Chapter 14
How are rational points distributed, really?

In §3.2 we found a description of all the points with rational coordinates on the unit circle $x^2 + y^2 = 1$. In this chapter we examine some topological and analytic properties of these rational points. In particular, we will show that points with rational coordinates are *equidistributed* with a respect to a natural measure on the unit circle centered at the origin. The starting point of our investigation is the concept of *equidistribution* on the real line, and addressing the equidistribution properties of rational numbers according to a natural measure on the real line. This requires introducing an ordering of the set of rational numbers. The ordering we use is determined by the *height* of the rational number. The proof of Theorem 14.3, while in principle straightforward, is very complicated. We end the first section of this chapter with a strengthening of the latter theorem, Theorem 14.4. The proof of this theorem uses some technical tools from analysis. We prove the equidistribution of rational points on the unit circle in the second section of the chapter. In the Notes, we state a general theorem of Bohl, Sierpiński, and Weyl, proved independently of each other, about the distribution of a sequence of numbers in the interval [0, 1]. We also make some comments about the general question of the equidistribution of rational points on higher dimensional spheres.

14.1 The real line

It is a well-known fact that the set of rational numbers is dense in the set of real numbers. Our first goal here is to quantify this density statement.

Definition 14.1. Suppose $I = (\alpha, \beta)$ is an interval in $\mathbb{R}$, and ϑ a Riemann integrable function on I. We say a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of I is *ϑ-equidistributed*, or *equidistributed with respect to the function ϑ*, if for each subinterval $J \subset I$ we have

$$\lim_{X\to\infty} \frac{\#\{n \leq X \mid x_n \in J\}}{X} = \int_J \vartheta(x)\, dx.$$

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If $\vartheta(x) = 1/(\beta - \alpha)$ for all $x \in I$, we simply say the sequence $\{x_n\}$ is *equidistributed* in I.

We note that if a sequence $\{x_n\}$ is equidistributed in the interval I it will be dense in the interval, but not vice versa. In fact, it is possible to construct sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ with the property that

$$\{x_n \mid n \in \mathbb{N}\} = \{y_n \mid n \in \mathbb{N}\}$$

with $\{x_n\}_n$ equidistributed, and $\{y_n\}_n$ not equidistributed; see Exercise 14.1. These examples also show that whether a sequence $\{x_n\}_n$ is equidistributed in an interval I depends strongly on the particular ordering of the elements of $\{x_n\}_n$.

We now turn our attention to the study of the distribution of rational numbers in real numbers. It is already an interesting problem to find a function ϑ such that the set of rational numbers is ϑ-equidistributed in the set of real numbers. As pointed out earlier, the function ϑ depends very much on the choice of the ordering of the set of rational numbers. Let us describe one such ordering which is particularly natural.

Definition 14.2. For a rational number $\gamma = r/s$ with $r, s \in \mathbb{Z}$ with $\gcd(r, s) = 1$, we define the *height* of γ by

$$H(\gamma) = \sqrt{r^2 + s^2}.$$

The motivation behind this definition is that we tend to think of the rational number

$$\frac{5000001473}{5000003010}$$

as a more *arithmetically complicated* rational number than $1.02 = 51/50$, even though both numbers are approximately 1. The height function quantifies this notion, in the sense that

$$H\left(\frac{5000001473}{5000003010}\right) = \sqrt{5000003010^2 + 5000001473^2}$$

is much bigger than

$$H(1.02) = \sqrt{51^2 + 50^2}.$$

An interesting property of our height function is that for all finite $B > 0$ the number of rational numbers γ with $H(\gamma) \leq B$ is finite. In fact, if $\gamma = r/s$ with $\gcd(r, s) = 1$, $H(\gamma) \leq B$ means $|r| \leq B$ and $|s| \leq B$. There are only finitely many such integers r and s. For example, the following rational numbers γ have the property $H(\gamma) \leq 4$:

$$0, \pm 1, \pm 2, \pm 1/2, \pm 3, \pm 1/3, \pm 2/3, \pm 3/2.$$

The proof of the following theorem occupies most of the remainder of this chapter:

Theorem 14.3. *Rational numbers ordered by their height are equidistributed in every interval (α, β), including unbounded intervals, with respect to*

$$\vartheta(t) = \frac{1}{\pi} \cdot \frac{1}{1+t^2}.$$

Proof. We need to compute the limit

$$S(\alpha, \beta) := \lim_{X\to\infty} \frac{\#\{\gamma \in \mathbb{Q} \cap (\alpha, \beta) \mid H(\gamma) \leq X\}}{\#\{\gamma \in \mathbb{Q} \mid H(\gamma) \leq X\}} \tag{14.1}$$

for each $\alpha < \beta$.

Two basic observations:

- For each $\alpha < \beta$, $S(\alpha, \beta) = S(-\beta, -\alpha)$;
- for each $\alpha < \beta < \delta$, we have $S(\alpha, \beta) + S(\beta, \delta) = S(\alpha, \delta)$.

These observations imply that it suffices to compute $S(\alpha, \beta)$ in the following three cases:

1. $0 < \alpha < \beta < 1$;
2. $1 < \alpha < \beta$.
3. $1 < \alpha$ and $\beta = +\infty$.

We will compute $S(\alpha, \beta)$ in each case.

First we find a formula for the denominator of the expression in Equation (14.1),

$$n(X) := \#\{\gamma \in \mathbb{Q} \mid H(\gamma) \leq X\}.$$

For a non-zero rational number $\gamma = m/n$ with $\gcd(m, n) = 1$, we have

$$H(\gamma) = H(-\gamma) = H(\gamma^{-1}) = H(-\gamma^{-1}) = \sqrt{m^2 + n^2}.$$

It is now not hard to see (Exercise 14.7) that

$$n(X) = \frac{1}{2} f(X^2) + O(1). \tag{14.2}$$

with f defined by

$$f(B) = \#\{(x, y) \neq (0, 0) \mid x, y \in \mathbb{Z}, \gcd(x, y) = 1, x^2 + y^2 \leq B\}.$$

By the computation of C_2 from §13.2 and Exercise 14.8, we have

$$n(X) = \frac{1}{2\zeta(2)} X^2 + O(X \log X). \tag{14.3}$$

Now we find an expression for

$$n_{\alpha,\beta}(X) := \#\{\gamma \in \mathbb{Q} \cap (\alpha, \beta) \mid H(\gamma) \leq X\}. \tag{14.4}$$

Suppose $0 < \alpha < \beta < 1$, and that we have a reduced fraction $n/m \in (\alpha, \beta)$ with $H(n/m) \leq X$. This means, $m, n \in \mathbb{N}$, $\gcd(m, n) = 1$, $m^2 + n^2 \leq X^2$, and $\alpha m < n < \beta m$.

Our strategy is to write $n_{\alpha,\beta}$ as a sum of 1's over the defining conditions on m, n, and then use the function μ from Chapter 13 to handle the coprimality condition on m, n. Eventually we will use the geometric method of the proof of Theorem 9.4 to finish the computation.

By Lemma 13.2 applied to $\gcd(m, n)$ we have

$$\sum_{d|\gcd(m,n)} \mu(d) = \begin{cases} 1 & \text{if } m, n \text{ coprime;} \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} n_{\alpha,\beta}(X) &= \sum_{\substack{m,n\in\mathbb{N}, \gcd(m,n)=1 \\ m^2+n^2\leq X^2, \alpha m<n<\beta m}} 1 = \sum_{\substack{m,n\in\mathbb{N} \\ m^2+n^2\leq X^2, \alpha m<n<\beta m}} \sum_{d|\gcd(m,n)} \mu(d) \\ &= \sum_{d\leq X} \mu(d) \sum_{\substack{m,n\in\mathbb{N}, d|m, d|n \\ m^2+n^2\leq X^2, \alpha m<n<\beta m}} 1 = \sum_{d\leq X} \mu(d) \sum_{\substack{m,n\in\mathbb{N} \\ m^2+n^2\leq X^2/d^2, \alpha m<n<\beta m}} 1. \end{aligned}$$

Consequently, if we set

$$\tilde{n}_{\alpha,\beta}(X) = \sum_{\substack{m,n\in\mathbb{N} \\ m^2+n^2\leq X^2, \alpha m<n<\beta m}} 1,$$

we have

$$n_{\alpha,\beta}(X) = \sum_{d\leq X} \mu(d) \tilde{n}_{\alpha,\beta}\left(\frac{X}{d}\right).$$

Our immediate task is to find a formula for $\tilde{n}_{\alpha,\beta}(X)$. For simplicity we will assume that α, β are irrational numbers; also since eventually we will be letting $X \to \infty$, we will assume that $\alpha^{-1}(1 + \beta^2) < X$.

We start by writing

$$\tilde{n}_{\alpha,\beta}(X) = \sum_{\substack{m\leq X \\ \max(\alpha m,1)\leq n\leq\min(\beta m,\sqrt{X^2-m^2})}} 1 = \sum_{\substack{m<\alpha^{-1} \\ 1\leq n\leq\min(\beta m,\sqrt{X^2-m^2})}} 1 + \sum_{\substack{\alpha^{-1}<m\leq X \\ \alpha m\leq n\leq\min(\beta m,\sqrt{X^2-m^2})}} 1.$$

In the first sum, since $X > \alpha^{-1}\sqrt{1 + \beta^2}$, we have

$$\beta m < \sqrt{X^2 - m^2}.$$

Hence,

$$\sum_{\substack{m<\alpha^{-1} \\ 1\le n\le \min(\beta m,\sqrt{X^2-m^2})}} 1 = \sum_{\substack{m<\alpha^{-1} \\ 1\le n\le \beta m}} 1 = O(1)$$

as the whole sum can be bounded independent of X.

So we have shown

$$\sum_{\substack{m<\alpha^{-1} \\ 1\le n\le \min(\beta m,\sqrt{X^2-m^2})}} 1 = O(1). \tag{14.5}$$

Now we examine the second sum

$$\sum_{\substack{\alpha^{-1}<m<X \\ \alpha m\le n\le \min(\beta m,\sqrt{X^2-m^2})}} 1.$$

We note that if

$$\alpha^{-1} < m \le \frac{X}{\sqrt{1+\beta^2}},$$

then

$$\beta m \le \sqrt{X^2-m^2},$$

and if

$$\frac{X}{\sqrt{1+\beta^2}} < m \le X,$$

then

$$\sqrt{X^2-m^2} \le \beta m.$$

As a result the sum is equal to

$$\sum_{\substack{\alpha^{-1}<m<X \\ \alpha m\le n\le \min(\beta m,\sqrt{X^2-m^2})}} 1 = \sum_{\substack{\alpha^{-1}<m\le \frac{X}{\sqrt{1+\beta^2}} \\ \alpha m\le n\le \beta m}} 1 + \sum_{\substack{\frac{X}{\sqrt{1+\beta^2}}<m\le X \\ \alpha m\le n\le \sqrt{X^2-m^2}}} 1.$$

We analyze each piece separately.

We have

$$\sum_{\substack{\alpha^{-1}<m\le \frac{X}{\sqrt{1+\beta^2}} \\ \alpha m\le n\le \beta m}} 1 = \sum_{\alpha^{-1}<m\le \frac{X}{\sqrt{1+\beta^2}}} ([\beta m]-[\alpha m])$$

$$= \sum_{\alpha^{-1}<m\le \frac{X}{\sqrt{1+\beta^2}}} (\beta-\alpha)m + O(1) = \frac{\beta-\alpha}{2(1+\beta^2)}X^2 + O(X),$$

after using Corollary A.6.

We have shown

$$\sum_{\substack{\alpha^{-1} < m \le \frac{X}{\sqrt{1+\beta^2}} \\ \alpha m \le n \le \beta m}} 1 = \frac{\beta - \alpha}{2(1+\beta^2)} X^2 + O(X). \tag{14.6}$$

Next, we consider the sum

$$\sum_{\substack{\frac{X}{\sqrt{1+\beta^2}} < m \le X \\ \alpha m \le n \le \sqrt{X^2 - m^2}}} 1.$$

The important point to note is that for some values of m, $\alpha m > \sqrt{X^2 - m^2}$, and for such values, the n-sum is empty. To have $\alpha m \le \sqrt{X^2 - m^2}$, we need to have $m \le X(1+\alpha^2)^{-1/2}$ as an easy computation shows. Consequently,

$$\begin{aligned}
\sum_{\substack{\frac{X}{\sqrt{1+\beta^2}} < m \le X \\ \alpha m \le n \le \sqrt{X^2 - m^2}}} 1 &= \sum_{\substack{\frac{X}{\sqrt{1+\beta^2}} < m \le \frac{X}{\sqrt{1+\alpha^2}} \\ \alpha m \le n \le \sqrt{X^2 - m^2}}} 1 = \sum_{\frac{X}{\sqrt{1+\beta^2}} < m \le \frac{X}{\sqrt{1+\alpha^2}}} ([\sqrt{X^2 - m^2}] - [\alpha m]) \\
&= \sum_{\frac{X}{\sqrt{1+\beta^2}} < m \le \frac{X}{\sqrt{1+\alpha^2}}} ([\sqrt{X^2 - m^2}] - \alpha m + O(1)) \\
&= \sum_{\frac{X}{\sqrt{1+\beta^2}} < m \le \frac{X}{\sqrt{1+\alpha^2}}} [\sqrt{X^2 - m^2}] - \alpha \sum_{\frac{X}{\sqrt{1+\beta^2}} < m \le \frac{X}{\sqrt{1+\alpha^2}}} m + O(X) \\
&= \sum_{\frac{X}{\sqrt{1+\beta^2}} < m \le \frac{X}{\sqrt{1+\alpha^2}}} [\sqrt{X^2 - m^2}] - \alpha \left(\frac{X^2}{2(1+\alpha^2)} - \frac{X^2}{2(1+\beta^2)} \right) + O(X).
\end{aligned}$$

The sum

$$\sum_{\frac{X}{\sqrt{1+\beta^2}} < m \le \frac{X}{\sqrt{1+\alpha^2}}} [\sqrt{X^2 - m^2}]$$

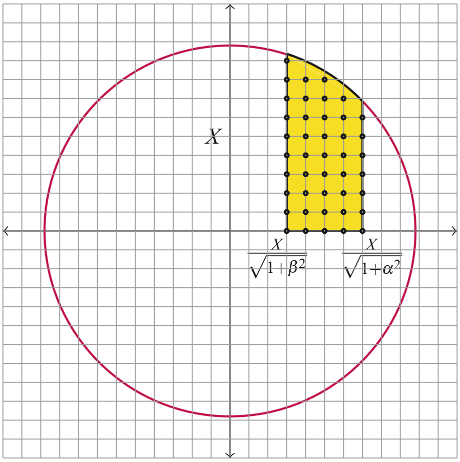
is the number of integral points (m, n) within the disk $x^2 + y^2 \le X^2$ with positive y-coordinates such that the x-coordinate is in the interval

$$\frac{X}{\sqrt{1+\beta^2}} < m \le \frac{X}{\sqrt{1+\alpha^2}}.$$

These are the points with integral coordinates in the yellow region, including the boundary, in Figure 14.1.

By an argument similar to proof of Theorem 9.4 (Exercise 14.9), this number is equal to

Fig. 14.1 The integral points (m, n) within the disk $x^2 + y^2 \leq X^2$ with positive y-coordinates such that the x-coordinate is in the interval $\frac{X}{\sqrt{1+\beta^2}} < m \leq \frac{X}{\sqrt{1+\alpha^2}}$



$$\int_{\frac{X}{\sqrt{1+\beta^2}}}^{\frac{X}{\sqrt{1+\alpha^2}}} \sqrt{X^2 - t^2}\, dt + O(X) = X^2 \int_{\frac{1}{\sqrt{1+\beta^2}}}^{\frac{1}{\sqrt{1+\alpha^2}}} \sqrt{1 - t^2}\, dt + O(X).$$

We have

$$\int \sqrt{1-t^2}\, dt = \frac{1}{2}\sin^{-1} t + \frac{1}{2} t\sqrt{1-t^2} + C.$$

Consequently,

$$\int_{\frac{1}{\sqrt{1+\beta^2}}}^{\frac{1}{\sqrt{1+\alpha^2}}} \sqrt{1-t^2}\, dt = \frac{1}{2}\sin^{-1}\frac{1}{\sqrt{1+\alpha^2}} - \frac{1}{2}\sin^{-1}\frac{1}{\sqrt{1+\beta^2}} + \frac{\alpha}{2(1+\alpha^2)} - \frac{\beta}{2(1+\beta^2)}.$$

So we have proved

$$\sum_{\substack{\frac{X}{\sqrt{1+\beta^2}} < m \leq X \\ \alpha m \leq n \leq \sqrt{X^2-m^2}}} 1 = X^2 \left(\frac{1}{2}\sin^{-1}\frac{1}{\sqrt{1+\alpha^2}} - \frac{1}{2}\sin^{-1}\frac{1}{\sqrt{1+\beta^2}} \right)$$

$$+X^2 \left(\frac{\alpha}{2(1+\alpha^2)} - \frac{\beta}{2(1+\beta^2)} \right) - \alpha \left(\frac{X^2}{2(1+\alpha^2)} - \frac{X^2}{2(1+\beta^2)} \right) + O(X).$$

Putting everything together, we have

$$\tilde{n}_{\alpha,\beta}(X) = \frac{\beta-\alpha}{2(1+\beta^2)} X^2 + X^2 \left(\frac{1}{2}\sin^{-1}\frac{1}{\sqrt{1+\alpha^2}} - \frac{1}{2}\sin^{-1}\frac{1}{\sqrt{1+\beta^2}} \right)$$

$$+X^2\left(\frac{\alpha}{2(1+\alpha^2)}-\frac{\beta}{2(1+\beta^2)}\right)-\alpha\left(\frac{X^2}{2(1+\alpha^2)}-\frac{X^2}{2(1+\beta^2)}\right)+O(X)$$

$$=X^2\left(\frac{1}{2}\sin^{-1}\frac{1}{\sqrt{1+\alpha^2}}-\frac{1}{2}\sin^{-1}\frac{1}{\sqrt{1+\beta^2}}\right)+O(X).$$

We set

$$\eta(t)=\frac{1}{2}\sin^{-1}\frac{1}{\sqrt{1+t^2}}.$$

We can now analyze $n_{\alpha,\beta}(X)$. We have

$$n_{\alpha,\beta}(X)=\sum_{d\le X}\mu(d)\tilde{n}_{\alpha,\beta}\left(\frac{X}{d}\right)$$

$$=\sum_{d\le X}\mu(d)\left\{(\eta(\alpha)-\eta(\beta))\left(\frac{X}{d}\right)^2+O\left(\frac{X}{d}\right)\right\}$$

$$=\frac{\eta(\alpha)-\eta(\beta)}{\zeta(2)}X^2+O(X\log X).$$

Finally,

$$S(\alpha,\beta)=\lim_{X\to\infty}\frac{n_{\alpha,\beta}(X)}{n(X)}=\lim_{X\to\infty}\frac{\frac{\eta(\alpha)-\eta(\beta)}{\zeta(2)}X^2+O(X\log X)}{\frac{\pi}{2\zeta(2)}X^2+O(X\log X)}=\frac{\eta(\alpha)-\eta(\beta)}{\pi}.$$

Hence we have proved for $0<\alpha<\beta<1$,

$$S(\alpha,\beta)=\frac{1}{\pi}\left(\sin^{-1}\frac{1}{\sqrt{1+\alpha^2}}-\sin^{-1}\frac{1}{\sqrt{1+\beta^2}}\right)=\frac{1}{\pi}\int_\alpha^\beta\frac{dt}{1+t^2}.$$

Now we handle the case where $1<\alpha<\beta$. In this case we have

$$n_{\alpha,\beta}(X)=\sum_{\substack{m,n\in\mathbb{N},\gcd(m,n)=1\\ m^2+n^2\le X^2,\alpha m<n<\beta m}}1=\sum_{\substack{m,n\in\mathbb{N},\gcd(m,n)=1\\ m^2+n^2\le X^2,\beta^{-1}n<m<\alpha^{-1}n}}1=n_{\beta^{-1},\alpha^{-1}}(X).$$

Consequently, if $1<\alpha<\beta$, we have

$$S(\alpha,\beta)=\frac{1}{\pi}\int_{\beta^{-1}}^{\alpha^{-1}}\frac{dt}{1+t^2}.$$

It is easy to see (Exercise 14.10) that the latter integral is equal to

$$\frac{1}{\pi}\int_\alpha^\beta\frac{dt}{1+t^2}.$$

Now we treat the third case, where $\beta = +\infty$. The argument in this case is very similar to the first case, so we only sketch the proof. In this case we abbreviate $n_{\alpha,+\infty}(X)$ and $\tilde{n}_{\alpha,+\infty}(X)$ to $n_\alpha(X)$ and $\tilde{n}_\alpha(X)$, respectively. As before, we have

$$n_\alpha(X) = \sum_{d \le X} \mu(d)\tilde{n}_\alpha\left(\frac{X}{d}\right).$$

We start by writing

$$\tilde{n}_\alpha(X) = \sum_{\substack{m \le X \\ \alpha m \le n \le \sqrt{X^2 - m^2}}} 1.$$

Since we want $\alpha m \le \sqrt{X^2 - m^2}$ we need to have

$$m \le \frac{X}{\sqrt{1+\alpha^2}}.$$

Thus,

$$\begin{aligned}
\tilde{n}_\alpha(X) &= \sum_{\substack{m \le \frac{X}{\sqrt{1+\alpha^2}} \\ \alpha m \le n \le \sqrt{X^2 - m^2}}} 1 = \sum_{m \le \frac{X}{\sqrt{1+\alpha^2}}} ([\sqrt{X^2 - m^2}] - [\alpha m]) \\
&= \sum_{m \le \frac{X}{\sqrt{1+\alpha^2}}} ([\sqrt{X^2 - m^2}] - \alpha \sum_{m \le \frac{X}{\sqrt{1+\alpha^2}}} m + O(X) \\
&= X^2 \int_0^{\frac{1}{\sqrt{1+\alpha^2}}} \sqrt{1-t^2}\, dt - \frac{\alpha}{2(1+\alpha^2)} X^2 + O(X) \\
&= X^2 \left(\frac{1}{2} \sin^{-1} \frac{1}{\sqrt{1+\alpha^2}} + \frac{\alpha}{2(1+\alpha^2)}) \right) - \frac{\alpha}{2(1+\alpha^2)} X^2 + O(X) \\
&= X^2 \left(\frac{1}{2} \sin^{-1} \frac{1}{\sqrt{1+\alpha^2}} \right) + O(X).
\end{aligned}$$

Again if we set

$$\eta(t) = \frac{1}{2} \sin^{-1} \frac{1}{\sqrt{1+t^2}},$$

we have proved

$$\tilde{n}_\alpha(X) = \eta(\alpha) X^2 + O(X).$$

We can now analyze $n_\alpha(X)$. We have

$$\begin{aligned}
n_\alpha(X) &= \sum_{d \le X} \mu(d)\tilde{n}_\alpha(\frac{X}{d}) = \sum_{d \le X} \mu(d) \left\{ \eta(\alpha) \left(\frac{X}{d}\right)^2 + O\left(\frac{X}{d}\right) \right\} \\
&= \frac{\eta(\alpha)}{\zeta(2)} X^2 + O(X \log X).
\end{aligned}$$

Finally,

$$S(\alpha, +\infty) = \lim_{X\to\infty} \frac{n_\alpha(X)}{n(X)} = \lim_{X\to\infty} \frac{\frac{\eta(\alpha)}{\zeta(2)} X^2 + O(X \log X)}{\frac{\pi}{2\zeta(2)} X^2 + O(X \log X)} = \frac{2\eta(\alpha)}{\pi}.$$

Hence we have proved for $1 < \alpha$,

$$S(\alpha, +\infty) = \frac{1}{\pi} \sin^{-1} \frac{1}{\sqrt{1+\alpha^2}} = \frac{1}{\pi} \int_\alpha^{+\infty} \frac{dt}{1+t^2}.$$

□

Theorem 14.3 has an interesting consequence. We call a real function f on $\mathbb{R}$ *locally Riemann integrable* if for each finite interval I, the restriction of f to I is Riemann integrable on I.

Theorem 14.4. *Let f be a bounded locally Riemann integrable function on $\mathbb{R}$. Then*

$$\lim_{X\to\infty} \frac{1}{\#\{\gamma \in \mathbb{Q} \mid H(\gamma) \leq X\}} \sum_{\gamma\in\mathbb{Q}, H(\gamma)\leq X} f(\gamma) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{1+t^2}\, dt.$$

We note that for a bounded locally Riemann integrable function f as in the theorem, the integral

$$\int_{-\infty}^{+\infty} \frac{f(t)}{1+t^2}\, dt$$

converges absolutely, Exercise 14.11.

Before we can start the proof of the theorem we need a general lemma:

Lemma 14.5. *A sequence $\{x_n\}$ is ϑ-equidistributed in a finite interval $I = (\alpha, \beta)$ if and only if for every Riemann integrable function f on I we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_\alpha^\beta f(x)\vartheta(x)\, dx. \tag{14.7}$$

Proof. The definition of ϑ-equidistribution is equivalent to the validity of (14.7) for the characteristic function of each subinterval of I. This shows the sufficiency of the condition.

Now suppose the sequence $\{x_n\}$ is ϑ-equidistributed. Then Equation (14.7) is valid for all characteristic functions of subintervals of I. Since the two sides of (14.7) are linear in the function f, we conclude that (14.7) is also true for all linear combinations of characteristic functions of subintervals, i.e., step functions.

Now let f be a Riemann integrable function. Fix $\varepsilon > 0$. By [41, Theorem 6.6] there are step functions f_1, f_2 on I such that for all $x \in I$, $f_1(x) \leq f(x) \leq f_2(x)$, and

$$\int_{\alpha}^{\beta} f_2(x)\vartheta(x)\,dx - \int_{\alpha}^{\beta} f_1(x)\vartheta(x)\,dx < \varepsilon. \tag{14.8}$$

Then

$$\int_{\alpha}^{\beta} f_1(x)\vartheta(x)\,dx = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_1(x_n) \le \liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n)$$

$$\le \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n) \le \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_2(x_n) = \int_{\alpha}^{\beta} f_2(x)\vartheta(x)\,dx.$$

Finally, (14.8) implies

$$\left|\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n)\right|$$

$$\le \quad \left|\int_{\alpha}^{\beta} f_2(x)\vartheta(x)\,dx - \int_{\alpha}^{\beta} f_1(x)\vartheta(x)\,dx\right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n) = \liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n).$$

Hence it makes sense to speak of the limit $\lim_N \sum_{n\le N} f(x_n)/N$. Revisiting the earlier inequalities gives

$$\int_{\alpha}^{\beta} f_1(x)\vartheta(x)\,dx \le \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n) \le \int_{\alpha}^{\beta} f_2(x)\vartheta(x)\,dx.$$

Since by definition

$$\int_{\alpha}^{\beta} f_1(x)\vartheta(x)\,dx \le \int_{\alpha}^{\beta} f(x)\,dx \le \int_{\alpha}^{\beta} f_2(x)\vartheta(x)\,dx,$$

we have

$$\left|\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_{\alpha}^{\beta} f(x)\vartheta(x)\,dx\right|$$

$$\le \left|\int_{\alpha}^{\beta} f_2(x)\vartheta(x)\,dx - \int_{\alpha}^{\beta} f_1(x)\vartheta(x)\,dx\right| < \varepsilon.$$

Again, since $\varepsilon > 0$ is arbitrary, the theorem follows. □

Now we can prove the theorem:

Proof of Theorem 14.4. Our first claim is that it suffices to prove the theorem for bounded locally Riemann integrable functions f which are nonnegative, i.e., $f(x) \ge 0$ for all $x \in \mathbb{R}$. In fact, for a function f, if we define the functions f_+, f_- by

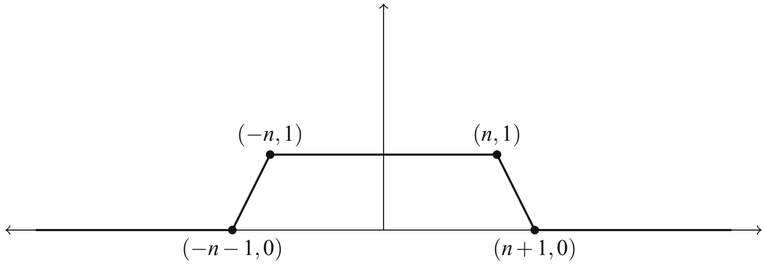


Fig. 14.2 The graph of u_k

$$f_+(x) = \max(f(x), 0), \quad f_-(x) = -\min(f(x), 0),$$

then by Exercise 14.12,

1. $f_+(x), f_-(x) \geq 0$ for all $x \in \mathbb{R}$;
2. f_+, f_- are locally Riemann integrable functions if f is.

It is clear that if we know the theorem for the nonnegative functions f_+, f_-, then we will know the result for the function f.

For reasons that will become clear in a moment, for a natural number k we define a function $u_k(x) : \mathbb{R} \to [0, 1]$ by

$$u_k(x) = \begin{cases} +1 & |x| < k; \\ k + 1 - |x| & k \leq |x| \leq k + 1; \\ 0 & |x| > k + 1. \end{cases}$$

The graph of the function u_k looks like the diagram in Figure 14.2.

Now fix a nonnegative bounded locally Riemann integrable function f on $\mathbb{R}$, and suppose for each $x \in \mathbb{R}$ we have $f(x) \leq C$ for some constant C. For each natural number n, define a function f_k by

$$f_k(x) = f(x)u_k(x).$$

Note that

- for $x \in \mathbb{R}$, $f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots$;
- $\lim_{k\to\infty} f_k(x) = f(x)$.
- for all k and all $x \in \mathbb{R}$, $f(x) - f_k(x) \leq C\chi_{[n,\infty]}(x)$.

For a function g and a real number X we set

$$S(g, X) = \frac{1}{\#\{\gamma \in \mathbb{Q} \mid H(\gamma) \leq X\}} \sum_{\gamma\in\mathbb{Q}, H(\gamma)\leq X} g(\gamma).$$

Note that $S(g, X)$ is linear and increasing in terms of g, meaning if $g_1(x) \le g_2(x)$ for all x, then $S(g_1, X) \le S(g_2, X)$.

By Lemma 14.5, for all k

$$\lim_{X\to\infty} S(f_k, X) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f_k(t)}{1+t^2}\, dt.$$

We have

$$S(f, X) = S(f_k, X) + S(f - f_k, X) \le S(f_k, X) + CS(\chi_{[k,+\infty)}, X).$$

Hence for all k and all X,

$$S(f_k, X) \le S(f, X) \le S(f_k, X) + CS(\chi_{[k,+\infty)}, X). \tag{14.9}$$

By Theorem 14.3 we have

$$\begin{aligned}\lim_{X\to\infty} S(\chi_{[k,+\infty)}, X) &= \lim_{X\to\infty} \frac{\#\{\gamma \in \mathbb{Q} \cap [k,\infty) \mid H(\gamma) \le X\}}{\#\{\gamma \in \mathbb{Q} \mid H(\gamma) \le X\}}\\ &= \frac{1}{\pi}\int_k^{+\infty} \frac{1}{1+t^2}\, dt < \frac{1}{\pi}\int_k^{+\infty} \frac{dt}{t^2} = \frac{1}{\pi k}.\end{aligned}$$

Now in (14.9) we let $X \to \infty$ to obtain

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f_k(t)}{1+t^2}\, dt &= \lim_{X\to\infty} S(f_k, X) \le \liminf_{X\to\infty} S(f, X)\\ \le \limsup_{X\to\infty} S(f, X) &\le \lim_{X\to\infty} S(f_k, X) + \frac{C}{\pi k} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f_k(t)}{1+t^2}\, dt + \frac{C}{\pi k}.\end{aligned}$$

Now we let $k \to \infty$. We obtain

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f_k(t)}{1+t^2}\, dt &\le \liminf_{X\to\infty} S(f, X)\\ \le \limsup_{X\to\infty} S(f, X) &\le \lim_{k\to\infty} \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f_k(t)}{1+t^2}\, dt.\end{aligned}$$

At this point we can simply use the Monotone Convergence Theorem [41, Theorem 11.28] to conclude

$$\lim_{k\to\infty} \int_{-\infty}^{+\infty} \frac{f_k(t)}{1+t^2} dt = \int_{-\infty}^{+\infty} \frac{f(t)}{1+t^2} dt, \tag{14.10}$$

but we will prove this using an elementary argument to avoid relying on measure theory. By the remark after the statement of the theorem, the integrals

$$\int_{-\infty}^{+\infty} \frac{f(t)}{1+t^2}\,dt, \quad \int_{-\infty}^{+\infty} \frac{f_k(t)}{1+t^2}\,dt, \quad \int_{-\infty}^{+\infty} \frac{f(t)-f_k(t)}{1+t^2}\,dt,$$

are all absolutely convergent. Hence we can safely write

$$\int_{-\infty}^{+\infty} \frac{f(t)}{1+t^2}\,dt = \int_{-\infty}^{+\infty} \frac{f_k(t)}{1+t^2}\,dt + \int_{-\infty}^{+\infty} \frac{f(t)-f_k(t)}{1+t^2}\,dt.$$

The functions $f(x)$ and $f_k(t)$ are equal on the interval $[-k,k]$, and for $|x|>k$, $0 \le f(x)-f_k(x) \le C$. Hence,

$$0 < \int_{-\infty}^{+\infty} \frac{f(t)-f_k(t)}{1+t^2}\,dt$$
$$\le C\int_{|t|>k} \frac{dt}{1+t^2} < C\int_{|t|>k} \frac{dt}{t^2} = \frac{2C}{k} = O(\frac{1}{k}).$$

Consequently,

$$\int_{-\infty}^{+\infty} \frac{f(t)}{1+t^2}\,dt = \int_{-\infty}^{+\infty} \frac{f_k(t)}{1+t^2}\,dt + O(\frac{1}{k}).$$

Letting $k \to \infty$ establishes (14.10), and the theorem is proved. □

14.2 The unit circle

We now turn our attention to rational points on the circle $S^1 : x^2+y^2=1$. Our first statement is the following easy proposition:

Proposition 14.6. *The set of points with rational coordinates is dense in the unit circle.*

Proof. It is clear that it suffices to show that rational points are dense among points with positive y-coordinates. The points P and Q on the circle with positive y-coordinates are "close" to each other if and only if their x-coordinates are close to each other. Now suppose $P=(\alpha,\beta)$, with $\beta>0$, is a point on the unit circle. Fix $\varepsilon>0$. We will show that there is a point of the form

$$P_m := \left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$$

with rational m such that the difference between the x-coordinates of P and P_m is less than ε. Without loss of generality assume $\alpha>0$. We also assume that ε is much smaller than α and β. Note there is an $m\in\mathbb{Q}$ such that

$$\alpha-\varepsilon < \frac{1-m^2}{1+m^2} < \alpha+\varepsilon.$$

Indeed, in order for these inequalities to hold we need

$$\sqrt{\frac{1-(\alpha+\varepsilon)}{1+\alpha+\varepsilon}} < m < \sqrt{\frac{1-(\alpha-\varepsilon)}{1+\alpha-\varepsilon}}$$

and there is certainly a rational number m satisfying these inequalities. □

Our purpose in the remainder of this chapter is to give a quantitative version of this density statement, and, as before, the concept that is central to our analysis is *equidistribution*.

In order to speak of equidistribution we need to have a notion of integral. In the case of the unit circle if we parametrize the circle as

$$(\cos\gamma, \sin\gamma), \quad 0 \le \gamma < 2\pi$$

then the natural integration will be relative to $d\gamma$, i.e., if f is a continuous function on the circle then we define

$$\int_{S^1} f := \frac{1}{2\pi}\int_0^{2\pi} f(\cos\gamma, \sin\gamma)\, d\gamma.$$

Definition 14.7. Suppose ϑ is a function on S^1. We say a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of S^1 is *ϑ-equidistributed*, or *equidistributed with respect to the function ϑ*, if for every continuous function f on S^1 we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_{S^1} f\vartheta.$$

If $\vartheta(x, y) = 1$ for all $(x, y) \in S^1$, we simply say the sequence $\{x_n\}$ is *equidistributed* on S^1.

Recall from §3.2 that we have an explicit parametrization of the points on the circle S^1:

$$\eta(t) := \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \tag{14.11}$$

with $t \in \mathbb{R}$, plus the point $(-1, 0)$ which corresponds to t being equal to "infinity." Also, recall that if $\gamma \in \mathbb{Q}$, then $\eta(\gamma)$ is a point with rational coordinates on the circle, and that the set of points $\eta(\gamma)$ for $\gamma \in \mathbb{Q}$ with the point $(-1, 0)$ is equal to the set of points with rational coordinates on the circle.

In order to speak of equidistribution of rational points on the circle, we need a notion of ordering. A natural way to order rational points $\eta(\gamma)$, $\gamma \in \mathbb{Q}$, is according to the height of the rational numbers γ. As an example, earlier in this chapter we determined all rational numbers γ with $H(\gamma) \le 4$:

$$0, \pm1, \pm2, \pm1/2, \pm3, \pm1/3, \pm2/3, \pm3/2.$$

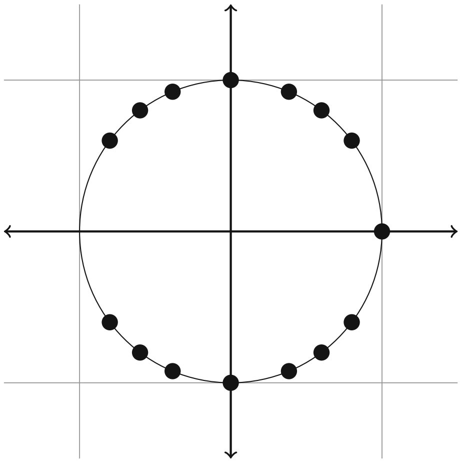


Fig. 14.3 Points of the form $\eta(\gamma)$ with $H(\gamma) \leq 4$

If we draw all points of the form $\eta(\gamma)$ for γ in the above list we obtain the following picture in Figure 14.3.

Theorem 14.8. *The rational points $h(\gamma)$, $\gamma \in \mathbb{Q}$, ordered according to the height of γ are equidistributed on the unit circle, i.e., for each arc ω with length t,*

$$\lim_{X \to \infty} \frac{\#\{\gamma \in \mathbb{Q} \mid H(\gamma) \leq X, \eta(\gamma) \in \omega\}}{\#\{\gamma \in \mathbb{Q} \mid H(\gamma) \leq X\}} = \frac{t}{2\pi}.$$

For a piecewise continuous function f on the unit circle,

$$\frac{1}{\#\{\gamma \in \mathbb{Q} \mid H(\gamma) \leq X\}} \sum_{H(\gamma) \leq X} f(\eta(\gamma)) \to \int_{S^1} f$$

as $X \to \infty$.

Proof. Since S^1 is a compact space and f is continuous, f is bounded. By Theorem 14.4, the limit is equal to

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}) \frac{dt}{1+t^2}.$$

A change of variable $t = \tan(\gamma/2)$ with $-\pi < \gamma < +\pi$ gives the result. The first statement of the theorem follows if we let f be the characteristic function of an arc.

□

Exercises

14.1 Construct examples of sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ with the property that

$$\{x_n \mid n \in \mathbb{N}\} = \{y_n \mid n \in \mathbb{N}\}$$

with $\{x_n\}_n$ equidistributed, and $\{y_n\}_n$ not equidistributed.

14.2 Prove that for $\eta, \xi \in \mathbb{R}$, if $\xi < \eta$ then

$$\sum_{\xi < n \leq \eta} 1 = [\eta] - [\xi]$$

14.3 Show that if $\xi \in \mathbb{R}$ and $\xi \geq 0$,

$$\sum_{0 \leq n \leq \xi} 1 = [\xi] + 1;$$

$$\sum_{-\xi \leq n \leq \xi} = 2[\xi] + 1.$$

14.4 Show that for all natural numbers n,

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{n} + \sqrt{n+2}];$$

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+2}].$$

14.5 Let n be a natural number. Define a set D_n to be the collection of pairs $(x, y) \in \mathbb{Z}^2$ such that

$$0 < x \leq n/2, \quad 0 < y \leq n/2, \quad n/2 \leq x + y < n.$$

Prove that

$$\#D_n = \begin{cases} \frac{(n-2)(n+8)}{8}, & 2 \mid n; \\ \frac{n^2-1}{8}, & 2 \nmid n. \end{cases}$$

14.6 Fix $n, r \in \mathbb{N}$. Find the number of solutions of

$$|x_1| + \cdots + |x_r| \leq n$$

in integers $x_1, \ldots, x_r$.

14.7 Prove Equation (14.2).

14.8 Prove Equation (14.3).

14.9 Suppose $u > v > 1$. Prove that the number of integral points (m, n) within the disk $x^2 + y^2 \leq X^2$ such that $n > 0$ and

$$\frac{X}{u} < m \leq \frac{X}{v}$$

is equal to

$$\int_{\frac{X}{u}}^{\frac{X}{v}} \sqrt{X^2 - t^2}\, dt + O(X).$$

14.10 Show that for each $0 < \alpha < \beta$ we have

$$\int_{\beta^{-1}}^{\alpha^{-1}} \frac{dt}{1+t^2} = \int_{\alpha}^{\beta} \frac{dt}{1+t^2}.$$

14.11 Prove that for a bounded locally Riemann integrable function f on $\mathbb{R}$, the integral

$$\int_{-\infty}^{+\infty} \frac{f(t)}{1+t^2}\, dt$$

converges absolutely.

14.12 For a function $f : \mathbb{R} \to \mathbb{R}$, we define the functions f_+, f_- by

$$f_+(x) = \max(f(x), 0), \quad f_-(x) = -\min(f(x), 0).$$

For all x, $f_+(x), f_-(x) \geq 0$. Show that f is locally Riemann integrable if and only if f_+, f_- are locally Riemann integrable functions.

14.13 We can define another, and perhaps more natural, height function on the set of rational numbers as follows. For a rational number $\gamma = r/s$ with $r, s \in \mathbb{Z}$, $\gcd(r, s) = 1$, we set

$$H'(\gamma) := \max(|r|, |s|).$$

a. List all rational numbers γ with $H'(\gamma) \leq 4$.
b. Show that there is a real number $C > 1$ such that

$$C^{-1}H(\gamma) \leq H'(\gamma) \leq CH(\gamma)$$

for all $\gamma \in \mathbb{Q}$.
c. Find asymptotic formulae for

$$N'(X) := \#\{\gamma \in \mathbb{Q} \mid H'(\gamma) \leq X\}.$$

and

$$N'_1(X) := \#\{\gamma \in \mathbb{Q} \cap [0, 1] \mid H'(\gamma) \leq X\}.$$

d. Show that for a continuous function f on $[0, 1]$ we have

$$\lim_{X\to\infty} \frac{1}{N'_1(X)} \sum_{\gamma\in\mathbb{Q}\cap[0,1]} f(\gamma) = \int_0^1 f(x)\, dx.$$

Hint. Prove the statement for a function of the form $f(x) = x^k$, and then use the Stone–Weierstrass Theorem (in fact, Weierstrass's Theorem [41, Theorem 7.26] is sufficient).
e. Find the function η with respect to which rational points listed according to their H' height are equidistributed.

f. Find the function θ' on the circle S^1 which respect to which the points $\eta(\gamma)$ listed according to the H' of γ are equidistributed.

14.14 (✠) Draw a unit circle. Mark the points $\eta(t)$ with η as in Equation (14.11) and t ranging over rational number $\frac{a}{b}$ with $|a|, |b| < 1000$ and $\gcd(a, b) = 1$.

14.15 (✠) For each integral point $(x, y) \in \mathbb{Z}^2$ with $(x, y) \neq (0, 0)$, define a point $\sigma(x, y) \in \mathbb{R}^2$ with

$$\sigma(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).$$

Show that $\sigma(x, y) \in S^1$. Draw three unit circles and on each one mark one of the following collections of points:

a. $\sigma(x, y)$, $(x, y) \in \mathbb{Z}^2$, $(x, y) \neq (0, 0)$, $|x|, |y| \leq 1000$;
b. $\sigma(x, y)$, $(x, y) \in \mathbb{Z}^2$, $(x, y) \neq (0, 0)$, $|x| + |y| \leq 1000$;
c. $\sigma(x, y)$, $(x, y) \in \mathbb{Z}^2$, $(x, y) \neq (0, 0)$, $\sqrt{x^2 + y^2} \leq 1000$.

Do you see any difference between the patterns you obtain?

14.16 (✠) Compare the patterns you obtain in the previous two exercises.

Notes

The theorem of Bohl, Sierpiński, and Weyl

Piers Bohl, Wacław Sierpiński, and Hermann Weyl proved the following important theorem around 1910 independently of each other: For each irrational α, the sequence $x_n = \{n\alpha\}$, $n \in \mathbb{N}$, is equidistributed in the interval $[0, 1]$, where here $\{n\alpha\}$ is the fractional part of the real number $n\alpha$. In 1916, Weyl proved the remarkable theorem that $\{n^2\alpha\}$, too, is equidistributed in $[0, 1]$, and that is how the theory of equidistribution started. Weyl also proved the following general criterion for the equidistribution of a sequence in the interval $[0, 1]$: Suppose $a_1, a_2, a_3, \ldots$ is a sequence of real numbers. Then the sequence $\{a_1\}, \{a_2\}, \{a_3\}, \ldots$ is equidistributed in the interval $[0, 1]$ if and only if for all non-zero integers m,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i m a_k} = 0.$$

See [33, Ch. 12] or [22, Ch. 1] for comments on the proofs of these statements. The book [22] is a useful collection of articles exploring the various ways in which equidistribution makes an appearance in number theory.

Rational points on the sphere

In this chapter we proved the equidistribution of rational points on the unit circle. Proving the equidistribution of rational points on higher dimensional spheres, even the standard sphere in $\mathbb{R}^3$, is much more difficult. In fact, Duke [72] proved the equidistribution of rational points on the standard sphere in $\mathbb{R}^3$ only in 1998 (!). See [87] for a contemporary treatment of these problems.