

Chapter 1

Introduction



In the first section of this opening chapter we review two different proofs of the Pythagorean Theorem, one due to Euclid and the other one due to a former president of the United States, James Garfield. In the same section we also review some higher dimensional analogues of the Pythagorean Theorem. Later in the chapter we define Pythagorean triples; explain what it means for a Pythagorean triple to be primitive; and clarify the relationship between Pythagorean triples and points with rational coordinates on the unit circle. At the end we list the problems that we will be interested in studying in the book. In the notes at the end of the chapter we talk about Pythagoreans and their, sometimes strange, beliefs. We will also briefly review the history of Pythagorean triples.

1.1 The Pythagorean Theorem

Proposition XLVII of Book II of Euclid's *Elements* [20] is the following theorem:

Theorem 1.1. *In a right triangle ABC the square on the hypotenuse AB is equal to the sum of the squares on the other sides AC and BC , that is,*

$$AB^2 = AC^2 + BC^2.$$

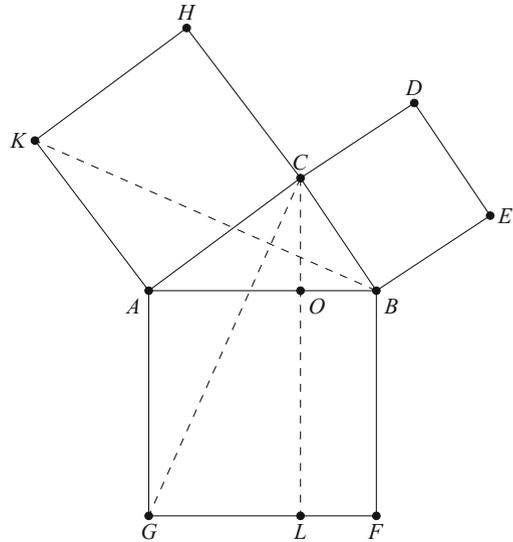
Theorem 1.1 is usually attributed to Pythagoras (580 BCE-500 BCE) or at least to the Pythagorean school, and for that reason the equation

$$x^2 + y^2 = z^2, \tag{1.1}$$

satisfied by the side lengths of a right triangle, is referred to as the *Pythagorean Equation*.

There are hundreds of proofs for the Pythagorean Theorem. We will momentarily give the proof contained in Euclid's *Elements*. The proof is truly geometric and very

Fig. 1.1 Euclid's proof of Theorem 1.1. The triangle ABC is a right angle triangle with C being the right angle



much in the Pythagorean tradition. In the argument, AB^2 is interpreted as the area of the square built on the edge AB , and the theorem is proved by showing that the area of the square built on AB is equal to the sum of the areas of the squares built on AC and BC .

Proof (Euclid). Draw squares $ACHK$, $CBED$, and $ABFG$ as in Figure 1.1. Pick a point O on AB such that $CO \perp AB$. Draw the altitude CO from C and extend it to intersect GF at L . Draw CG and KB .

Since $ABFG$ is a square, $AG = AB$. Similarly, $AC = AK$. Since $\angle GAB$ and $\angle CAK$ are right angles, $\angle GAC = \angle BAK$. Putting these facts together, we conclude $\triangle KAB \simeq \triangle CAG$. In particular the areas of these triangles are equal.

Since ACB and HCA are both right angles, the line segment HB passes through C . Consequently, the area of KAB is half the area of the square $ACHK$. Next, the area of CAG is half the area of the rectangle $OLGA$ as the shapes share the same base AG and have equal heights. Hence, the area of $ACHK$ is equal to the area of $OLGA$. A similar argument shows that the area of the square $CBED$ is equal to the area of the rectangle $OLFB$. Finally, the sum of the areas of $OLGA$ and $OLFB$ is the area of the square $ABFG$. \square

This is by no means the easiest proof of the Pythagorean Theorem. Here we record a famous proof published by James Garfield, the 20th president of the United States, five years before he took office. This proof appeared in the *New England Journal of Education* in 1876.

Proof (Garfield). Suppose a , b , c are the sides of a right triangle. Consider the trapezoid in Figure 1.2.

Fig. 1.2 President James Garfield's proof of the Pythagorean Theorem

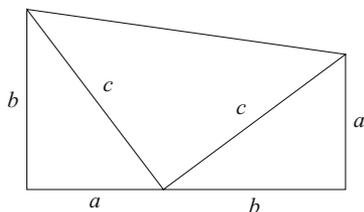
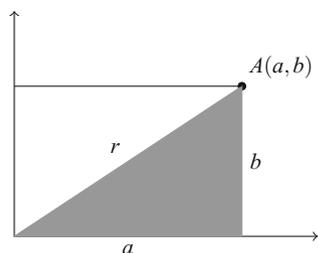


Fig. 1.3 Applying the Pythagorean Theorem to analytic geometry



We calculate the area of the trapezoid in two different ways. First recall the standard formula for the area of a trapezoid: If the parallel sides of a trapezoid of height h have lengths x, y , then the area is equal to $h(x + y)/2$. By this formula, the area of our trapezoid is $(a + b)^2/2$. On the other hand, the trapezoid is the union of three right triangles: two with legs equal to a, b , and one with legs equal to c . For this reason the area of the trapezoid is equal to

$$2 \cdot \frac{1}{2}ab + \frac{1}{2}c^2.$$

Setting the two expressions for the area equal to each other gives

$$2 \cdot \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}(a + b)^2.$$

Expanding and simplifying the sides of the equality gives the Pythagorean Equation. \square

The Pythagorean Theorem is a fundamental theorem with many applications. For example, the main identity of trigonometry, that for each angle θ

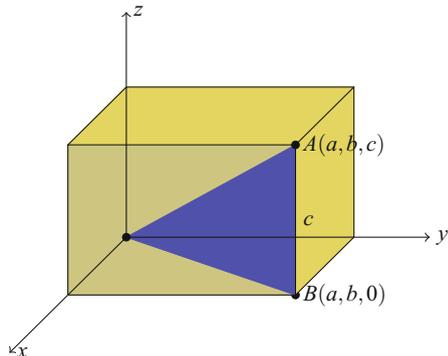
$$\cos^2 \theta + \sin^2 \theta = 1,$$

is nothing but the Pythagorean Theorem in a right triangle with hypotenuse of length 1. The theorem has an interesting interpretation in analytic geometry. Suppose we have a point A with coordinates (a, b) in the xy -plane as in Figure 1.3.

If r is the distance from A to the origin, then applying the Pythagorean Theorem to the gray right triangle gives

$$r^2 = a^2 + b^2.$$

Fig. 1.4 Applying the Pythagorean Theorem to three-dimensional analytic geometry



Suppose, on the other hand, we have a fixed number $r > 0$ and we want to identify all points (x, y) which have distance r to the origin. This is of course the circle of radius r centered at the origin with equation

$$x^2 + y^2 = r^2.$$

This picture can be generalized to higher dimensions. Suppose we have a point $A(a, b, c)$ in the three-dimensional space \mathbb{R}^3 as in Figure 1.4.

Again let r be the distance from the point $A(a, b, c)$ to the origin $O(0, 0, 0)$. Applying the Pythagorean Theorem to the blue triangle gives

$$r^2 = OB^2 + c^2 = a^2 + b^2 + c^2.$$

As an application, we find that the equation of the sphere of radius r centered at the origin is

$$x^2 + y^2 + z^2 = r^2.$$

Similarly, if we have a point with coordinates (x_1, \dots, x_n) in \mathbb{R}^n , its distance r to the origin satisfies

$$r^2 = x_1^2 + \dots + x_n^2. \quad (1.2)$$

We can use this result to write down the equation of a sphere in \mathbb{R}^n of radius r centered at the origin.

1.2 Pythagorean triples

In this book we are interested in those solutions of the Pythagorean Equation which are interesting from the number theoretic perspective. This means we will work with solutions x, y, z of Equation (1.1) which are elements of particular subsets of the real numbers, e.g., natural numbers, integers, or rational numbers. In general, a *Diophantine equation* is an equation of the form

$$f(x_1, x_2, \dots, x_n) = 0$$

where we search for solutions $(x_1, \dots, x_n) \in \mathbb{Z}^n$, though in some situations we may seek solutions in other sets, e.g., \mathbb{N} , \mathbb{Q} , $\mathbb{Z}[i]$.

A *Pythagorean triple* is a triple of natural numbers x, y, z satisfying Equation (1.1). A *primitive* Pythagorean triple is one where the three numbers do not share any non-trivial common factors. Such triples are called primitive because if (a, b, c) is some Pythagorean triple, there is a primitive Pythagorean triple (a', b', c') and an integer d such that

$$(a, b, c) = (da', db', dc').$$

The most famous Pythagorean triple is $(3, 4, 5)$, and one can easily check that $5^2 = 25 = 9 + 16 = 3^2 + 4^2$. The next few Pythagorean triples are $(5, 12, 13)$, $(7, 24, 25)$, $(8, 15, 17)$. We will determine all primitive Pythagorean triples in §3.1. A right triangle whose side lengths form a Pythagorean triple is called an *integral right triangle*. We call an integral right triangle *primitive* if its side lengths form a primitive Pythagorean triple.

We can also study the solutions of the Pythagorean Equation in integers x, y, z . Again, we call an integral solution *primitive* if x, y, z do not share any common factors other than $+1$ or -1 . If (x, y, z) satisfies the Pythagorean Equation, then we have $x^2 + y^2 = z^2$. If $z \neq 0$, then we divide by z^2 to obtain

$$\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = 1,$$

i.e., the point $(x/z, y/z)$ is a point with rational coordinates on the circle of radius 1 centered at the origin. For example, $(3/5, 4/5)$ is a point on the unit circle centered at the origin obtained from the Pythagorean triple $(3, 4, 5)$. In fact the triple $(3, 4, 5)$ gives rise to eight different points on the circle:

$$(\pm 3/5, \pm 4/5), \quad (\pm 4/5, \pm 3/5), \quad (\pm 3/5, \mp 4/5), \quad (\pm 4/5, \mp 3/5).$$

Though we have not yet developed the tools to prove this statement rigorously, the reader should convince herself that there is a correspondence between primitive integral solutions (x, y, z) of the Pythagorean Equation with $z > 0$ and points with rational coordinates on the unit circle center at the origin. We can make similar definitions for higher dimensional Pythagorean Equations

$$x_1^2 + \dots + x_n^2 = z^2, \tag{1.3}$$

and relate integral solutions to points with rational coordinates on the higher dimensional unit spheres centered at the origin.

1.3 The questions

Understanding the integral solutions of the Pythagorean Equation and exploring the fine properties of integral right triangles have been great sources of inspiration for mathematicians throughout the history of mathematics in general, and number theory in particular. Our purpose in this book is to explore some number theoretic problems that have arisen in relation to right triangles. As we saw a moment ago the study of right triangles and solutions to the Pythagorean Equation is intimately connected with the study of points with rational (or integral) coordinates on circles and spheres. These are some of the questions we address in this book:

1. What are the primitive solutions of the Pythagorean Equation? Does geometry have anything to do with finding the solutions? We study these questions in Chapter 3.
2. What integers are areas of integral right triangles? This is the subject matter of Chapter 4.
3. What numbers are edges of integral right triangles? This question is answered in Chapter 5.
4. How many solutions are there to the Pythagorean Equation modulo various integers? We answer this question in Chapter 8. For what it means to speak of a number modulo an integer, see Chapter 2.
5. How are integral points distributed on big spheres? Some results in this direction are obtained in Chapters 9 and 10.
6. Approximately, how many Pythagorean triples (x, y, z) are there with $z < B$, for a larger number B ? The answer to this question occupies Chapter 13.
7. How are points with rational coordinates distributed on the unit circle centered at the origin in \mathbb{R}^2 ? This is discussed in Chapter 14.

The rest of the book is devoted to developing background material for these results, or exploring related topics.

Exercises

- 1.1 Let a, b, c be the side lengths of a right angle triangle with c the length of the hypotenuse. Use the dissection in Figure 1.5 of a $c \times c$ square into four triangles and a square to give a proof of the Pythagorean Theorem. This proof is due to the famous 12th century Indian mathematician Bhaskara, [9, §3.3].
- 1.2 Suppose a, b, c are the side lengths of a right triangle. Use Figure 1.6 to give a proof of the Pythagorean Theorem. In the diagram, the three triangles are similar to the original triangle with scaling factors a, b , and c .
- 1.3 Here is an alternative formulation of the idea exploited in Garfield's proof. Again, suppose a, b, c are the sides of a right triangle. Use Figure 1.7 to give one more proof of the Pythagorean Theorem.

Fig. 1.5 The dissection in Problem 1.1

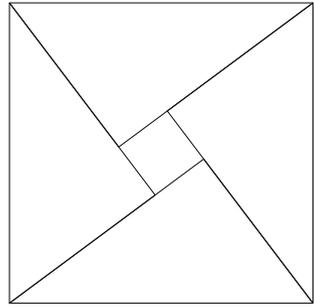


Fig. 1.6 Figure for Problem 1.2

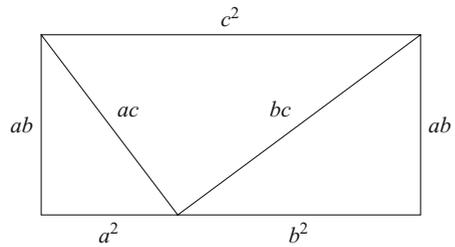
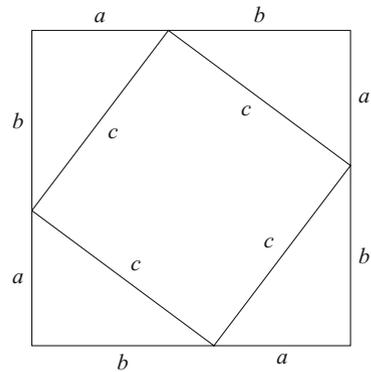


Fig. 1.7 The diagram for Problem 1.3



1.4 Let ABC be a triangle. Show that

$$\operatorname{sgn}(\angle A + \angle B - \angle C) = \operatorname{sgn}(BC^2 + AC^2 - AB^2).$$

Here sgn is the following function:

$$\operatorname{sgn}(x) = \begin{cases} +1 & x > 0; \\ 0 & x = 0; \\ -1 & x < 0. \end{cases}$$

1.5 (✂) List all Pythagorean triples (a, b, c) , with $a \leq b < c \leq 100$.

- 1.6 (✕) Let $N(B)$ be the number of Pythagorean triples (a, b, c) , with $a, b, c < B$. Compute $N(B)$ for some large values of B like 1000, 15000, 100000. Does $N(B)/B$ approach a limit as B gets large? We will investigate this limit in Chapter 13.

Notes

Pythagoreans

Pythagoreans certainly deserve a good deal of credit for their contributions to mathematics, if nothing else for their formalization of the concept of proof. While they may have in fact been the first people in history to have written down a formal proof of Theorem 1.1, there is no doubt that the theorem itself was known much earlier. For example, the Babylonian clay tablet Plimpton 322 described in [9, §2.6], dated between 1900 and 1600 BCE, contains fifteen pairs of fairly large natural numbers x, z , every one of which is the hypotenuse and a leg of some right triangle with integer sides. Even though the tablet does not contain a diagram showing a right triangle, it is hard to imagine these numbers would have appeared in a context other than the Pythagorean Theorem. Furthermore, given the sizes of the entries, 8161 and 18541, among others, it is only natural to assume that these numbers were not the result of random guesswork, and that the Babylonian mathematicians responsible for the content of the tablet actually had a method to produce integral solutions.

Mathematicians in Egypt too were certainly aware of the Pythagorean Theorem. The Cairo Mathematical Papyrus, described again in [9, §2.6], contains a variety of problems, some of them fairly sophisticated, dealing directly with the Pythagorean Theorem. There is also evidence to suggest that the theorem and something resembling a geometric proof of it were known to Chinese mathematicians some 300 years before Euclid, c.f. [9, §3.3]. Dickson [16, Ch. IV] reports that the Indian mathematicians, Baudhayana and Apastamba, had obtained a number of solutions to the Pythagorean Equation independently of the Greeks around 500 BCE.

At any rate, Pythagoreans were led to irrational numbers from the Pythagorean Theorem. Kline [29, Ch. 3] writes: “The discovery of incommensurable ratios [irrational numbers] is attributed to Hippasus of Metapontum (5th cent. B.C.). The Pythagoreans were supposed to have been at sea at the time and to have thrown Hippasus overboard for having produced an element in the universe which denied the Pythagorean doctrine that all phenomena in the universe can be reduced to whole numbers or their ratios.”

This most likely refers to the discovery of $\sqrt{2}$. Some historians dispute the story that Hippasus was thrown overboard. The basic argument seems to be that the drowning of the discoverers sounds unlikely—which considering the fact that at the time of this writing fundamentalism in all of its shapes and forms has been eradicated in the world, the skepticism of these historians is justified. There is apparently no historical

evidence that Pythagoras himself ever knew of irrational numbers—which, as little as we know of the life of the man, this is not surprising. The earliest reference to irrational numbers is in Plato’s *Theaetetus* [38, Page 200] where it is said of Theodorus: “was writing out for us something about roots, such as the roots of three or five, showing that they are incommensurable by the unit: he selected other examples up to seventeen—there he stopped.”

Since Theodorus skips over 2 then presumably this means that the irrationality of root 2 must have already been known. In fact there is mention of this in passing in Aristotle’s *Prior Analytics* [3, §23] and this appears to be the first place this is written down somewhere: “prove the initial thesis from a hypothesis, when something impossible results from the assumption of the contradictory. For example, one proves that the diagonal is incommensurable because odd numbers turn out to be equal to even ones if one assumes that it is commensurable.”

To learn more about Pythagoras and his school, we refer the reader to [9], especially Chapter 3. For the philosophical contributions of the Pythagoreans, see Russell’s fantastic book [42]. For Greek mathematics in general, see Artman [5]. To see some original writings by the Greek masters, see Thomas [51].

Pythagorean triples throughout history

Proclus, in his commentary on Euclid, states that Pythagoras had obtained the family of Pythagorean triples

$$\begin{cases} x = 2\alpha + 1, \\ y = 2\alpha^2 + 2\alpha, \\ z = 2\alpha^2 + 2\alpha + 1, \end{cases}$$

for α a natural number, c.f. [16, §IV]. As we will see in §3.1 this family does not cover all solutions. Euclid obtained the solutions

$$\begin{cases} x = \alpha\beta\gamma, \\ y = \frac{1}{2}\alpha(\beta^2 - \gamma^2), \\ z = \frac{1}{2}\alpha(\beta^2 + \gamma^2). \end{cases}$$

Diophantus may have been the first person to write the solutions as

$$\begin{cases} x = m^2 - n^2, \\ y = 2mn, \\ z = m^2 + n^2. \end{cases} \quad (1.4)$$

Dickson [16, §IV] mentions an anonymous Arabic text from the tenth century where necessary and sufficient conditions are derived for the integers m, n so that the triple (1.4) is primitive. The same reference contains numerous other works by many

mathematicians which provide various formulations of the solutions of the Pythagorean Equation.

Our purpose here is not to review the history of Pythagorean Equation in its entirety—the references [9, 16] do an impressive job at reviewing the history of the subject, though, see *Historical References* in Notes to Chapter 2. Our goal in mentioning the above isolated anecdotes is to highlight the fact that mathematics, as all other branches of human knowledge, progresses very slowly—and sometimes what in hindsight looks completely obvious, takes years, centuries, and sometimes millennia, to develop and mature. We sometimes feel smarter than our predecessors because we have learned their works, but in reality the mathematicians of the antiquity were every bit as brilliant and hardworking as the best of us.