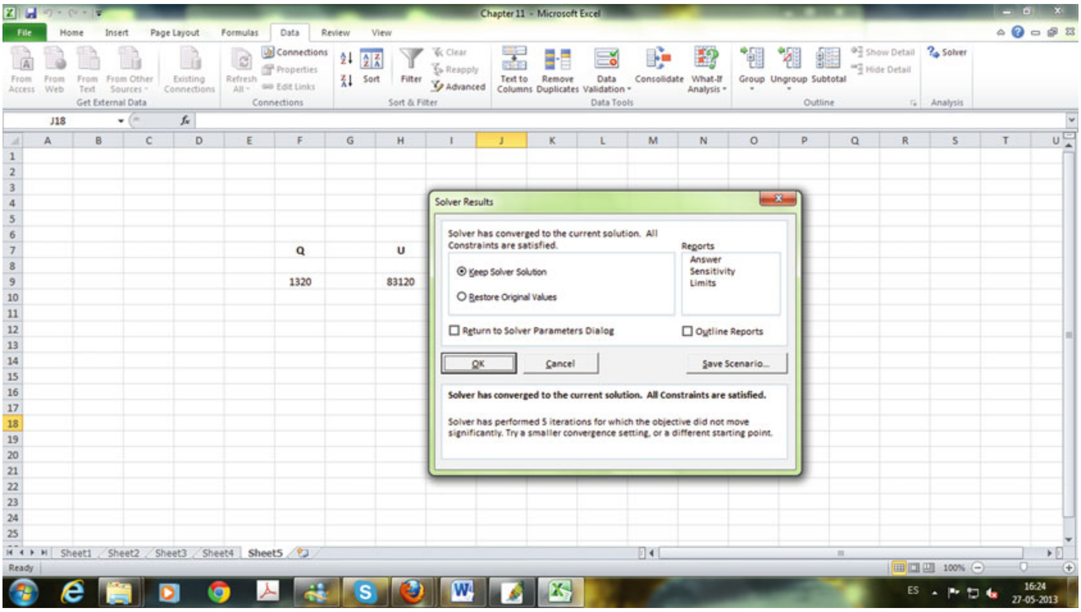
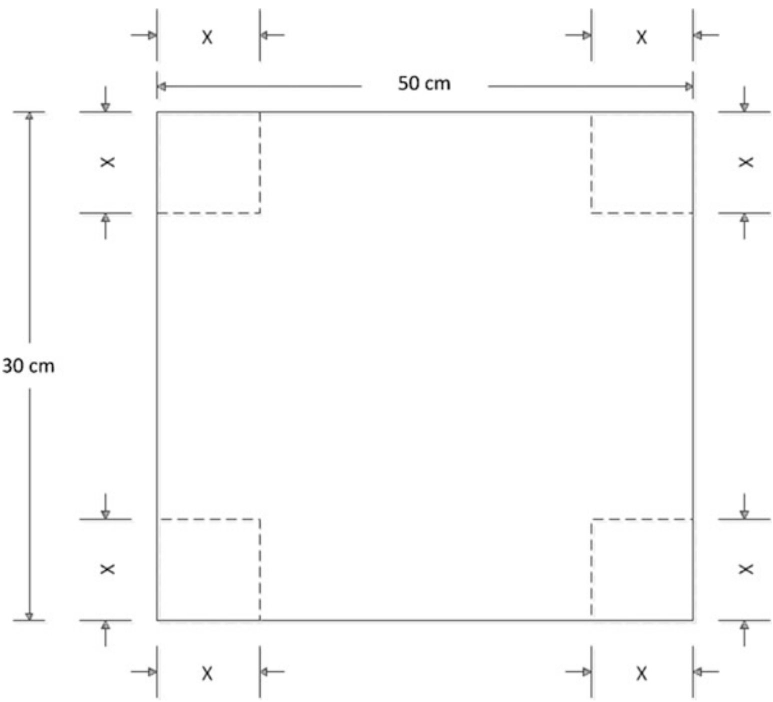



**Fig. 11.30** Final screen depicting the solution found by Solver tool

**Fig. 11.31** Parallelepiped box from a sheet with the following dimensions: length 50 cm and width 30 cm

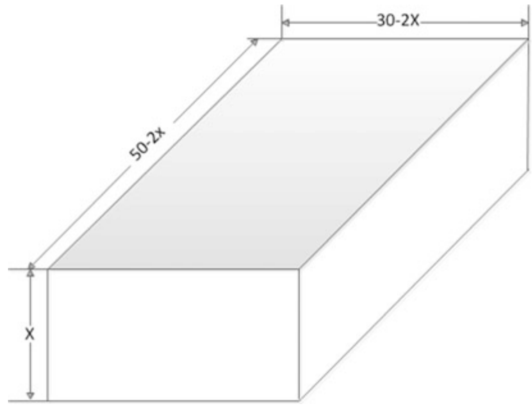



**Step II**

(a) Graphical solution

Before creating a graph, by inspection we can see that the value of $x$ is in the range of 0–15. Then creating a table for $x$ and $V$ taking values of $x$ from 0 to 15, we get

**Fig. 11.32** Parallelepiped box




| $x$ [cm] | $V$ [cm$^3$] |
|---|---|
| 0 | 0.0 |
| 1 | 1,344 |
| 2 | 2,392 |
| 3 | 3,168 |
| 4 | 3,696 |
| **5** | **4,000** |
| **6** | **4,104** |
| **7** | **4,032** |
| 8 | 3,808 |
| 9 | 3,456 |
| 10 | 3,000 |
| 11 | 2,464 |
| 12 | 1,872 |
| 13 | 1,248 |
| 14 | 616.0 |
| 15 | 0.0 |

Then the maximum volume of the box is for 5 [cm] $< x <$ 7 [cm]. After several data refinements we get

| $X$ [cm] | $V$ [cm$^3$] |
|---|---|
| 6.00 | 4,104.0 |
| 6.02 | 4,104.2 |
| 6.04 | 4,104.3 |
| **6.06** | **4,104.4** |
| 6.08 | 4,104.4 |
| 6.10 | 4,104.3 |

Then, according to this approximate procedure, a value of $x = 6.06$ [cm] will maximize the volume of the box (approximately 4.1 [L]).

(b) Solver

We assign $x$ to cell E5 [with a starting value of 6, following the result obtained in (a)], and we write the objective function ($V$) in cell G5. Then in the Solver tool box we set G5 as the objective function,

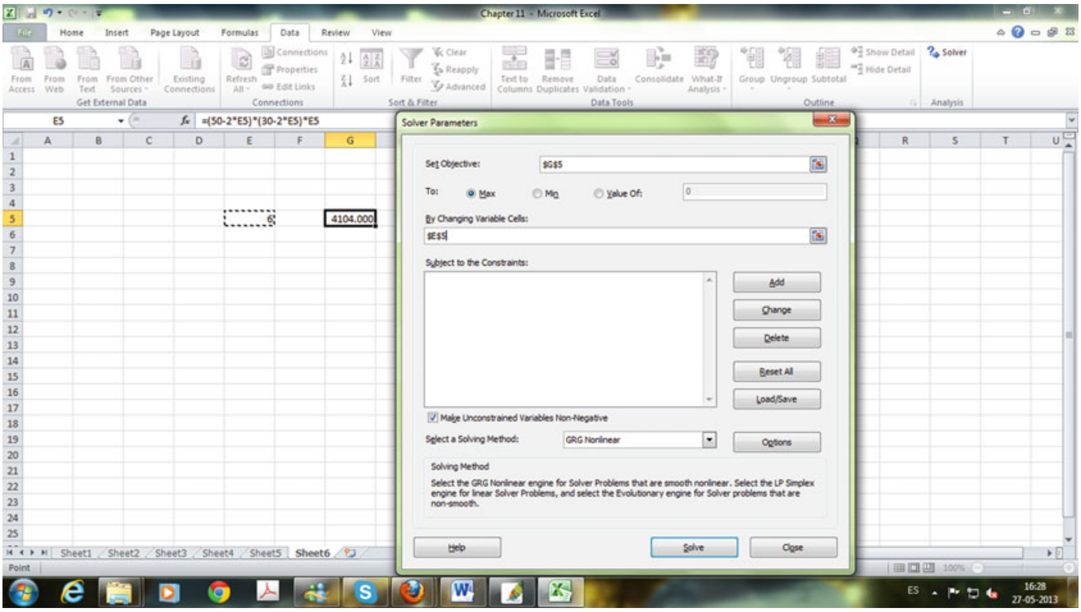



**Fig. 11.33** Solver window where cell E5 has been assigned for $x$ (with a starting value of 6)

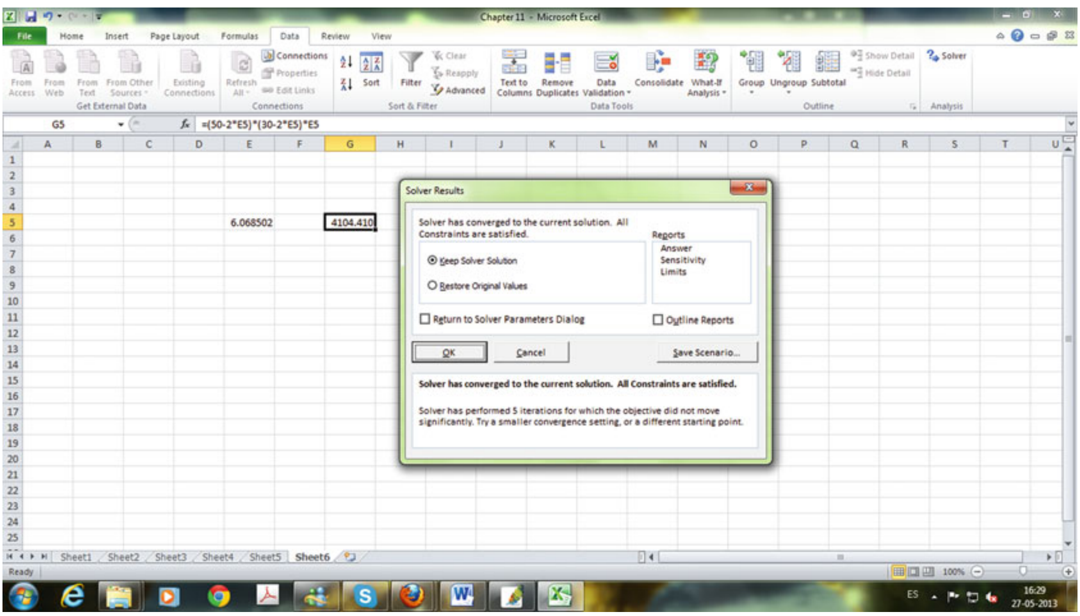



**Fig. 11.34** Final screen depicting the solution found by Solver tool

Max, for maximization and E5 as a changing variable cell. Then the screen of the spreadsheet will look like this (Fig. 11.33):

Clicking on Solve we get (Fig. 11.34) where the optimum value for $x$ is 6.0685 [cm] and $V = 4{,}104.4$ [$cm^3$]. Again the approximate solution given in (a) is close to the "exact" solution obtained with the Solver tool. Solver is not only more accurate but also simpler, although the graphical (or table) solution is much more visually friendly.

### 11.7.2 Operations Research Problems

**4. Bakery [6]**. A bakery has 140 hundredweight of flour to make two types of bread, premium and regular. Regular bread costs \$35 per hundredweight, while premium bread costs \$65 per hundredweight. The bakery has available \$7,200 of capital to make bread. Each hundredweight of regular bread requires 1 h of work, and each hundredweight of premium bread requires 2.5 h. The maximum hours available for work are 330. If the bakery expects to make a profit of \$130 per hundredweight of regular bread and \$180 per hundredweight of premium breads, how many hundredweight go to regular and premium bread, respectively, to maximize the benefit to the bakery?

**Solution**

**Step I**

**Variable definition and codification**

$R$: Hundredweight of flour for regular bread
$P$: Hundredweight of flour for premium bread
$B$: Total benefit to bakery in dollars

**Step II**

**Formulation of objective function**

The total benefits ($B$) will be $B = 130 \times R + 180 \times P$.

**Step III**

**Constraints**

We have three constraints, 140 hundredweights of flour, \$7,200 of capital, and 330 h of labor:

| | |
|---|---|
| Flour (hundredweights): | $R + P \leq 140$, |
| Capital [\$]: | $35 \times R + 65 \times P \leq 7{,}200$, |
| Labor [h]: | $R + 2.5 \times P \leq 330$. |

In addition, all decision variables should be integers and equal to or greater than 0 ($R$ and $P$ integers and $\geq 0$).

**Step IV**

As detailed and explained in the warm-up example (11.5.5) we will use the Solver tool from Microsoft Excel. The following screen shows the objective function (cell L6), the variables $R$ and $P$ (cells I9 and I10, respectively), and the constraints for Flour, Capital and Hundredweights (cells D9, D10 and D11 respectively). We tentatively start with initial values of $R = 50$ and $P = 30$ (Fig. 11.35).

Then, as shown in the following screen, in the Solver tool box we include objective function, changing variables and constraints (Fig. 11.36).

Clicking on Solve we obtain (Fig. 11.37) where the optimum solution is $R = 63.3$ hundredweight for regular bread and $P = 76.7$ hundredweight for premium bread, and with a total benefit of $B = \$22{,}033$.

**5. Catalyst [5]**. Chemical reactions must be catalyzed, and it has been found that the best combination of catalysts is to use at least 1.5 [mg] of catalyst $C_1$ and 8.5 [mg] of catalyst $C_2$. Two products ($P_1$ and $P_2$) are sold that contain these two catalysts. The following table provides information on the concentrations of catalysts $C_1$ and $C_2$ per gram of each product and, in addition, the cost of these products. Because the products are very expensive, you are asked to determine how many grams must be purchased of each product to minimize the total cost?

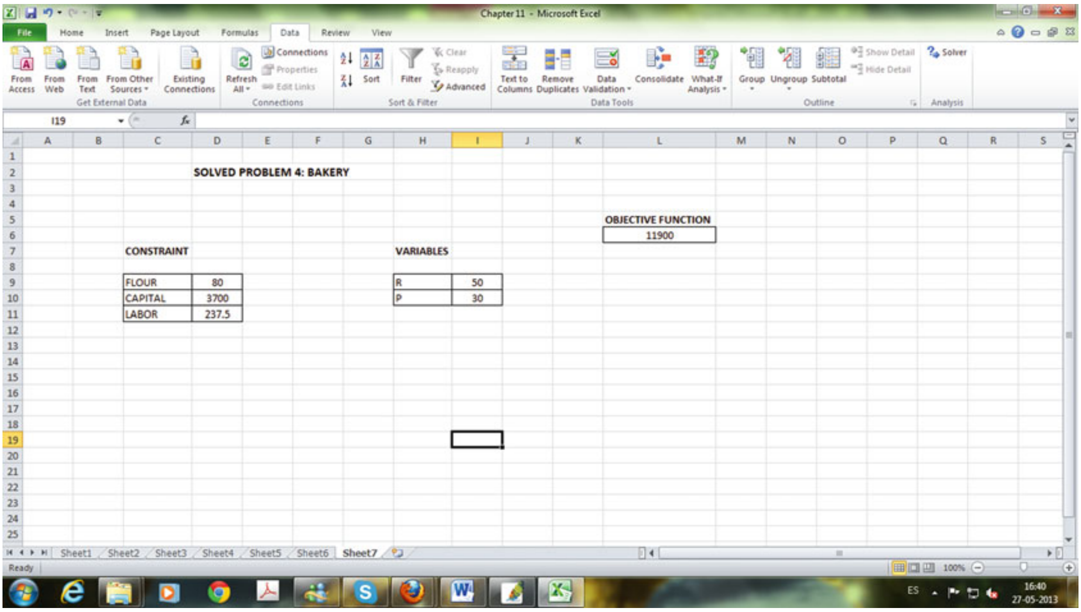



**Fig. 11.35** Microsoft Excel screen showing the Objective Function (cell L6), the variables $R$ and $P$ (cells I9 and I10 respectively) and the constraints for Flour, Capital and Hundredweights (cells D9, D10 and D11 respectively)

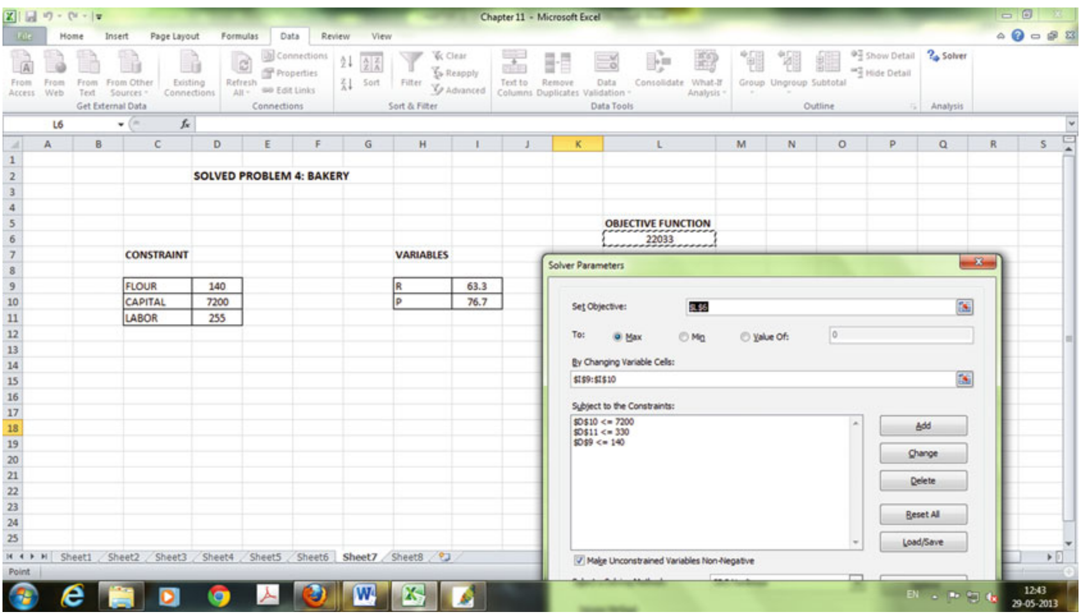



**Fig. 11.36** Solver tool box where variables $R$ and $P$ should be integer (I9 and I10 = integer)

| | Catalyst $C_1$ [mg/g] | Catalyst $C_2$ [mg/g] | Price [$/g] |
|---|---|---|---|
| Product $P_1$ | 0.12 | 0.4 | 200 |
| Product $P_2$ | 0.13 | 1.4 | 240 |

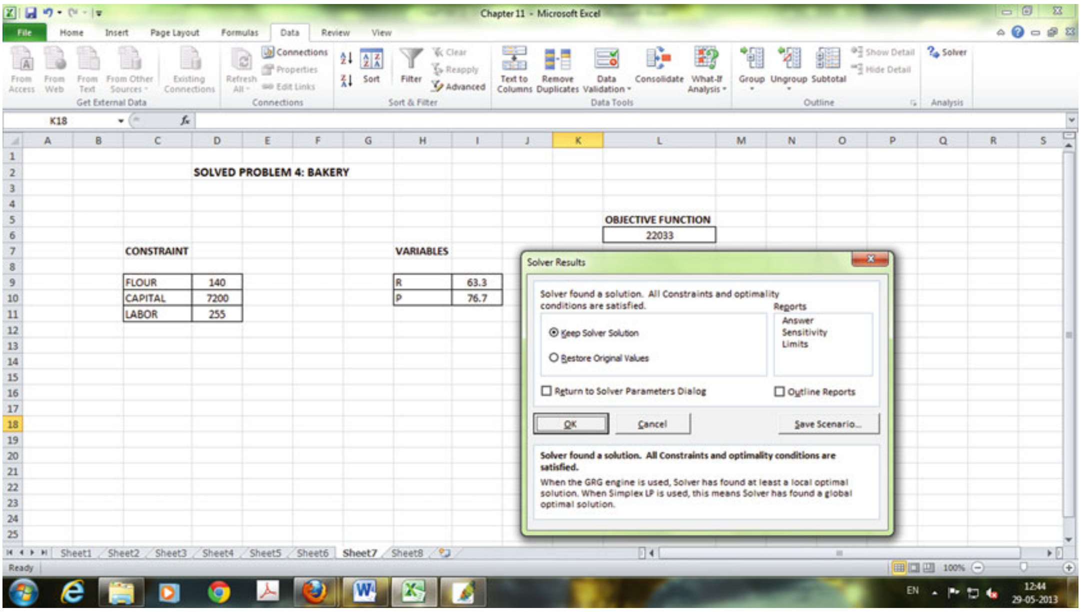



**Fig. 11.37** Final screen depicting the solution found by Solver tool

**Solution**

**Step I**

**Variable definition and codification**

$G_{P1}$: Grams of product $P_1$ [g]

$G_{P2}$: Grams of product $P_2$ [g]

TC: Total cost [$]

**Step II**

**Formulation of objective function**

The total cost (TC) will be TC $= 200 \times G_{P1} + 240 \times G_{P2}$.

**Step III**

**Constraints**

We have two constraints, 1.5 [mg] of catalyst $C_1$ and 8.5 [mg] of catalyst $C_2$:

CATALYST $C_1$ : $0.12 \times G_{P1} + 0.13 \times G_{P2} \geq 1.5$,

CATALYST $C_2$ : $0.4 \times G_{P1} + 1.4 \times G_{P2} \geq 8.5$.

In addition, all decision variables should be equal to or greater than 0 ($G_{P1}$ and $G_{P2} \geq 0$).

**Step IV**

As detailed and explained in the previous example, we will use the Solver tool from Excel. The following screen shows the objective function (cell J5), the variables $G_{P1}$ and $G_{P2}$ (cells I9 and I10, respectively), and the constraints for catalyst $C_1$ and catalyst $C_2$ (cells D11 and D12, respectively). We tentatively start with initial values of $G_{P1} = 8$ [g] and $G_{P2} = 5$ [g] (as shown in the screenshot, this is a feasible solution) (Fig. 11.38):

Then, as shown in the following screen, in the Solver tool box we include the objective function (J5), Min to minimize, I9 and I10 as the changing cells and constraints indicating that catalyst $C_1$ should be $\geq$1.5 [mg] and catalyst $C_2$ should be $\geq$8.5 [mg] (Fig. 11.39):

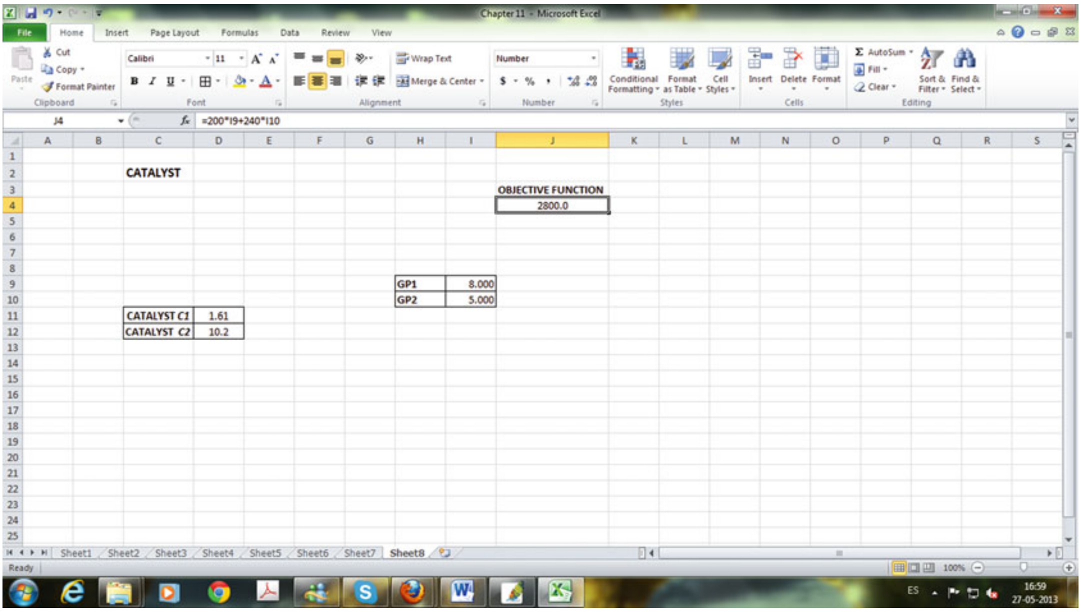



**Fig. 11.38** Microsoft Excel screen showing the Objective Function (cell J5), the variables $G_{P1}$ and $G_{P2}$ (cells I9 and I10 respectively) and the constraints for catalyst $C_1$ and catalyst $C_2$ (cells D11 and D12 respectively)

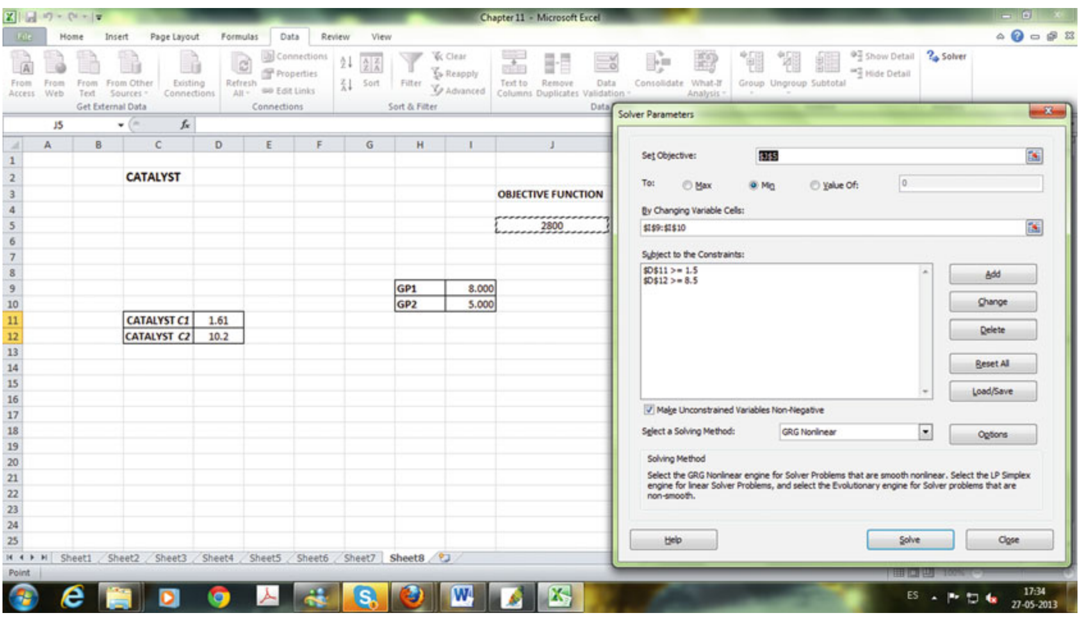



**Fig. 11.39** Microsoft Excel screen including the Solver window

Clicking on Solve we obtain (Fig. 11.40)

Where the optimum solution is $G_{P1} = 8.578$ [g] and $G_{P2} = 3.621$ [g] and the minimum total cost is TC $=$ \$2,584.5.

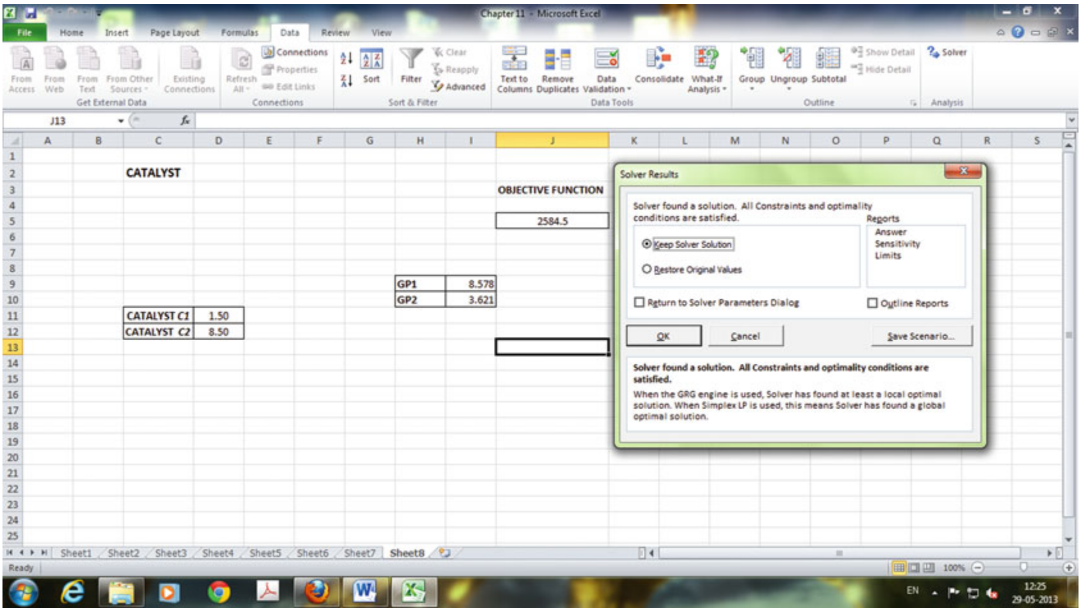



**Fig. 11.40** Final screen depicting the solution found by Solver tool

## 11.8 Proposed Problems

### 11.8.1 Maximum, Minimum, and Applications

1. **Maximizing and minimizing $Y$ [4]**. Check for different values of $a$, $b$, and $c$ that for the following equation the value of $X$ that maximizes $Y$ is equal to $-b/2a$ when $a$ is negative and minimizes $Y$ when $a$ is positive:

$$Y = aX^2 + bX + c.$$

2. **Finding $X$ [3]**. Find the value of $X$ that minimizes the value of $Y$ in the following equations: $(a)\ Y = 5X^2 - 20X + 15;\ (b)\ Y = 7X^2 - 35X + 18;\ (c)\ Y = 9X^2 - 9X + 10$.
   **A:** (a) $X = 2$; (b) $X = 2.5$; (c) $X = 0.5$
3. **Finding $X$ [3]**. Find the value of $X$ that maximizes the value of $Y$ in the following equations: $(a)\ Y = 12 + 6X - 6X^2;\ (b)\ Y = 12 + 10X - 5X^2;\ (c)\ Y = 19 + 36X - 6X^2$.
   **A:** (a) $X = 0.5$; (b) $X = 1$; (c) $X = 3$
4. **Maximum product [4]**. Find two positive real numbers whose sum is 100 and whose product is a maximum.
   **A:** 50 and 50
5. **Airplane [6]**. A cargo plane is arriving in Santiago from southern Chile at a speed of 600 [km/h]. When the cargo plane is 200 [km] from Santiago, a Cessna airplane departs from Santiago to Argentina at a speed of 300 [km/h]. When will the minimum distance be reached between the cargo and the Cessna airplanes after the departure of the Cessna airplane? <u>**Assumption**</u>: The two airplanes are moving at an angle of 90° from each other.
   **A:** After 16 min (0.267 h)

6. **Pencils [5]**. A company that sells pencils is going through a critical time. The manager of the company, Mr. Johnson, decides to hire an engineer to help resolve the problem. Based on an economic analysis, the engineer determines the functions of sales and costs for the company as follows:

$$Q = 5,000 - 100P,$$
$$C = 35,000 + 15Q,$$

where $Q$ is the quantity of pencils produced, $P$ is the selling price of each pencil ($), and $C$ is the total costs of producing $Q$ pencils ($).
Using these equations, the engineer determines that the company is operating in suboptimal conditions and suggests that a new price and quantity should be established by the company. (a) What is the optimum price for the pencils? (b) What is the optimum number of pencils to manufacture? (c) Could the company earn a profit? **Hint**: Profits = Input − Output.
**A:** (a) $17.5; (b) 3,250 pencils; (c) Yes, $21,612.50.
7. **Optimum speed [10]**. The owner of an aircraft wants to start doing commercial trips. For this he needs to hire a pilot, who requires a monthly salary of $10,000 plus an allowance per flight hour of $30. We also know that legally, the pilot cannot fly more than 100 h/month.
For each flight, the plane needs maintenance to verify proper operation, which means an average cost of $2,000 per trip. The aircraft has a fuel consumption that is proportional to the flying speed, where 0.1 + $v$/2,000 [L/km], determined under windless weather conditions, and the price for the fuel is $3/L. The owner plans to do regular trips between Columbus, Ohio, and New York City, a distance of 600 [km]. What is the optimal flying speed on the flight to New York City, given that, on average, the wind has a speed of 15 [km/h] and moves from east to west?
**A:** 205.7 [km/h]
8. **Cartons [$10^+$]**. You are assigned to construct cartons having a volume of 1 [$m^3$]. You have a machine that performs cuts of 0.2 × 0.2 [m] in the corners of the carton. What dimensions of the carton sheet would minimize consumption of cardboard?
**A:** 2.636 [m] each side
9. **Canned food [6]**. You have been charged with designing a cylindrical jar for a new canned food product. Try out different can volumes (e.g., 0.3, 0.5, and 0.8 [L]) whereby the optimal ratio between height ($H$) and diameter ($D$), in order to minimize the amount of material (area including both lids), is when $H = D$.
10. **Storage tank [6]**. You have been assigned to design a stainless steel cylindrical storage tank of 200 [$m^3$] without a lid. What are the optimal dimensions [diameter ($D$) and height ($H$)] of the tank to minimize the amount of stainless steel used?
**A:** $D \sim 4$ [m] and $H \sim 3.98$ [m]
11. **Editorial [7]**. An editorial company wishes to publish a booklet of 25,000 words. According to the technical department, two words occupy a space of 1 [$cm^2$] of printed material. Standard policy of the editorial company is that each page should have margins of 1 in per side and 1.5 in at the top and bottom of the page. If the editorial wants to print the booklet in 200 pages, what are the optimum dimensions of each page in order to minimize the consumption of paper?
**A:** 8.60 and 13.18 [in.]
12. **Riverside [5]**. A chemical plant is located by a river that is 1 [km] wide. In addition, 2 [km] upstream is a power station. A cable needs to be run from the chemical plant to the power station at minimum cost. It costs $4 to place each meter of cable through the land and $5 through the river.

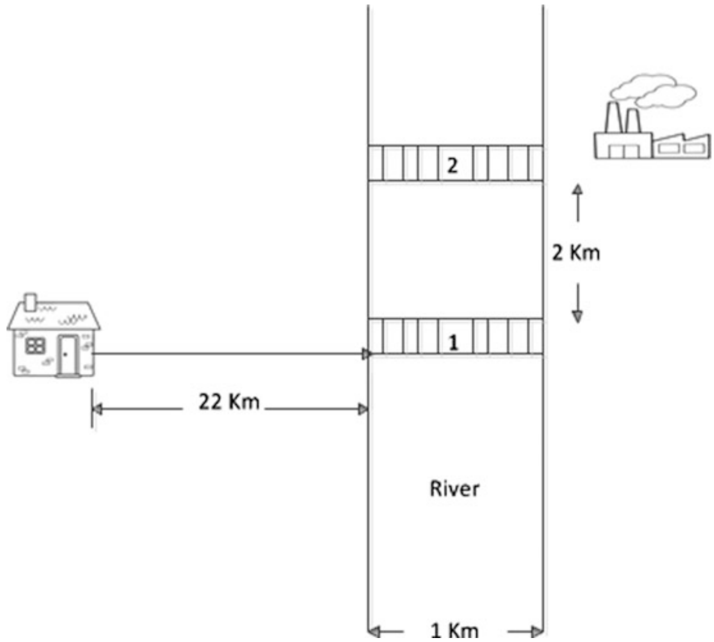



**Fig. 11.41** Schematic representation of the bioprocessing plant that is located at the other side of the river

How much cable should you run through the land and the river? **Assumption**: Ignore the cost of the cable.

**A:** 2/3 [km] through the land and 1.67 [km] through the river

13. **Going to work [5]**. Every day, you take the highway to work at the bioprocessing plant. The plant is located on the other side of the river (Fig. 11.41). Your average speed is 100 [km/h], and it takes you 15 [min] to get to the processing plant. In addition, there are two bridges you could take to get to work; call them bridge 1 and bridge 2. Because you have an all-wheel-drive vehicle, at any time on the way to work you can bypass bridge 1 through the country and go directly to bridge 2, but the speed limit along that route is just 20 [km/h]. One day, bridge 1 is under construction and you are forced to take bridge 2 to get to work. (a) What is the optimum route? (b) How long will the trip take?

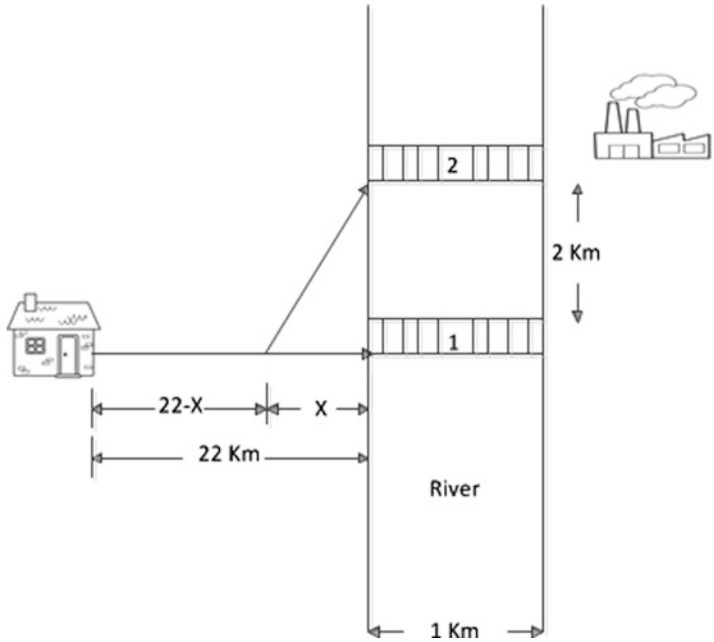
    **A:** (a) You should go approximately 21.6 [km] on the highway, then use the back roads (on a diagonal; see Fig. 11.42) to reach bridge 2. Go one more kilometer to cross the river. (b) Approximately 20 min 2 s.

14. **Businessman [6]**. An important businessman is planning a long car trip from Columbus, Ohio, to Auburn, Alabama (approximately 1,200 [km]). The businessman rents a car and is told by the car rental company that the car's gas consumption is directly proportional to the average speed of the car. At an average speed of 90 [km/h], the fuel efficiency is 12 [km/L] and at 100 [km/h], it is 10 [km/L]. In addition, the businessman will hire a driver who charges $15/h of driving. The rental car company charges the businessman $100 to deliver the car at the destination in Auburn. Also, the businessman must pay the driver a fixed amount for the hotel, meals, and trip back to Columbus. If the speed limit on the highway is 110 [km/h], what is the optimum average speed to minimize the total cost of the trip?

    **A:** 95.1 [km/h]

15. **Designing a new flag [9]**. Oregon State University is designing a new, small, and modern flag for its basketball team. As a designer, you receive the following instructions: (a) The flag should

**Fig. 11.42** Potential optimal route using bridge 2




have three vertical stripes, two orange (same dimensions) and one black. (b) The perimeter of the flag should be 40 [cm]. (c) The two orange stripes should have a total area of 20 [cm$^2$]. What should the dimensions of the stripes be to minimize the amount of fabric?
**A:** Two orange stripes of 10 × 1 [cm] and one black stripe of 10 × 8 [cm]

16. **Piece of wire [9]**. A piece of wire will be cut in two to make a square and a circumference. The total area (square plus the circumference) should be 16 [cm$^2$]. What is the maximum length ($L$) of wire that can be used?
**A:** $L = 21.39$ [cm]
17. **University openings [7]**. The planning department of a nonprofit university is trying to find the number of spots to open up next year and at the same to minimize tuition costs per student. This year, the university had 6,000 students, but because of graduation and students dismissed as a result of poor performance, it will be finishing the year with 5,400 students. The state government gives $30 million annually to the university. On the other hand, the fixed costs are $50 million annually. The monthly salary of professors is $6,000 and university policy is to have 1 professor for every 25 students. Additionally, there is a cost for support of $1/professor × student per month. (a) What should be the price of the tuition? (b) What will be the number vacancies to offer next year? (c) What will be the number of professors (must be integer)? (d) What will be the university budget?
**A:** (a) $7,256; (b) 1,671; (c) 283 [(5,400 + 1,671)/25]; (d) ~$81.3 million
18. **Silo [5]**. You are designing and building a silo of 1,000 [m$^3$]. The silo will be cylindrical with a hemispherical cap. Due to construction techniques, the hemispherical cap costs twice as much as the cylindrical part per square meter. (a) What radius ($R$) and height ($H$) of the cylinder would minimize the cost of construction? (b) If the cost of the cylinder and the hemispherical cap were the same, then would the optimal radius be lower or higher than in answer (a)? (c) What radius and height of the cylinder would minimize the cost of construction in question (b)? (d) Explain why, when the cost of the hemispherical cap is double, the radius of the silo is shorter.

**A:** (a) $R = 4.3$ [m], $H = 17.2$ [m]. (b) Higher. (c) $R = 5.42$ [m], $H = 10.84$ [m]. (d) Because the higher the cost of the hemispherical cap, the more the silo will be such as to minimize the amount of material needed in this part of the construction (shorter radius).

19. **Math test [7]**. You love math and, in addition, you have a natural talent for it, but unfortunately you are also a little bit anxious about it. There is no doubt that with the right amount of effort, you can get an $A^+$ (close to 100), but if you study too much (all day and night), you get a bit anxious about not visiting your devoted girlfriend, and your grade drops. Because of your vast experience taking tests and with your girlfriend, you have developed the following equation that relates your test grades ($G$, from 0 to 100) as a function of your effort ($E$, days of study) and anxiety ($A$): $G = E - A + C$, where $C$ is a constant indicating that if you decide not to study ($t = 0$), you will get a score of $G = 30$.
In addition, as mentioned, effort ($E$) and anxiety ($A$) are related to time ($t$) as follows:

$$E = 28t \text{ and } A = 3t^2.$$

At this time you have 7 days to study and to visit your girlfriend. (a) How many days will you study to maximize your test score and what will your grade be? (b) Regarding your loving girlfriend, what is the best option?
**A:** (a) 4.67 days and $G = 95.34$. (b) A good option for getting a good grade and having the necessary time to share with your loving girlfriend is to study 4 days and get a 94!

20. **Apples [4]**. The owner of some land with apple trees estimates that if he plants 45 trees per hectare, each tree will produce 550 apples annually. For every additional tree that is planted per hectare, the number of apples produced per tree will decrease by eight. How many trees (integer) should be planted per hectare to maximize the production of apples?
**A:** 57 trees

21. **Microorganisms [6]**. The microbial growth rate can be expressed by the following rate equation:

$$v_r = kN(N_m - N), \text{ where } k = 0.4[\text{g cfu/h}] \text{ and } N_m = 10^7[\text{cfu/g}].$$

(a) What is the maximum growth rate ($V_r$)? (b) What is the microorganism's concentration ($N$) at the maximum growth rate?
**A:** (a) $V_r = 10^{13}$ [cfu/g h]. (b) $0.5 \times 10^7$ [cfu/g] ($N_m/2$)

22. **Tubular reactor [8]**. You have been commissioned to design and determine the dimension sizing of a pilot plant cylindrical tubular reactor with semispherical lids. Preliminary estimates indicate that the total volume should be 50 [L]. If we want to minimize the amount of steel used: (a) What should the dimensions of the reactor be (radius and height)? (b) What should the dimensions of the reactor be if the lids cost four times as much as the cylindrical part due to construction technicalities and fitting devices?
**A:** (a) Radius $= 0.363$ [m] and height $= 0$. Interestingly, according to the results, the reactor is spherical. This is because the geometric figure that encloses the minimum area per unit volume is the sphere. (b) Radius $= 0.21$ [m] and height $= 1.13$ [m]

23. **Two or one pressure tanks [9]?** You are responsible for the purchase of a pressure storage tank of 100 [m$^3$]. The manufacturer indicates that it is much more convenient to buy two tanks of 50 [m$^3$] each. Although the two 50 [m$^3$] tanks are more expensive than one tank of 100 [m$^3$], the manufacturer indicates that maintenance costs for these two tanks will be significantly lower, so the offer is very appealing. You have some doubts because it is known that this manufacturer has no pressure tanks of 100 [m$^3$]. But, in fact, you have information that the maintenance costs of

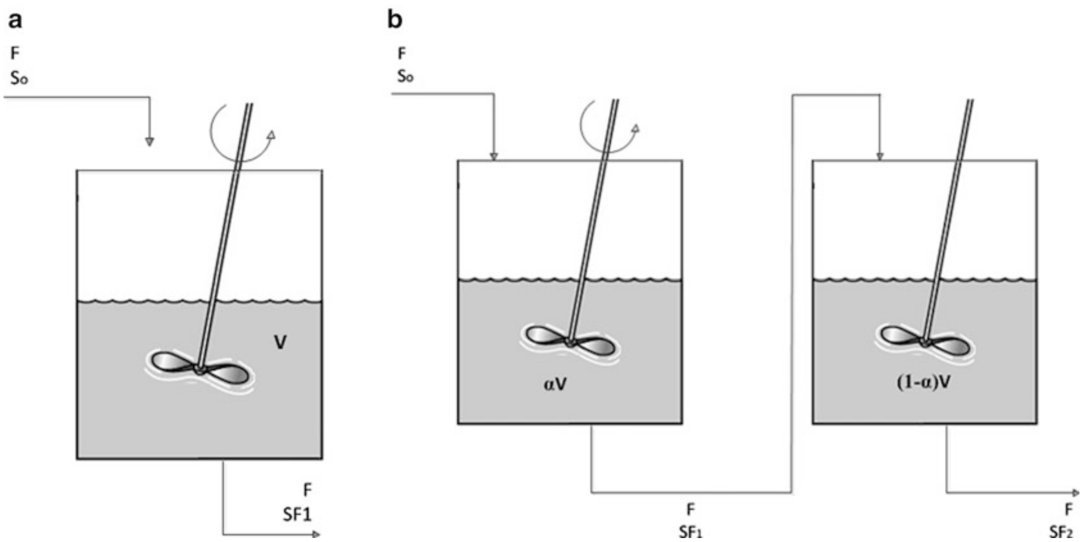



**Fig. 11.43** (**a**) Continuous stirred-tank reactor (CSTR) (**b**) Two Continuous stirred-tank reactors (CSTR) in series

these two tanks will be lower than for one tank of 100 [m$^3$]. Therefore, to be clear, you ask the manufacturer for information regarding the price of the tanks according to the volume and about maintenance costs. The manufacturer gives you complete information as detailed below:
Pressure tank price ($P$): $P = 10{,}000 \times (V)^a$, where $V$ is volume in cubic meters and $a$ is a constant equal to 0.6. The shelf life of the pressure tank is 10 years.
Maintenance costs ($M_c$): $M_c = 18 \times (V)^b$, where $V$ is volume in cubic meters and $b$ is a constant equal to 2. The maintenance is for the whole shelf life of the pressure tank (10 years).
After receiving all the information, you carry out an optimization analysis to decide what to do. Your options are to buy one pressure tank of 100 [m$^3$], two pressure tanks of 50 [m$^3$] each, or two pressure tanks of different volumes, but with the condition that the total volume be 100 [m$^3$]. What will you do to minimize your costs?
**A:** The best option is to buy two pressure tanks of equal size (50 [m$^3$] each).

24. **Two reactors? [9]**. A young engineer is proposing to replace an old continuous stirred tank reactor (CSTR) of volume $V$ for two CSTR reactors (of volume $V/2$ each) arranged in series. The manager agrees to replace the old reactor but disagrees with your proposition. The manager, who has no training in chemical engineering, wants to replace the old one for a similar one, but with up-to-date technology, instead of buying two new ones of volume $V/2$ each. According to Chap. 8, and considering that the disappearance of the substrate follows a first-order reaction, you derive the following equations for one reactor of volume $V$ (Fig. 11.43a) and for two reactors (total volume $V$) in series (Fig. 11.43b):

$$1 \text{ reactor}: \qquad S_{F1} = FS_0/(F + kV), \tag{11.5}$$

$$2 \text{ reactors in series}: \qquad S_{F2} = F^2S_0/[(F + \alpha kV)(F + (1-\alpha)kV)], \text{ where } 0 < \alpha < 1, \tag{11.6}$$

where $F = 20$ [L/h], $S_0 = 12$ [g/L], $k = 0.4$ [1/h], $V = 100$ [L], and $S_{F1}$ and $S_{F2}$ are the outputs for one reactor and two reactors in series, respectively. With a simple calculation, say $\alpha = 0.5$, you can check that $S_{F2} < S_{F1}$. Therefore, it is better to have two CSTR reactors in series than just

one big CSTR reactor. (a) Derive and check (11.5) and (11.6). (b) Determine the value of $\alpha$ that maximizes the disappearance of the substrate ($S_F$ minimum).
**A:** $\alpha = 0.5$

25. **Continuous culture [6]**. The biomass productivity ($Q_x$) of a continuous culture is calculated using the following equation:

$$Q_x = \left[ y_{x/s} \left( S_0 - \frac{DK_s}{\mu_m - D} \right) \right] D.$$

Determine the value of $D$ that maximizes the biomass productivity using the following data: $Y_{x/s} = 0.42$, $S_0 = 20$ [g/L], $K_s = 0.01$ [g/L], and $\mu_m = 0.5$ [1/h]. Assume that the dilution rate, $D$, has the restriction that it must be less than the maximum specific growth rate ($D < \mu_m$).
**A:** $D = 0.4888$ [1/h]

26. **Extraction process [6]**. In an extraction process, the extract concentration (g/L) can be described by the following equation as a function of time ($t$):

$$C = -3.75t^2 + 13.75t + 30.$$

Furthermore, the flow rate (L/min) can be expressed by

$$Q = 15t.$$

After how much time does it reach the maximum rate of product extracted ($C \times Q$ [g/min])?
**A:** 3.26 min

27. **Fuel mixture [7]**. A specialized laboratory conducts studies to determine the cost per kilometer of a fuel that is made from a mixture of diesel and bioethanol. Project engineers estimate that the model that describes the behavior of the cost against the fraction of each component is

$$C = 55.2D^3 + 65.2B^2 + 24.8,$$

where $D$ ($0 < D < 1$) is the fraction of diesel and $B$ ($0 < B < 1$) the fraction of ethanol. Determine the fraction of diesel ($D$) and ethanol ($B$) that minimizes the cost per kilometer of fuel. Remember: $D + B = 1$.
**A:** $D = 0.58$ and $B = 0.42$

28. **Piping [10$^+$]**. A long pipe is going to be installed to carry a compound from building B1, where it is made, to building B2, where it will be used. Both buildings are constructed in parallel beside each other with a separation of 8 [m] between them, and each one has a length of 60 [m]. In addition, both buildings are oriented north–south. The output of the pipe in B1 is located 5 [m] to the south end of the building, and the beginning of the process line in B2 *i*s located 5 [m] from the north end of building B2. The pipe installed between and outside the two buildings requires an additional coating; therefore, the cost of installation is 75 % higher than for one installed inside the buildings. Engineers have proposed the following diagram for the pipe installation (Fig. 11.44). Determine the angle ($\theta$) that minimizes the cost of installation.
**A:** ~34.85°

29. **New package design [5]**. A natural-juice factory seeks to minimize the amount of material to use in its design of parallelepiped-shaped boxes. The designer has set one condition: one side must be

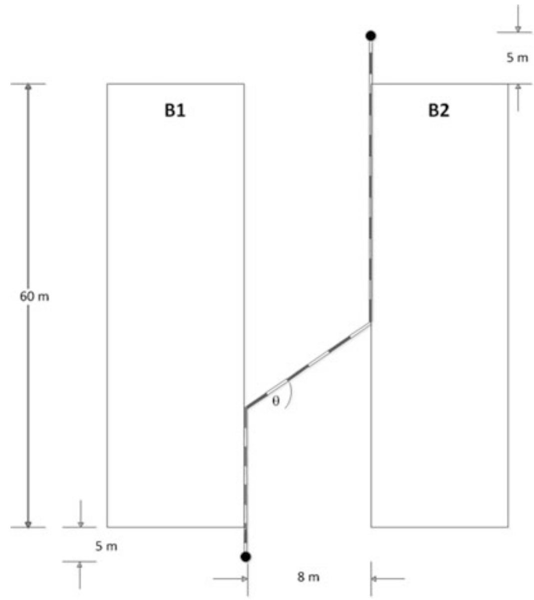



**Fig. 11.44** Long pipe installed to carry a compound from the building *B1* where it is made to the building *B2* where it will be used

twice the length of another side and, in addition, the volume should be 1 [L]. What are the dimensions that minimize the use of material?
**A:** 7.211 [cm], 14.422 [cm], and 9.615 [cm]

30. **Activated enzyme [6]**. In an enzyme recovery process, engineers were able to model the recovery of protein [$P(t)$ %] and the enzyme activity [$\alpha(t)$ UI/mg] as a function of time with the following equations:

$$P(t) = 0.001t^3 - 0.207t^2 + 7.508t + 5.387,$$

$$\alpha(t) = 0.001t^2 - 0.0166t + 9.945,$$

where $t$ is in minutes.

What extraction time maximizes the product of these two functions (known as activated enzyme)?
**A:** 22.27 min

31. **Pipe insulation [8]**. A pipe 15 [m] long and 3 [in.] in diameter carries steam, and the process engineer recommends insulating the pipe to reduce heat loss and, thus, reduce steam condensation. The cost of steam condensation is 1 [MU/kg*]. The insulation has a cost of 0.012 [MU/cm$^3$] and a thermal conductivity ($k$) of 0.04 [W/m K]. Neglecting the thickness of the pipe, determine the thickness of the insulation material that delivers the lowest annual cost. Because the cost of insulation is a fixed cost (once), it should be multiplied by 0.16 to be expressed as an annual cost. Therefore, the total cost to be minimized is

$$\begin{aligned}\text{Annual cost} = {} & (0.16 \times \text{insulation cost})\ [\text{MU/year}] \\ & + \text{Annual cost of steam condensation}\ [\text{MU/year}].\end{aligned}$$

The following equation can be used to determine steam loss (kg/s):

$$\mathrm{SL} = \frac{2\pi L r \Delta T k}{x\omega},$$

where SL is steam losses (kg/s), $L$ is the length of the pipe (m), $r$ is the radius of the pipe (m), $\Delta T$ is the temperature difference (80 K), $k$ is the thermal conductivity (W/m K), $\omega$ is the heat of vaporization of water 2,257 × $10^3$ (J/kg), and x is the thickness of insulation (m).
The plant operates 24 h/day, 330 days a year. What thickness of the insulation material (cm) would minimize the annual cost?
**A:** 8.2 [cm]

32. **Canned food design [9]**. A small cannery is developing a new seafood product to be packed in a cylindrical can of 400 [$cm^3$]. The designer wants to minimize the distance from the center of the can to the center of the lid plus the distance from the center of the can to the wall. (a) What are the radius and height of the can? (b) What should the radius and height of the can be if the distance from the center of the can to the middle of the lid plus the distance from the center of the can to the border of the lid should be minimized?
**A:** (a) Radius = height = 5.031 [cm]. (b) Radius = 5.65 [cm], height = 3.9885 [cm]
33. **Extraction costs [6]**. In a pilot plant the engineer estimates the cost (MU) of a concentrated product obtained from roots. The model is expressed as a function of time through the following equation:

$$\text{Cost} = \left(-0.0012t^3 + 0.04t^2 + 0.9167t\right)/(10t).\ \text{(time in min)}$$

If the concentration, $C$ (kg/$m^3$) against time is

$$C = -0.0012t^3 + 0.04t^2 + 0.9167t,$$

then: (a) What is the extraction time at which the extract is obtained at a lower cost? (b) What is the concentration of the extract?
**A:** (a) 16.67 [min], (b) ~ 20.84 [kg/$m^3$]
34. **Free energy of formation [10⁺]**. The free energy of formation of a spherical crystal ($\Delta G_T$) can be described as the change in free energy of the liquid to the solid phase ($\Delta G_P$) multiplied by the volume of the spherical crystal plus the free energy for creating the solid–liquid interface ($\Delta G_i$). Thus,

$$\Delta G_T = V\Delta G_P + \Delta G_i,$$

where $\Delta G_i = A\gamma$, $\Delta G_P = \Delta H_f \times \frac{\Delta T}{T_f}$, $A$ is the area of the sphere ($m^2$), $\gamma$ is the surface free energy (J/$m^2$) ($\gamma_{(H_2O)} = 0.072$ [J/$m^2$]), $\Delta H_f$ is the heat of fusion (J/$m^3$) ($\Delta H_{fH_2O} = 3.344 \times 10^8$[J/$m^3$]), $\Delta T$ is the degrees of subcooling (K), $V$ is the volume of the spherical crystal ($m^3$), and $T_f$ is the temperature of fusion (K) ($T_{fH_2O} = 273$[K]).
The radius of the crystal to where the free energy of formation of the spherical crystal ($\Delta G_T$) is at its maximum is known as the critical radius; this proves to be the minimum size to begin the formation of a crystal. Determine the critical radius for water with 10C ($\Delta T = 10$ [K]) of subcooling. **HINT:** Search around 1.1 × $10^{-8}$ and 1.2 × $10^{-8}$ [m].
**A:** 1.1756 × $10^{-8}$ [m] (0.011756 [μm])

### 11.8.2 Operation Research Problems

35. **Trucks [6]**. A food distribution company has three types of trucks, $T_1$, $T_2$, and $T_3$. $T_1$ has no refrigeration capacity and a volumetric capacity of 25 [m$^3$]. $T_2$ has a modern refrigeration system and a total capacity of 45 [m$^3$], but just 35 [m$^3$] are equipped with the cooling and the remaining 10 [m$^3$] are not refrigerated. Finally, $T_3$ is a smaller truck with a capacity of 15 [m$^3$] and without a refrigeration system. Wal-Mart is located 100 [km] from the distribution company and is ready to launch a mega store and places a big order of 2,500 [m$^3$] of refrigerated foods and 3,800 [m$^3$] of nonrefrigerated foods. If the cost per kilometer of using each truck is $0.20 for the $T_1$ truck, $0.26 for $T_2$, and $0.23 for $T_3$, then in order to minimize the distribution costs, how many trucks of each type must use the distributor?
    **A:** 124 $T_1$ trucks, 72 $T_2$ trucks, and no $T_3$ trucks.
36. **A bus trip [7]**. Our chemical and bioprocess engineering department is preparing a trip to a scientific meeting to be held 250 [miles] away. The department contacted a bus company that provides good service at a reasonable price. We think 420 students will attend meeting. The bus company has eight buses with 45 seats ($B_1$) and six buses with 35 seats ($B_2$). In addition, the company has just 11 drivers available. If the company charges $950 and $750 for the large and small buses, respectively, then in order to minimize costs, how many buses of each type should the department hire?
    **A:** Seven large buses, three small buses.
37. **Chemical products [10]**. You are a sales manager of a small chemical company. You are interested in selling the last stock of two chemicals, $A_1$ and $B_Z$. Each chemical is stored in 1 [kg] jars. The inventory indicates that you have 200 jars of chemical $A_1$ and 100 jars of chemical $B_Z$. Hoping to sell the entire inventory you decide to offer two types of packages. The first package is one jar of $A_1$ and one jar of $B_Z$ for $60. The second package is three jars of $A_1$ and one jar of $B_Z$ for $100. You expect to sell at least 30 packages and, in addition, that you will sell twice as many of the first package as the second package. To maximize your sales, how many packages will you need to sell of each type?
    **A:** 66 of the first type and 33 of the second type
38. **Candies and chocolates [9]**. A candy company receives, out of the blue, an urgent order for its most popular candies, Candit and Choc. Table 11.3 shows the current availability of each of these products in the manufacturing plant.
    To fulfill this order, two trucks are used with the following characteristics (Table 11.4):
    The trucks are expected to arrive at the destination 3 h late, so each truck has to carry at least 2 [tons] of each product. How much Candit and Choc does each truck have to carry to maximize profits?
    **A:** Truck 1: 2 [tons] of Candit and 13 [tons] of Choc; Truck 2: 6 [tons] of Candit and 6 [ton] of Choc.

**Table 11.3**

| Product | Amount [ton] | Specific volume [m$^3$/ton] | Profit [$/ton] |
|---|---|---|---|
| Candit | 22 | 4 | 550 |
| Choc | 24 | 5.2 | 650 |

**Table 11.4**

| Truck | Weight capacity [ton] | Volume capacity [m$^3$] |
|---|---|---|
| 1 | 15 | 70 |
| 2 | 12 | 55 |

39. **Small food processing company [6]**. A small company has two food processing plants and has received two orders, one for 450 [kg] of product from Corvallis, Oregon, and the other for 300 [kg] of product from Olympia, Washington. The company has one processing plant in Portland, Oregon, and a second one in Salem, Oregon. The product availability in Portland is 500 [kg] and in Salem 400 [kg]. It costs \$0.50/kg to ship the product from Salem to Corvallis but \$1.50/kg to ship it to Olympia. On the other hand, it costs \$0.80/kg to ship the product from Portland to Corvallis and \$1.20/kg to ship it from Portland to Olympia. The minimum order to be shipped from each processing plant is 10 [kg]. How many kilograms of product should the company ship from each plant to Corvallis and Olympia to minimize the cost of the order?

    **A:** To minimize the cost of the order, from the plant in Portland the company must send 61 [kg] to Corvallis and 290 [kg] to Olympia; from the plant in Salem it must send 389 [kg] to Corvallis and 10 [kg] to Olympia.

40. **Pet food [7]**. Although you are young and also a new engineer at the Pet Food Company you are sure that the cost of the pellets can be minimized. So far, the pellets (1 [g]) are manufactured from three base products, A, B, and C, with unit costs of \$0.50, \$1.25, and \$2.25 per kilogram, respectively. Each pellet must contain at least 3 [mg] of vitamin M and 4 [mg] of vitamin N. It is known that for every gram the base products A, B, and C contain 1, 4, and 4 [mg] of vitamin M and 1, 2, and 6 [mg] of vitamin N, respectively. Your argument to change the actual recipe is that the current product is not using a filler or excipient. The filler does not contain vitamins N and M but is very cheap (\$0.0625/kg). (a) What is the actual cost per kilogram and composition of the pellets? (b) What is the cost per kilogram and composition of the pellets if you include filler? (c) What is the minimum cost of the pellets if they must meet the vitamins requirements but can weigh less than 1 [g]?

    **A:** (a) ~\$1.5834 and using 0.34 [g] of A, 0.0834 [g] of B and 0.5834 [g] of C per pellet; (b) \$1.5781 and using 0 [g] of A, 0.125 [g] of B, 0.625 [g] of C, and 0.25 [g] of filler per pellet. So you were right, using filler reduces the cost, but just slightly. (c) \$1.5625 and using 0 [g] of A, 0.125 [g] of B, 0.625 [g] of C, and 0 [g] of filler per pellet.

41. **4 × 100 [m] relay [10]**. Following the team's defeat in the 4 × 100 [m] relay at the Olympics in London, the US men's trainer has been conducting an interesting study to determine the best way to arrange each sprinter in the relay. The performance of each sprinter at the different stages of the 4 × 100 [m] relay has been carefully measured by the trainer. The following table shows the time of each sprinter in each stage of the relay.

| | Stage 1 | Stage 2 | Stage 3 | Stage 4 |
|---|---|---|---|---|
| Justin Gatlin | 9.9 | 9.3 | 9.5 | 9.2 |
| Tyson Gay | 9.7 | 9.5 | 9.3 | 9.5 |
| Darvis Patton | 10.0 | 9.3 | 9.2 | 9.1 |
| Jeff Demps | 9.8 | 9.2 | 9.1 | 9.3 |

In addition, the trainer has noticed that the way he arranges his sprinters really matters. For example, if he puts Justin Gatlin first, Tyson Gay second, Darvis Patton third, and Jeff Demps last, the time for the 4 × 100 [m] relay is 37.9 s. On the other hand, if Tyson Gay goes first, Justin Gatlin second, Darvis Patton third, and again Jeff Demps last, the time is 37.5 s. The trainer is smart and knows that there must be an order that minimizes the time. Help him find the best order of runners and determine what the time will be for this optimum arrangement.

**A:** The optimum arrangement is Tyson Gay first, Justin Gatlin second, Jeff Demps third, and Darvis Patton last, and the time is 37.2 s

42. **Feeding pigs [8]**. Some farmers are planning to feed their pigs with a mixture of three products: crushed meal, a special food, and vitamins. They have several thousand pigs, and each pig should eat at least 8 [kg] of the mixture per day. The daily dietary requirements and associated costs are presented in the following table:

| | Calories | Vitamin (mg) | | | |
|---|---|---|---|---|---|
| Minimum daily requirement | 3,500 | A | B | C | |
| | | 6 | 10 | 22 | |
| Content per product | | | | | Costs |
| Crushed meal [kg] | 450 | 0.5 | – | – | 0.70 [$/kg] |
| Special food [kg] | 290 | 0.5 | – | 1 | 0.80 [$/kg] |
| Vitamin concentrate[a] [bottle] | – | 0.5 | 7 | 14 | 1.5 [($/Bottle)] |

[a]The vitamin concentrate weighs 100 [g] per bottle. In addition, the number of bottles could be fractional.

(a) What is the optimal mixture of the three products to minimize the cost of the diet? (b) What is the cost of the daily diet per pig? (c) Your veterinarian argues that for health and productivity reasons, it is very important to restrict the amount of the mixture to no more than 10 [kg/day]. What is the optimum result if you follow the veterinarian's recommendation? (d) What is the cost of this new daily diet per pig?

**A:** (a) 10.429 [kg] of crushed meal and 1.571 bottles of vitamin concentrate, (b) $9.657, (c) 9.778 [kg] of crushed meal and 2.222 bottles of vitamin concentrate, (d) $10.178.

43. **Optimizing distribution [9]**. A company that manufactures animal feed has two processing plants. To fulfill a contract with three supermarkets, it produces 400,000 bags of food a week at plant 1 and 200,000 at plant 2. These bags will be transported to two packaging companies whose capacities are 300,000 bags each. Finally, the bags must be sent to the three supermarkets (200,000 bags to each supermarket). The following table shows the transport costs (arbitrary units) per bag of food. How many packages should be sent to each packaging company and to each supermarket to minimize costs?

| | Packaging company 1 | Packaging company 2 | Supermarket 1 | Supermarket 2 | Supermarket 3 |
|---|---|---|---|---|---|
| Plant 1 | 50 | 90 | – | – | – |
| Plant 2 | 120 | 150 | – | – | – |
| Packaging company 1 | – | – | 200 | – | 165 |
| Packaging company 2 | – | – | – | 180 | 195 |

**A:** The following table shows the number of packages sent to each packaging store and to each supermarket.

| | Packaging company 1 | Packaging company 2 | Supermarket 1 | Supermarket 2 | Supermarket 3 |
|---|---|---|---|---|---|
| Plant 1 | 300,000 | 100,000 | – | – | – |
| Plant 2 | – | 200,000 | – | – | – |
| Packaging company 1 | – | – | 200,000 | – | 100,000 |
| Packaging company 2 | – | – | – | 200,000 | 100,000 |

44. **Optimal investment [10]**. A large chemical company is considering the implementation of four projects, designated P1, P2, P3, and P4. The following table shows the capital requirements for each of these projects for the next 3 years. The same table shows the NPV (net present value) of each project and the annual availability of capital that the company will have over the next 3 years.

In addition, the table uses an arbitrary currency and the money values corresponding to the year in question.

| | NPV (at beginning of year 1) | Annual capital requirements per project | | | |
|---|---|---|---|---|---|
| | | Year 1 | Year 2 | Year 3 | Observation |
| P1 | 61 | −22 | −5 | −21 | |
| P2 | 77 | −31 | −31 | 0 | Runs during first 2 years |
| P3 | 103 | 0 | −38 | −37 | Runs in second and third years |
| P4 | 120 | −20 | −12 | −40 | |
| Available capital per year | | 65 | 60 | 82 | |

Moreover, projects P2 and P4 are mutually exclusive, i.e., if one of them is carried out, the other is not, although perhaps neither is carried out. The policy of the board and the CEO is to run more than one project. Finally, P1 can be executed only if P3 is executed. (a) Which projects should the company go with? (b) Based on the answer to (a), do you have any comments?

**A:** (a) Projects P3 and P4. (b) If each year some money will be made available (45, 10, and 5, respectively), then after the first year, the board and CEO might reevaluate the situation (new projects?).

45. **Cookies [6]**. An industrial bakery receives from a supermarket chain an urgent order for 1,000 [kg] of high-protein cookies. The cost of the ingredients should be minimized and the mixture must meet the following minimum requirements: 360 [kg] protein, 225 [kg] fat, 240 [kg] carbohydrates, and 45 [kg] sugar. In addition, the moisture content of the cookies should be lower than 3.5 % w/w.

The cookies are made based on the mixture of four ingredients R, S, T, and U, with associated costs of \$2, \$0.50, \$0.75, and \$0.25/kg, respectively. The composition of the ingredients is given in the following table:

| | Protein [% w/w] | Fat [%w/w] | Carbohydrate [% w/w] | Sugar [% w/w] | $H_2O$ [% w/w] | Filler [% w/w] |
|---|---|---|---|---|---|---|
| R | 48 | 30 | 14 | 5 | 3 | 0 |
| S | 10 | 15 | 48 | 13 | 4 | 10 |
| T | 28 | 5 | 29 | 10 | 3 | 25 |
| U | 0 | 5 | 5 | 28 | 2 | 60 |

How many kilograms of each ingredient do you need to use to minimize the cost?

**A**: R = 607.69 [kg], S = 230.77, T = 161.54, U = 0.

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