

Chapter 14

Compressible Flow: Nozzles and Diffusers

14.1 Sub- and Supersonic Flows

Gas flows through nozzles and diffusers show an interesting behavior when their speed reaches, or is above, the speed of sound of the gas. This behavior must be well understood for the proper design of devices and engines. In this section we study the laws that govern the transition from sub- to supersonic flow and vice-versa, and draw relevant conclusions for the design of nozzles and diffusers, and for rocket motors. We will see that subsonic nozzles have a converging cross section, while supersonic nozzles have a converging-diverging cross section.

14.2 Speed of Sound

The ear reacts to small pressure oscillations,¹ that is sound is a pressure wave. The speed of sound is the speed with which such a small pressure disturbance travels through a medium. To compute its value, we consider a simple experiment, shown in Fig. 14.1 (left): In a long pipe, a wave is created by a small push of a piston. This wave travels through the pipe with the speed of sound, a .

To analyze the wave, and to compute the speed of sound, it is best to consider the wavefront from a co-moving frame, as depicted on the right of the figure. The wave travels into undisturbed fluid with properties p, ρ, h, s , and the observer travelling with the wave sees fluid coming in at the speed of sound a . The fluid left behind by the wave has slightly altered properties $p + dp, \rho + d\rho, h + dh, s + ds$, and the observer on the wave sees it leaving with speed $a - dV$.

Since the wave travels fast, there is no time for heat exchange, thus we can assume that the wave is adiabatic. Moreover, we shall ignore irreversibilities through friction, that is we consider reversible wave propagation.

¹ Large pressure difference can damage the ear!

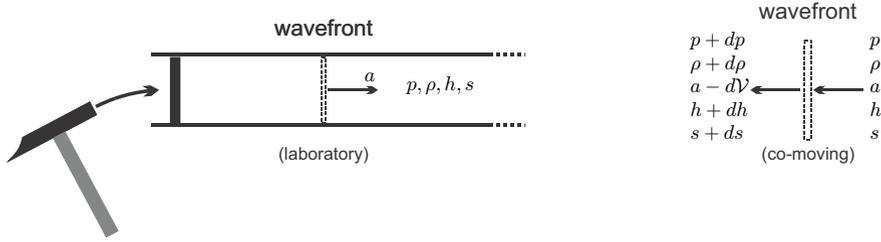


Fig. 14.1 A soundwave travelling through a pipe. Left: In the laboratory frame. Right: In a co-moving frame.

Indeed, sound waves in air are only very weakly damped. That we hear noises from far away only weakly is due to the spherical propagation of sound waves, where the sound energy is distributed over shells of surface $4\pi R^2$ when R is the distance from the sound source. Thus local sound energy, which is what we hear, is inversely proportional to the square of the distance from the source, that is proportional to $1/R^2$. In a pipe, however, the propagation is one-dimensional, waves travel as plane waves with unchanged wave surface area, which is equal to the pipe cross section.

Applying the mass balance to the wave front yields

$$\dot{m} = \rho A \mathcal{V} = \text{const.} \tag{14.1}$$

Since the cross section A does not change, this yields

$$(\rho + d\rho)(a - d\mathcal{V}) = \rho a . \tag{14.2}$$

The changes $d\mathcal{V}$ and $d\rho$ are so small that their product can be ignored, so that the above reduces to

$$\frac{1}{\rho}d\rho = \frac{1}{a}d\mathcal{V} . \tag{14.3}$$

Since the wave is adiabatic, and does not exchange any work, the first law simply gives

$$\dot{m} \left(h + \frac{1}{2}\mathcal{V}^2 \right) = \text{const.} , \tag{14.4}$$

which, with $\dot{m} = \text{const.}$, reduces to

$$h + dh + \frac{1}{2}(a - d\mathcal{V})^2 = h + \frac{1}{2}a^2 , \tag{14.5}$$

so that, with (14.3)

$$dh = a d\mathcal{V} = \frac{a^2}{\rho} d\rho . \tag{14.6}$$

Since the wave is adiabatic and reversible, the second law gives

$$\dot{m}s = \text{const.} \quad (14.7)$$

or, since $\dot{m} = \text{const.}$,

$$s = \text{const.} \quad \text{or} \quad ds = 0; \quad (14.8)$$

the entropy remains unchanged, the flow is isentropic.

We insert the above equations (14.3, 14.6, 14.8) into the Gibbs equation $Tds = dh - \frac{1}{\rho}dp$ to find

$$0 = a^2 \frac{1}{\rho} d\rho - \frac{1}{\rho} dp. \quad (14.9)$$

Solving for a we find

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}, \quad (14.10)$$

where the subscript s indicates that the derivative must be taken at constant entropy.

14.3 Speed of Sound in an Ideal Gas

The speed of sound depends on the material, and for its computation from (14.10) one needs to know the thermal and caloric equations of state. Here, we determine the speed of sound for an ideal gas. Starting point is, again, the Gibbs equation, together with the caloric equation of state, $h = h(T) = \int c_p dT$,

$$Tds = dh - \frac{1}{\rho}dp = c_p dT - \frac{1}{\rho}dp. \quad (14.11)$$

With $ds = 0$ and the ideal gas law written as $T = \frac{p}{\rho R}$, we obtain

$$0 = c_p d\left(\frac{p}{\rho R}\right) - \frac{1}{\rho}dp = \frac{c_p}{\rho R} dp - \frac{c_p p}{\rho^2 R} d\rho - \frac{1}{\rho}dp. \quad (14.12)$$

With $c_v = c_p - R$ this yields

$$\frac{c_v}{\rho R} dp = \frac{c_p p}{\rho^2 R} d\rho, \quad (14.13)$$

or

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{c_p p}{c_v \rho} = kRT. \quad (14.14)$$

Thus, the speed of sound in an ideal gas depends on temperature as

$$a = \sqrt{kRT}. \quad (14.15)$$

Keep in mind that k depends on temperature: For air ($R = 0.287 \frac{\text{kJ}}{\text{kg K}}$) at $T = 298$ we have $k = 1.4$ and thus $a = 346 \frac{\text{m}}{\text{s}}$, while at $T = 1200$ K we have $k = 1.314$ and find $a = 672.8 \frac{\text{m}}{\text{s}}$.

The speed of sound depends on the type of gas through the ratio of specific heats, k , and through the gas constant, $R = \frac{\bar{R}}{M}$. Gases with smaller molar mass M have larger speed of sound. As example we consider helium ($M_{\text{He}} = 4 \frac{\text{kg}}{\text{kmol}}$, $k_{\text{He}} = \frac{5}{3}$), argon ($M_{\text{Ar}} = 39.95 \frac{\text{kg}}{\text{kmol}}$, $k_{\text{Ar}} = \frac{5}{3}$), and carbon dioxide ($M_{\text{CO}_2} = 44 \frac{\text{kg}}{\text{kmol}}$, $k_{\text{CO}_2} = \frac{4}{3}$) at 298 K, for which we find $a_{\text{He}} = 1016 \frac{\text{m}}{\text{s}}$, $a_{\text{Ar}} = 321.5 \frac{\text{m}}{\text{s}}$ and $a_{\text{CO}_2} = 274 \frac{\text{m}}{\text{s}}$.

14.4 Area-Velocity Relation

The main design parameter for nozzles and diffusers is the change of cross section, and we ask how flow properties, in particular velocity and pressure, change with the cross section. Figure 14.2 shows an adiabatic and reversible, i.e., isentropic, flow through a duct with varying cross section. We consider a small slice of the duct of width dx , and apply the balances of mass, energy and entropy, similar to what we did for the wave in the pipe above.

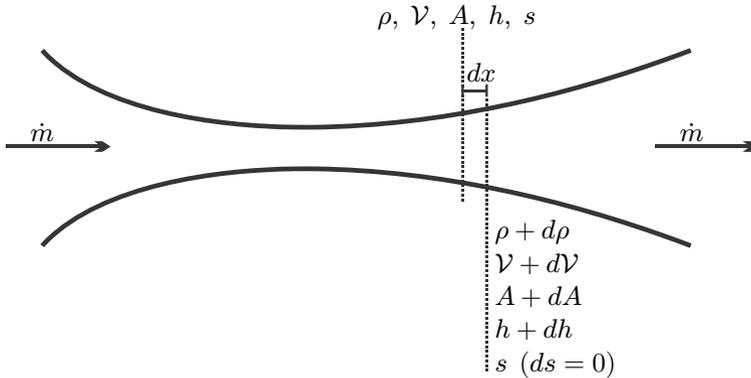


Fig. 14.2 Adiabatic reversible, i.e., isentropic, flow through a duct with changing cross section

The mass balance yields

$$\dot{m} = \rho \nu A = \text{const.} \quad (14.16)$$

and thus

$$\frac{d\dot{m}}{\dot{m}} = \frac{dA}{A} + \frac{d\nu}{\nu} + \frac{d\rho}{\rho} = 0. \quad (14.17)$$

The first law gives at first

$$h + \frac{1}{2}\mathcal{V}^2 = h_0 = \text{const.} \quad (14.18)$$

and thus

$$dh + \mathcal{V}d\mathcal{V} = 0. \quad (14.19)$$

Here, h_0 is the stagnation enthalpy, defined as the enthalpy the flow would obtain when brought to rest adiabatically. Since the flow is isentropic, we have $ds = 0$, and the Gibbs equation gives

$$Tds = dh - \frac{1}{\rho}dp = 0. \quad (14.20)$$

Elimination of enthalpy between the last two equations yields the relation between pressure and velocity changes in isentropic nozzles and diffusers,

$$\frac{1}{\rho}dp = -\mathcal{V}d\mathcal{V}, \quad (14.21)$$

which we discussed already in Sec. 9.11. We use this to eliminate \mathcal{V} from the mass balance to find

$$\frac{dA}{A} = \frac{1}{\mathcal{V}^2} \frac{1}{\rho} dp - \frac{d\rho}{\rho} = \frac{1}{\rho} dp \left(\frac{1}{\mathcal{V}^2} - \frac{1}{\left(\frac{\partial p}{\partial \rho}\right)_s} \right), \quad (14.22)$$

where the subscript indicates isentropic flow.

To proceed, we introduce the Mach number

$$\text{Ma} = \frac{\mathcal{V}}{a},$$

which compares flow velocity to speed of sound: $\text{Ma} < 1$ for subsonic flows, $\text{Ma} > 1$ for supersonic flows, and $\text{Ma} = 1$ for sonic flows. Flows with $\text{Ma} \gg 1$ are called hypersonic and flows with $\text{Ma} \simeq 1$ are called transonic.

With the definition (14.10) of the speed of sound we thus can write the relation (14.22) as

$$\frac{dA}{A} = \frac{1}{\rho\mathcal{V}^2} dp (1 - \text{Ma}^2), \quad (14.23)$$

or, by eliminating pressure,

$$\frac{dA}{A} = -\frac{d\mathcal{V}}{\mathcal{V}} (1 - \text{Ma}^2). \quad (14.24)$$

Equations (14.23) and (14.24) are the area-pressure relation and the area-velocity relation for isentropic duct flows. Both relations carry the factor $(1 - \text{Ma}^2)$ which has different sign for subsonic and supersonic flows.

Accordingly, a change of cross section has different effect when applied to sub- and supersonic flows.

Subsonic Flows ($Ma < 1$): For a converging duct we have from (14.23, 14.24)

$$dA < 0 \implies dp < 0, dV > 0;$$

the flow is accelerated while pressure drops; this is a nozzle.

For a diverging duct, on the other hand, we have the opposite signs,

$$dA > 0 \implies dp > 0, dV < 0;$$

pressure grows, and the flow decelerates; this is a diffuser.

Supersonic Flows ($Ma > 1$): For a converging duct we have from (14.23, 14.24)

$$dA < 0 \implies dp > 0, dV < 0;$$

pressure grows, and the flow decelerates; this is a diffuser.

For a diverging duct, on the other hand, we have the opposite signs,

$$dA > 0 \implies dp < 0, dV > 0;$$

the flow is accelerated while pressure drops; this is a nozzle.

In other words, a converging duct acts as a nozzle in subsonic flow, but as a diffuser in supersonic flow. A diverging duct acts as a diffuser in subsonic flow, but as a nozzle in supersonic flow. Figure 14.3 shows a summary.

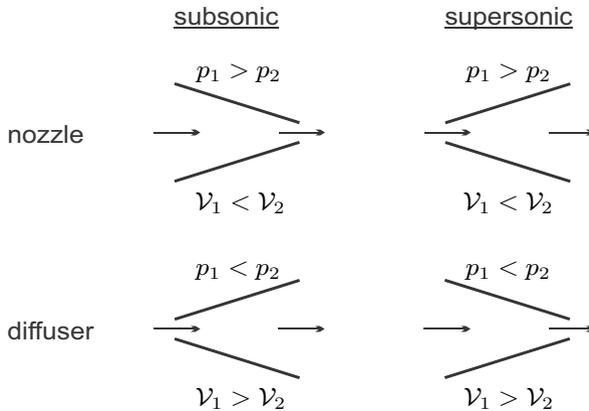


Fig. 14.3 Nozzles and diffuser cross section variation in sub- and supersonic flow

14.5 Nozzle Flows

Rocket motors and some jet engines expel supersonic flows for propulsion, and thus need appropriate nozzle geometries. To accelerate a subsonic flow to supersonic speed requires a converging-diverging nozzle, where the flow is accelerated to sonic speed in the converging part, and then to supersonic speed in the diverging part. After its inventor Gustaf de Laval (1845-1913), such a nozzle is called Laval nozzle.

We will almost exclusively deal with isentropic flows. So that we can perform analytical calculations, we restrict the treatment to ideal gases with constant specific heats. We shall discuss flows through purely converging nozzles, and through converging-diverging nozzles. In both cases, the balances of mass, energy and entropy reduce to

$$\begin{aligned} \dot{m} &= \rho \mathcal{V} A = \text{const.} , \\ h + \frac{1}{2} \mathcal{V}^2 &= h_0 = \text{const.} , \\ \frac{T}{p^{\frac{k-1}{k}}} &= \frac{T_0}{p_0^{\frac{k-1}{k}}} , \quad \frac{p}{\rho^k} = \frac{p_0}{\rho_0^k} , \end{aligned} \tag{14.25}$$

where ρ, \mathcal{V}, T, h are the properties at a given cross section of the nozzle, and T_0, p_0, h_0 are stagnation properties. The stagnation state is defined as the hypothetical state that is reached by bringing the flow to rest isentropically.

With $h - h_0 = c_p (T - T_0)$ and $c_p = \frac{k}{k-1} R$ we find from the above the local velocity as

$$\mathcal{V} = \sqrt{2(h_0 - h)} = \sqrt{\frac{2kRT_0}{k-1} \left(1 - \frac{T}{T_0}\right)} = \sqrt{\frac{2kRT_0}{k-1}} \sqrt{1 - \left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}} . \tag{14.26}$$

With the isentropic relation for density, we can write the mass flow through the nozzle as

$$\dot{m} = \rho_0 \sqrt{\frac{2kRT_0}{k-1}} A \left[\left(\frac{p}{p_0}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}} \right] = \text{const.} \tag{14.27}$$

The mass flow is a product of three factors: The constant $\rho_0 \sqrt{\frac{2kRT_0}{k-1}}$ which is fixed by the stagnation state (ρ_0, T_0) , the cross section A , and the flow function ψ , which we define as

$$\psi \left(\frac{p}{p_0}\right) = \left(\frac{p}{p_0}\right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}} . \tag{14.28}$$

Figure 14.4 shows the flow function for $k = 1.33, 1.4, 1.67$.

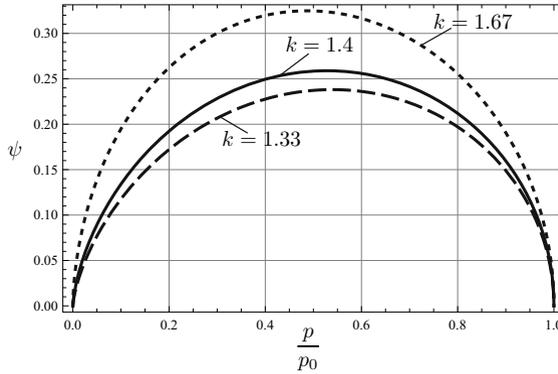


Fig. 14.4 Flow function $\psi(p/p_0)$ for $k = 1.33, 1.4, 1.67$. The function has a maximum at $\frac{p^*}{p_0} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} = 0.540, 0.528, 0.488$.

The argument $\left(\frac{p}{p_0}\right)$ of the flow function assumes values between 0 and 1, and the curve exhibits a maximum with the critical values

$$\frac{p^*}{p_0} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}, \quad \psi^* = \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \sqrt{\frac{k-1}{k+1}}; \tag{14.29}$$

for the k values in the figure, the critical pressure assumes the values $\frac{p^*}{p_0} = 0.534, 0.528, 0.487$.

The condition of constant mass flow is equivalent to

$$A \psi \left(\frac{p}{p_0}\right) = const. , \tag{14.30}$$

and this relation will be used now to understand converging and converging-diverging nozzle flows. For this, we study the outflow from a large container into a nozzle, where the gas in the container is in the constant stagnation state (T_0, p_0) . The flow is driven by the difference between the back pressure outside the nozzle, p_b , and the stagnation pressure p_0 , as indicated in Fig. 14.5. No flow occurs when $p_b = p_0$, and we now study what happens when p_b is lowered gradually.

14.6 Converging Nozzle

As the back pressure p_b is lowered a bit, the flow develops. The cross section A is decreasing along the nozzle coordinate x , see Fig. 14.6. According to Fig. 14.4, the flow function is hill-shaped, and the nozzle feed state is on the right foot of that hill, at $p = p_0, \psi = 0$. Since $A \psi \left(\frac{p}{p_0}\right)$ is constant, for

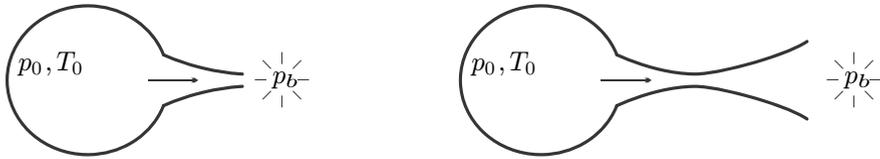


Fig. 14.5 Nozzle flows from stagnation. Left: Converging nozzle. Right: Converging-diverging nozzle (Laval nozzle).

decreasing cross section A the flow function must grow, i.e., go uphill. The flow function grows along x , and reaches its largest value in the smallest cross section A_{th} , the throat of the nozzle, which is at the end of the converging nozzle. As ψ grows along the nozzle coordinate, the pressure decreases, until it assumes the pressure $p_e = p_b$ in the end cross section of the nozzle. Further decrease of the back pressure leads to lower pressures along the nozzle, and larger values of the flow function.

When the back pressure p_b assumes the critical value p^* , the flow function in the exit is at its maximum ψ^* . No further growth of ψ is possible when the back pressure is lowered further. Thus, the exit state remains at $\psi_e = \psi^*$ and $p_e = p^*$, even when the back pressure is lowered below p^* . Figure 14.6 visualizes this behavior.

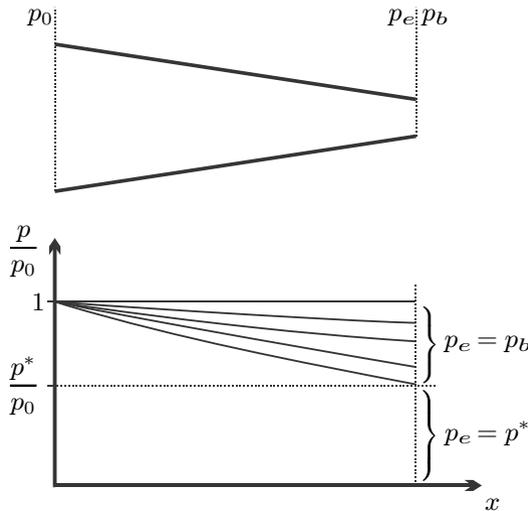


Fig. 14.6 Converging nozzle: Geometry and pressure profile

When the pressure p^* is reached in the nozzle throat, one speaks of choked flow. Indeed, under this condition, the mass flow obtains a maximum value,

$$\dot{m}^* = \rho_0 \sqrt{kRT_0} A_{th} \left(\frac{2}{k+1} \right)^{\frac{k+1}{2k-2}}, \quad (14.31)$$

where A_{th} is the cross section at the nozzle end. No further increase of the mass flow through the nozzle is possible. To understand this behavior, we determine temperature and velocity in the exit at choked conditions. From the adiabatic relation and (14.26, 14.29) we find

$$T^* = T_0 \left(\frac{p^*}{p_0} \right)^{\frac{k-1}{k}} = \frac{2T_0}{k+1} \quad \text{and} \quad \mathcal{V}^* = \sqrt{\frac{2kRT_0}{k+1}} = \sqrt{kRT^*} = a^*. \quad (14.32)$$

Thus, for choked flow, the exit speed is just the local speed of sound, a^* . We recall that the speed of sound is the velocity of a pressure disturbance. When the back pressure is lowered, the information on the pressure change travels with the speed of sound relative to the gas. Since the gas leaves with just the same speed, the information on pressure change is not transmitted into the nozzle, and no changes can occur inside, the exit velocity and the mass flow are limited.

With $p_b < p^*$, a pressure discontinuity occurs at the nozzle exit, which contributes to thrust for the airplane or rocket. The exhaust expands outside the nozzle to the back pressure, and accelerates, and this expansion is somewhat irreversible, more so with bigger pressure differences.² Most commercial airplanes have converging nozzles and subsonic outflow, to reduce noise.

14.7 Example: Safety Valve

The mass flow limitation must be considered for the design of safety valves. As an example we consider a steam boiler that produces 10t/h of saturated vapor at $p_0 = 15$ bar (so that $T_0 = 471$ K). In case of emergency, all steam produced must be discharged through a safety valve. We compute the minimum cross section the valve can have under the assumption that the steam can be described as an ideal gas with constant specific heats and $k = 1.135$. Solving (14.31) for the cross section gives, with the ideal gas law $p_0 = \rho_0 RT_0$,

$$A_{th} = \frac{\dot{m}^*}{p_0} \sqrt{\frac{RT_0}{k}} \left(\frac{k+1}{2} \right)^{\frac{k+1}{2k-2}} = 13.5 \text{ cm}^2.$$

² In the discussion of jet engines we have considered adiabatic expansion to the back pressure, which, for $p_b < p^*$, is only possible in Laval nozzles as discussed below. This simplification ignores the external expansion; however, the associated irreversibilities can be included into the nozzle efficiency. A full discussion of the exterior expansion, and the related thrust is beyond the scope of this book.

14.8 Laval Nozzle

To reach exit velocities above the speed of sound requires converging-diverging nozzles, which will be discussed next. The cross section A is first decreasing until it reaches its smallest value A_{th} in the throat of the nozzle, and then it is increasing to the exit cross section A_e . Again we consider the pressure distribution in the nozzle which results from lowering the back pressure p_b , see Fig. 14.7.

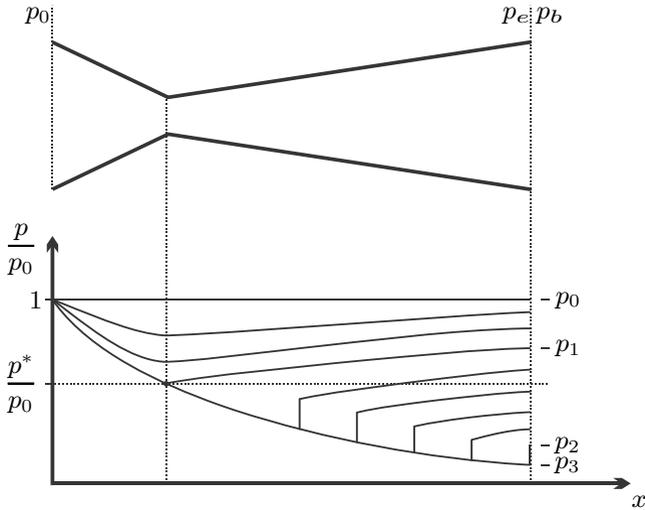


Fig. 14.7 Geometry and pressure profiles for a Laval nozzle. Pressure profiles depend on the back pressure as discussed in the text.

When $p_b = p_0$, there is no flow and homogenous pressure p_0 throughout the nozzle. When the back pressure is lowered somewhat, a flow occurs. According to Fig. 14.4, the flow function is hill-shaped, and the nozzle feed state is on the right foot of that hill, at $p = p_0, \psi = 0$. Since $A\psi\left(\frac{p}{p_0}\right)$ is constant, the flow function must grow—go uphill—in the converging part of the nozzle, and it must decrease—go downhill—in the diverging part.

We first consider relatively large back pressures in the pressure range $p_1 < p_b < p_0$ in the figure. As the nozzle converges, the flow state climbs uphill until a value $p_{th} > p^*, \psi_{th} < \psi^*$ is reached in the throat. This point is to the right of the maximum. As the cross section grows in the diverging part, the flow function must decrease, and this is only possible by returning to higher pressures, that is by going back downhill towards the right. The flow is accelerated in the converging part of the nozzle, and decelerated in the diverging part, the outflow velocity is relatively low, and subsonic. The extreme case of this flow type is reached for $p_b = p_1$, when the air is in the

critical state in the throat—on top of the hill with sonic speed—but then is decelerated again.

When the flow reaches the critical state in the throat, the flow function can decrease by going down the left side of the hill, towards lower pressures, and higher, i.e., supersonic, velocities. As indicated in the figure, this requires low back pressures $p_2 < p_b \leq 0$. In this range the flow is isentropic inside the nozzle. If the back pressure is just at p_3 , the end pressure is equal to the back pressure, and no external irreversibilities occur. If the back pressure is in the range $p_2 < p_b < p_3$, the end pressure is below the back pressure and pressure is equilibrated through oblique shocks outside the nozzle. For lower back pressures $p_b < p_3$, the end pressure is above the back pressure and pressure is equilibrated through external expansion waves.

In the range $p_2 < p_b < p_1$, isentropic flow inside the nozzle is not possible. The flow will follow isentropic flow conditions for a while, accelerating to supersonic flow behind the throat, and then a irreversible normal shock will occur, that is a sudden jump from low pressure supersonic flow to high pressure subsonic flow. Behind the shock, the flow will be isentropic again, and the gas will expand to the back pressure.

We shall not further discuss normal and oblique shocks, but stress that they are strongly irreversible, and thus reduce nozzle efficiency substantially. In a shock, the flow changes from supersonic to subsonic, which means significant reduction of thrust. Nozzle geometry must be carefully designed so that the flow conditions are optimal. Some supersonic aircraft have nozzles with variable geometry, to adjust for the wide range of back pressures encountered between take-off and high altitude flight.

14.9 Rockets, Ramjet and Scramjet

Jet engines are air-breathing, that is they carry the fuel on board, and burn it with oxygen from the ambient air that passes through the engine. Rockets carry the oxygen on board, either as liquefied oxygen, or in the form of a compound. Thus, rockets are independent of ambient air, and can fly at extremely high altitude, and in space. In a rocket motor, fuel and oxidizer are burnt at high pressure in the combustion chamber and then expand through a Laval nozzle, which provides large supersonic exit velocities, and thus large thrust, see Fig. 14.8.

The oxygen a rocket has to carry on board increases the take-off weight, and reduces the payload that can be carried along. Some military applications require fast transport of payload through the atmosphere,³ and rockets are used because they offer extremely high velocities.

Ramjet and scramjet are conceptually simple air-breathing engines for supersonic flight, hence they do not require to carry the heavy oxidizer on board.

³ ... It is left to the reader to fathom what kind of delivery would be that urgent ...

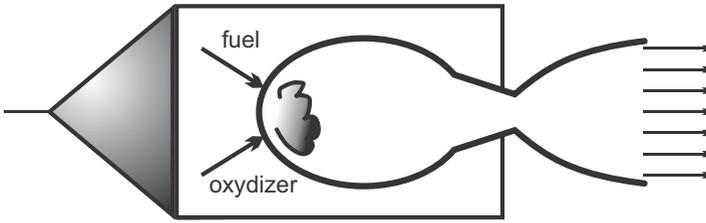


Fig. 14.8 A rocket

In both pressure build-up in the engine is affected only through diffusers, see Fig. 14.9 for sketches.

In a ramjet, the incoming air is slowed down to subsonic speed and high pressure by means of a converging-diverging diffuser, which operates on the same principles as a Laval nozzle, only inversely. Fuel is injected into the compressed air and burned, the hot combustion product then expands through a Laval nozzle to high velocities. Ramjets can be used for supersonic flight with speeds up to $Ma = 6$.

In a scramjet (supersonic combustion ramjet), the flow stays supersonic at all times. A supersonic diffuser slows down the flow and pressure increases. Fuel is injected into the flow and burned, and the hot pressurized combustion product leaves through a supersonic nozzle. Flight speeds could be up to $Ma = 15$ or so. Due to the high air velocity at the burner, it is quite difficult to maintain stable combustion of the fuel. To get the scramjet engine started it must be accelerated to supersonic speed first, so that the converging diffuser leads to pressure build-up.

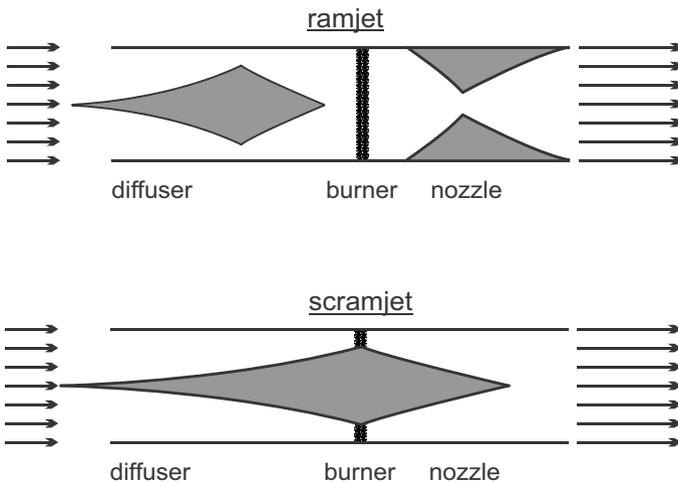


Fig. 14.9 Ramjet and scramjet

14.10 Example: Ramjet

A ramjet is to fly at $\text{Ma}_I = 3$ in high altitude, where the temperature is $T_I = 200 \text{ K}$ and the pressure is $p_I = 0.3 \text{ bar}$. The diffuser inlet, diffuser outlet, and nozzle inlet all have the cross section $A_R = 0.1 \text{ m}^2$. We determine the throat cross sections of diffuser and nozzle, and the nozzle exit cross section for the case that the nozzle expands isentropically to the outside pressure. Moreover, we will compute exit velocity, thrust, and propulsive power. To simplify proceedings, we consider the working gas air as ideal gas with constant specific heats, $R = 0.287 \frac{\text{kJ}}{\text{kg K}}$, $k = 1.4$, $c_p = \frac{k}{k-1} R$.

Diffuser: The incoming flow is at T_I, p_I and Ma_I . The inlet density and velocity are

$$\rho_I = \frac{p_I}{RT_I} = 0.523 \frac{\text{kg}}{\text{m}^3}$$

$$\mathcal{V}_I = \text{Ma}_I a_I = \text{Ma}_I \sqrt{kRT_I} = 850.4 \frac{\text{m}}{\text{s}} .$$

This gives the mass flow

$$\dot{m} = \rho_I \mathcal{V}_I A_R = 44.45 \frac{\text{kg}}{\text{s}} .$$

To be able to use the equations for nozzles, which also hold for diffusers, we first need to determine the stagnation state for the diffuser. The stagnation temperature T_{D0} follows from the first law (adiabatic deceleration), and the corresponding stagnation pressure p_{D0} follows from the isentropic relation (with $h(T) = c_p(T - T_0)$):

$$h_{in} + \frac{1}{2} \mathcal{V}_{in}^2 = h_{D0} \implies T_{D0} = \frac{c_p T_{in} + \frac{1}{2} \mathcal{V}_{in}^2}{c_p} = 560.2 \text{ K} ,$$

$$p_{D0} = p_{in} \left(\frac{T_{D0}}{T_{in}} \right)^{\frac{k}{k-1}} = 11.03 \text{ bar} .$$

The critical state for the diffuser is determined from (14.29, 14.32) as

$$p^{D*} = p_{D0} \left(\frac{2}{k+1} \right)^{\frac{k}{k-1}} = 5.82 \text{ bar} , \quad \psi^{D*} = \left(\frac{2}{k+1} \right)^{\frac{1}{k-1}} \sqrt{\frac{k-1}{k+1}} = 0.2588 ,$$

$$T^{D*} = T_{D0} \left(\frac{p_{D0}}{p^{D*}} \right)^{\frac{k-1}{k}} = 466.7 \text{ K} , \quad \mathcal{V}^{D*} = \sqrt{kRT^{D*}} = 433.0 \frac{\text{m}}{\text{s}} ,$$

$$\rho^{D*} = \frac{p^{D*}}{RT^{D*}} = 4.35 \frac{\text{kg}}{\text{m}^3} .$$

The throat cross section follows from the mass flow,

$$A^{D*} = \frac{\dot{m}}{\rho^{D*} \mathcal{V}^{D*}} = 0.0236 \text{ m}^2 .$$

The diffuser exit area is given, and the corresponding exit pressure p_D follows from the constant mass flow relation $A\psi = \text{const.}$ by first determining the exit flow function ψ_D and then finding the corresponding pressure; this yields

$$\psi_D = \psi^{D*} \frac{A^{D*}}{A_R} = 0.611 \quad \Rightarrow \quad p_D = 0.987 p_{D0} = 10.87 \text{ bar} .$$

The pressure can either be found from the plot of the flow function (Fig. 14.4), or by numerical solution of (14.28); note that there are two solutions, the larger one refers to subsonic flow. Temperature, density and velocity at the diffuser outlet are

$$T_D = T^{D*} \left(\frac{p_D}{p^{D*}} \right)^{\frac{k-1}{k}} = 557.9 \text{ K} , \quad \rho_D = \frac{p_D}{RT_D} = 6.79 \frac{\text{kg}}{\text{m}^3} ,$$

$$\mathcal{V}_D = \frac{\dot{m}}{\rho_D A_R} = 65.4 \frac{\text{m}}{\text{s}} .$$

Burner: In the burner, fuel is injected into the compressed air and isobarically burned. As always, we ignore the mass of the fuel added, and treat the combustion product as air. With the temperature after the burner at $T_B = 1300 \text{ K}$ and the pressure $p_B = p_D$, we find density and velocity as

$$\rho_B = \frac{p_D}{RT_D} = 2.91 \frac{\text{kg}}{\text{s}} , \quad \mathcal{V}_B = \frac{\dot{m}}{\rho_B A_R} = 152.5 \frac{\text{m}}{\text{s}} .$$

The total heat supplied is

$$\dot{Q}_B = \dot{m} c_p (T_B - T_D) = 33.1 \text{ MW} .$$

Nozzle: The inlet state for the nozzle is the exit state of the burner ($\rho_B, T_B, \mathcal{V}_B$). For the computation of the nozzle, we must first determine its stagnation state:

$$T_{N0} = \frac{c_p T_B + \frac{1}{2} \mathcal{V}_B^2}{c_p} = 1311.6 \text{ K} , \quad p_{N0} = p_B \left(\frac{T_{N0}}{T_B} \right)^{\frac{k}{k-1}} = 11.22 \text{ bar} .$$

Critical data in the nozzle throat are

$$\begin{aligned}
 p^{N*} &= p_{N0} \left(\frac{2}{k+1} \right)^{\frac{k}{k-1}} = 5.92 \text{ bar} , \quad \psi^{N*} = 0.2588 , \\
 T^{N*} &= T_{N0} \left(\frac{p_{N0}}{p^{N*}} \right)^{\frac{k-1}{k}} = 1093.0 \text{ K} , \quad \mathcal{V}^{N*} = \sqrt{kRT^{N*}} = 622.7 \frac{\text{m}}{\text{s}} , \\
 \rho^{N*} &= \frac{p^{N*}}{RT^{N*}} = 1.89 \frac{\text{kg}}{\text{m}^3} , \quad A^{N*} = \frac{\dot{m}}{\rho^{N*}\mathcal{V}^{N*}} = 0.0355 \text{ m}^2 .
 \end{aligned}$$

The computation of the throat cross section from critical data and the mass flow is already included in the above list.

The last process to consider is the isentropic expansion in the diverging part of the nozzle to the outside pressure $p_I = 0.3 \text{ bar}$. We find the following data for the nozzle exit:

$$\begin{aligned}
 \psi_E &= \left(\frac{p_I}{p_{N0}} \right)^{\frac{1}{k}} \sqrt{1 - \left(\frac{p_I}{p_{N0}} \right)^{\frac{k-1}{k}}} = 0.0604 , \quad A_E = A^{N*} \frac{\psi^{N*}}{\psi_E} = 0.152 \text{ m}^2 , \\
 T_E &= T^{N*} \left(\frac{p_I}{p^{N*}} \right)^{\frac{k-1}{k}} = 466.1 \text{ K} , \quad \rho_E = \frac{p_I}{RT_E} = 0.224 \frac{\text{kg}}{\text{m}^3} .
 \end{aligned}$$

Exit velocity and Mach number are

$$\mathcal{V}_E = \frac{\dot{m}}{\rho_E A_E} = 1303.3 \frac{\text{m}}{\text{s}} , \quad \text{Ma}_E = \frac{\mathcal{V}_E}{\sqrt{kRT_E}} = 3.01 .$$

Power and Efficiency: All cross sections and all property data for reversible operation were computed above. From the given data we find thrust and propulsive power as

$$F = \dot{m} (\mathcal{V}_E - \mathcal{V}_{in}) = 201.12 \text{ kN} , \quad \dot{W}_P = F\mathcal{V}_{in} = 17.12 \text{ MW} .$$

This corresponds to the thermal propulsive efficiency and the Froude efficiency

$$\eta_P = \frac{\dot{W}_P}{\dot{Q}_B} = 0.517 , \quad \eta_F = \frac{2\mathcal{V}_{in}}{\mathcal{V}_{in} + \mathcal{V}_E} = 0.79 .$$

Problems

14.1. Speed of Sound

Determine the speed of sound in helium (based on constant specific heats) and air (based on variable specific heats, tables) at 300 K and 1500 K. Compute the corresponding Mach numbers for a velocity of $290 \frac{\text{m}}{\text{s}}$.

14.2. Speed of Sound in R-134a

Determine the speed of sound in refrigerant R-134a at 1 MPa, 60 °C. Use table data!

14.3. Security Valve

A steam boiler produces saturated vapor at 17.5 bar. The security valve has a smallest free area of 20 cm^2 . Determine the maximum mass flow that can be produced so that no pressure is building up when the valve is open.

14.4. Laval Nozzle

A Laval nozzle is to be designed such that it delivers $4 \frac{\text{kg}}{\text{s}}$ of air at 10°C and 1 bar at twice the speed of sound. The air that expands in the nozzle is delivered by an isentropic compressor that draws air at 1 bar and 10°C .

Consider air as ideal gas with constant specific heats, $R = 0.287 \frac{\text{kJ}}{\text{kg K}}$, $c_p = 1.004 \frac{\text{kJ}}{\text{kg K}}$.

1. Determine the cross section at the end of the nozzle.
2. Determine temperature, pressure, mass density and velocity in the throat.
3. Determine stagnation pressure and stagnation temperature for the nozzle flow.
4. Determine the power consumed by the compressor.
5. Determine the heat that must be withdrawn from the flow between compressor and nozzle.

14.5. Nozzle Flow

Consider a Laval nozzle for rocket propulsion. Pressure and temperature in the combustion chamber are 10 bar and 2500 K, respectively. The mass flow through the nozzle is $30 \frac{\text{kg}}{\text{s}}$ of combustion product (ideal gas, constant specific heats, $R = 0.287 \frac{\text{kJ}}{\text{kg K}}$, $k = 1.4$), the cross section at the end of the nozzle is $A_e = 700 \text{ cm}^2$ and the flow is isentropic throughout the nozzle. The environmental pressure is 0.9 bar.

1. Do you expect supersonic or subsonic flow at the outlet? Why?
2. Compute the area of the throat of the nozzle.
3. Find pressures and gas velocities at throat and end.
4. Discuss the flow behind the nozzle

14.6. Rocket Engine

A converging-diverging nozzle is fed from a combustion chamber at temperature $T_0 = 2200 \text{ K}$. The flow through the nozzle is isentropic, and the outflow is supersonic with the velocity $v = 1400 \frac{\text{m}}{\text{s}}$. The pressure in the throat is measured as $p^* = 4 \text{ bar}$.

The gas flowing through the nozzle can be considered as an ideal gas with constant specific heats: $c_p = 0.98 \frac{\text{kJ}}{\text{kg K}}$, $k = c_p/c_v = 1.4$.

1. Determine the pressure in the combustion chamber.
2. Determine temperature and pressure at the nozzle exit.
3. Determine the speed of sound at the nozzle exit.