CHAPTER 22

The Homogeneous Self-Dual Method

In Chapter 18, we described and analyzed an interior-point method called the path-following algorithm. This algorithm is essentially what one implements in practice but as we saw in the section on convergence analysis, it is not easy (and perhaps not possible) to give a complete proof that the method converges to an optimal solution. If convergence were completely established, the question would still remain as to how fast is the convergence. In this chapter, we shall present a similar algorithm for which a complete convergence analysis can be given.

1. From Standard Form to Self-Dual Form

As always, we are interested in a linear programming problem given in standard form

$$
\begin{array}{ll}
\text{maximize} & c^T x \\
\text{subject to} & Ax \leq b \\
& \;\; x \geq 0
\end{array}
\tag{22.1}
$$

and its dual

$$
\begin{array}{ll}
\text{minimize} & b^T y \\
\text{subject to} & A^T y \geq c \\
& \;\;\; y \geq 0.
\end{array}
\tag{22.2}
$$

As we shall show, these two problems can be solved by solving the following problem, which essentially combines the primal and dual problems into one problem:

$$
\begin{array}{llcccl}
\text{maximize} & & & 0 & & \\
\text{subject to} & & -\, A^T y & +\, c\phi & \leq & 0, \\
& Ax & & -\, b\phi & \leq & 0, \\
& -c^T x \;+ & b^T y & & \leq & 0, \\
& & x,\; y, & \phi & \geq & 0.
\end{array}
\tag{22.3}
$$

Note that, beyond combining the primal and dual into one big problem, one new variable (ϕ) and one new constraint have been added. Hence, the total number of variables in (22.3) is $n + m + 1$ and the total number of constraints is $n + m + 1$. Furthermore, the objective function and the right-hand sides all vanish. Problems with such right-hand sides are called *homogeneous*. Also, the constraint matrix for problem (22.3) is skew symmetric. That is, it is equal to the negative of its transpose. Homogeneous linear programming problems having a skew symmetric constraint matrix are called *self-dual*.

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In the next section, we shall give an algorithm for the solution of homogeneous self-dual linear programming problems. But first, let's note that a solution to (22.3) in which $\phi > 0$ can be converted into solutions for (22.1) and (22.2). Indeed, let $(\bar{x}, \bar{y}, \bar{\phi})$ be an optimal solution to problem (22.3). Suppose that $\bar{\phi} > 0$. (The algorithm given in the next section will guarantee that this property is satisfied whenever (22.1) and (22.2) have optimal solutions.[1]) Put

$$x^* = \bar{x}/\bar{\phi} \quad \text{and} \quad y^* = \bar{y}/\bar{\phi}.$$

Then the constraints in (22.3) say that

$$\begin{array}{rcrcrcl} & - & A^T y^* & + & c & \leq & 0, \\ Ax^* & & & - & b & \leq & 0, \\ -c^T x^* & + & b^T y^* & & & \leq & 0. \end{array}$$

Also, x^* and y^* are both nonnegative. Therefore, x^* is feasible for (22.1) and y^* is feasible for (22.2). From the weak duality theorem together with the third inequality above, we get

$$c^T x^* = b^T y^*.$$

Therefore, x^* is optimal for the primal problem (22.1) and y^* is optimal for the dual problem (22.2). As we will see later, the case where $\bar{\phi} = 0$ corresponds to infeasibility of either the primal or the dual problem (or both).

2. Homogeneous Self-Dual Problems

Consider a linear programming problem in standard form

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

and its dual

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c \\ & y \geq 0. \end{array}$$

Such a linear programming problem is called *self-dual* if $m = n$, $A = -A^T$, and $b = -c$. The reason for the name is that the dual of such a problem is the same as the primal. To see this, rewrite the constraints as less-thans and then use the defining properties for self-duality to get

$$A^T y \geq c \quad \Leftrightarrow \quad -A^T y \leq -c \quad \Leftrightarrow \quad Ay \leq b.$$

Similarly, writing the objective function as a maximization, we get

$$\min b^T y \quad = \quad -\max -b^T y \quad = \quad -\max c^T y.$$

Hence, ignoring the (irrelevant) fact that the dual records the negative of the objective function, the primal and the dual are seen to be the same. A linear programming problem in which the right-hand side vanishes is called a *homogeneous* problem.

[1]The astute reader might notice that setting all variables to 0 produces an optimal solution.

It follows that if a problem is homogeneous and self-dual, then its objective function must vanish too.

For the remainder of this section, we assume that the problem under consideration is homogeneous and self-dual. Since the case $m = n = 1$ is trivial ($A = 0$ in this case), we assume throughout this section that $n \geq 2$. Also, since the dual is the same problem as the primal, we prefer to use the letter z for the primal slacks (instead of the usual w). Hence, the primal can be written as

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & Ax + z = 0 \\ & \quad x, z \geq 0. \end{array} \tag{22.4}$$

The following theorem establishes some of the important properties of homogeneous self-dual problems.

THEOREM 22.1. *For homogeneous self-dual problem (22.4), the following statements hold:*

(1) *It has feasible solutions and every feasible solution is optimal.*
(2) *The set of feasible solutions has empty interior. In fact, if (x, z) is feasible, then $z^T x = 0$.*

PROOF. (1) The trivial solution, $(x, z) = (0, 0)$, is feasible. Since the objective function is zero, every feasible solution is optimal.

(2) Suppose that (x, z) is feasible for (22.4). The fact that A is skew symmetric implies that $\xi^T A \xi = 0$ for every vector ξ (see Exercise 16.1). In particular, $x^T Ax = 0$. Therefore, multiplying $Ax+z = 0$ on the left by x^T, we get $0 = x^T Ax+x^T z = x^T z$. This completes the proof. □

Part (2) of the previous Theorem tells us that homogeneous self-dual problems do not have central paths.

2.1. Step Directions. As usual, the interior-point method we shall derive will have the property that the intermediate solutions it produces will be infeasible. Hence, let

$$\rho(x, z) = Ax + z$$

denote the infeasibility of a solution (x, z). Also, let

$$\mu(x, z) = \frac{1}{n} x^T z.$$

The number $\mu(x, z)$ measures the degree of noncomplementarity between x and z. When x and z are clear from context, we shall simply write ρ for $\rho(x, z)$ and μ for $\mu(x, z)$.

Step directions $(\Delta x, \Delta z)$ are chosen to reduce the infeasibility and noncomplementarity of the current solution by a given factor δ, $0 \leq \delta \leq 1$. Hence, we consider the nonlinear system that would make the infeasibility and noncomplementarity of $(x + \Delta x, z + \Delta z)$ be δ times that of (x, z):

$$A(x + \Delta x) + (z + \Delta z) = \delta(Ax + z),$$
$$(X + \Delta X)(Z + \Delta Z)e = \delta\mu(x, z)e.$$

As usual, this system is nonlinear in the "delta" variables. Dropping the nonlinear term (appearing only in the second equation), we get the following linear system of equations for the step directions:

$$A\Delta x + \Delta z = -(1 - \delta)\rho(x, z), \tag{22.5}$$
$$Z\Delta x + X\Delta z = \delta\mu(x, z)e - XZe. \tag{22.6}$$

With these step directions, we pick a step length θ and step to a new point:

$$\bar{x} = x + \theta\Delta x, \qquad \bar{z} = z + \theta\Delta z.$$

We denote the new ρ-vector by $\bar{\rho}$ and the new μ-value by $\bar{\mu}$:

$$\bar{\rho} = \rho(\bar{x}, \bar{z}) \quad \text{and} \quad \bar{\mu} = \mu(\bar{x}, \bar{z}).$$

The following theorem establishes some of the properties of these step directions.

THEOREM 22.2. *The following relations hold:*

(1) $\Delta z^T \Delta x = 0$.
(2) $\bar{\rho} = (1 - \theta + \theta\delta)\rho$.
(3) $\bar{\mu} = (1 - \theta + \theta\delta)\mu$.
(4) $\bar{X}\bar{Z}e - \bar{\mu}e = (1 - \theta)(XZe - \mu e) + \theta^2 \Delta X \Delta Z e$.

PROOF. (1) We start by multiplying both sides of (22.5) on the left by Δx^T:

$$\Delta x^T A\Delta x + \Delta x^T \Delta z = -(1 - \delta)\Delta x^T \rho. \tag{22.7}$$

The skew symmetry of A (i.e., $A = -A^T$) implies that $\Delta x^T A\Delta x = 0$ (see Exercise 16.1). Hence, the left-hand side of (22.7) simplifies nicely:

$$\Delta x^T A\Delta x + \Delta x^T \Delta z = \Delta x^T \Delta z.$$

Substituting the definition of ρ into the right-hand side of (22.7), we get

$$-(1 - \delta)\Delta x^T \rho = -(1 - \delta)\Delta x^T (Ax + z).$$

Next, we use the skew symmetry of A to rewrite $\Delta x^T Ax$ as follows:

$$\Delta x^T Ax = (Ax)^T \Delta x = x^T A^T \Delta x = -x^T A\Delta x.$$

Assembling what we have so far, we see that

$$\Delta x^T \Delta z = -(1 - \delta)(-x^T A\Delta x + z^T \Delta x). \tag{22.8}$$

To proceed, we use (22.5) to replace $A\Delta x$ with $-(1 - \delta)\rho - \Delta z$. Therefore,

$$\begin{aligned} -x^T A\Delta x + z^T \Delta x &= x^T ((1 - \delta)\rho + \Delta z) + z^T \Delta x \\ &= (1 - \delta)x^T \rho + x^T \Delta z + z^T \Delta x. \end{aligned} \tag{22.9}$$

Again using the definition of ρ and the skew symmetry of A, we see that

$$x^T \rho = x^T (Ax + z) = x^T z.$$

The last two terms in (22.9) can be simplified by multiplying both sides of (22.6) on the left by e^T and then using the definition of μ to see that

$$z^T \Delta x + x^T \Delta z = \delta \mu n - x^T z = (\delta - 1) x^T z.$$

Making these substitutions in (22.9), we get

$$-x^T A \Delta x + z^T \Delta x = (1-\delta) x^T z + (\delta - 1) x^T z = 0.$$

Hence, from (22.8), we see that $\Delta x^T \Delta z$ vanishes as claimed.

(2) From the definitions of $\bar{x}$ and $\bar{z}$, we see that

$$\begin{aligned} \bar{\rho} &= A(x + \theta \Delta x) + (z + \theta \Delta z) \\ &= Ax + z + \theta(A\Delta x + \Delta z) \\ &= (1 - \theta + \theta\delta)\rho. \end{aligned}$$

(3) From the definitions of $\bar{x}$ and $\bar{z}$, we see that

$$\begin{aligned} \bar{x}^T \bar{z} &= (x + \theta\Delta x)^T (z + \theta \Delta z) \\ &= x^T z + \theta(z^T \Delta x + x^T \Delta z) + \theta^2 \Delta z^T \Delta x. \end{aligned}$$

From part (1) and (22.6), we then get

$$\bar{x}^T \bar{z} = x^T z + \theta(\delta \mu n - x^T z).$$

Therefore,

$$\bar{\mu} = \frac{1}{n} \bar{x}^T \bar{z} = (1-\theta)\mu + \theta\delta\mu.$$

(4) From the definitions of $\bar{x}$ and $\bar{z}$ together with part (3), we see that

$$\begin{aligned} \bar{X}\bar{Z}e - \bar{\mu}e &= (X + \theta\Delta X)(Z + \theta \Delta Z)e - (1 - \theta + \theta\delta)\mu e \\ &= XZe + \theta(Z\Delta x + X \Delta z) + \theta^2 \Delta X \Delta Z e - (1 - \theta + \theta\delta)\mu e. \end{aligned}$$

Substituting (22.6) into the second term on the right and recollecting terms, we get the desired expression. □

2.2. Predictor-Corrector Algorithm. With the preliminaries behind us, we are now ready to describe an algorithm. We shall be more conservative than we were in Chapter 18 and define the algorithm in such a way that it keeps the components of XZe close to each other. Indeed, for each $0 \le \beta \le 1$, let

$$\mathcal{N}(\beta) = \left\{ (x, z) > 0 : \| XZe - \mu(x,z)e \| \le \beta \mu(x, z) \right\}.$$

Shortly, we will only deal with $\mathcal{N}(1/4)$ and $\mathcal{N}(1/2)$ but first let us note generally that $\beta < \beta'$ implies that $\mathcal{N}(\beta) \subset \mathcal{N}(\beta')$. Hence, as a function of β, the $\mathcal{N}(\beta)$'s form an increasing family of sets. Also, $\mathcal{N}(0)$ is precisely the set of points (x, z) for which XZe has all equal components.

The algorithm alternates between two types of steps. On the first iteration and subsequently on every other iteration, the algorithm performs a *predictor step*. Before a predictor step, one assumes that

$$(x, z) \in \mathcal{N}(1/4).$$

Then step directions are computed using $\delta = 0$ (i.e., with no centering) and the step length is calculated so as not to go outside of $\mathcal{N}(1/2)$:

$$\theta = \max\{t : (x + t\Delta x, z + t\Delta z) \in \mathcal{N}(1/2)\}. \tag{22.10}$$

On the even iterations, the algorithm performs a *corrector step*. Before a corrector step, one assumes that

$$(x, z) \in \mathcal{N}(1/2)$$

(as is guaranteed by the predictor step's step length). Then step directions are computed using $\delta = 1$ (i.e., pure centering) and the step length parameter θ is set to 1.

The following theorem shows that the result of each step satisfies the precondition for the next step of the algorithm and that μ decreases on predictor steps while it stays the same on corrector steps.

THEOREM 22.3. *The following statements are true:*

(1) *After a predictor step,* $(\bar{x}, \bar{z}) \in \mathcal{N}(1/2)$ *and* $\bar{\mu} = (1 - \theta)\mu$.
(2) *After a corrector step,* $(\bar{x}, \bar{z}) \in \mathcal{N}(1/4)$ *and* $\bar{\mu} = \mu$.

PROOF OF PART (1). The formula for $\bar{\mu}$ follows from part (3) of Theorem 22.2 by putting $\delta = 0$. The fact that $(\bar{x}, \bar{z}) \in \mathcal{N}(1/2)$ is an immediate consequence of the choice of θ. □

Before proving Part (2) of the Theorem, we need to introduce some notation and prove a few technical results. Let

$$\begin{aligned}
p &= X^{-1/2} Z^{1/2} \Delta x, \\
q &= X^{1/2} Z^{-1/2} \Delta z, \\
r &= p + q \\
&= X^{-1/2} Z^{-1/2} (Z \Delta x + X \Delta z) \\
&= X^{-1/2} Z^{-1/2} (\delta \mu e - X Z e).
\end{aligned} \tag{22.11}$$

The technical results are summarized in the following lemma.

LEMMA 22.4. *The following statements are true:*

(1) $\|PQe\| \le \frac{1}{2}\|r\|^2$.
(2) *If* $\delta = 0$, *then* $\|r\|^2 = n\mu$.
(3) *If* $\delta = 1$ *and* $(x, z) \in \mathcal{N}(\beta)$, *then* $\|r\|^2 \le \beta^2 \mu / (1 - \beta)$.

PROOF. (1) First note that $p^T q = \Delta x^T \Delta z = 0$ by Theorem 22.2(1). Hence,

$$\|r\|^2 = \|p + q\|^2 = p^T p + 2 p^T q + q^T q = \sum_j (p_j^2 + q_j^2).$$

Therefore,

$$\begin{aligned}
\|r\|^4 &= \left(\sum_j (p_j^2 + q_j^2)\right)^2 \\
&\geq \sum_j (p_j^2 + q_j^2)^2 \\
&= \sum_j \left((p_j^2 - q_j^2)^2 + 4p_j^2 q_j^2\right) \\
&\geq 4 \sum_j p_j^2 q_j^2 \\
&= 4\|PQe\|^2.
\end{aligned}$$

Taking square roots yields the desired inequality.

(2) Putting $\delta = 0$ in (22.11), we see that $r = -X^{1/2}Z^{1/2}e$. Therefore, $\|r\|^2 = z^T x = n\mu$.

(3) Suppose that $(x, z) \in \mathcal{N}(\beta)$. Whenever the norm of a vector is smaller than some number, the magnitude of each component of the vector must also be smaller than this number. Hence, $|x_j z_j - \mu| \leq \beta\mu$. It is easy to see that this inequality is equivalent to

$$(1 - \beta)\mu \leq x_j z_j \leq (1 + \beta)\mu. \tag{22.12}$$

Now putting $\delta = 1$ in (22.11), we get

$$\|r\|^2 = \sum_j \frac{(x_j z_j - \mu)^2}{x_j z_j}.$$

Therefore, using the lower bound given in (22.12), we get the following upper bound:

$$\|r\|^2 \leq \frac{1}{(1 - \beta)\mu} \sum_j (x_j z_j - \mu)^2.$$

Finally, since $(x, z) \in \mathcal{N}(\beta)$, we see that the above sum is bounded by $\beta^2\mu^2$. This gives the claimed inequality. □

PROOF OF THEOREM 22.3(2). Since $\theta = 1$ in a corrector step, it follows from Theorem 22.2(4) that $\bar{X}\bar{Z}e - \bar{\mu}e = \Delta X \Delta Z e = PQe$. Therefore, parts (1) and (3) of Lemma 22.4 imply that

$$\begin{aligned}
\|\bar{X}\bar{Z}e - \bar{\mu}e\| &= \|PQe\| \\
&\leq \frac{1}{2}\|r\|^2 \\
&\leq \frac{1}{2}\frac{(1/2)^2}{1 - 1/2}\mu \\
&= \frac{1}{4}\mu.
\end{aligned} \tag{22.13}$$

We also need to show that $(\bar{x}, \bar{z}) > 0$. For $0 \leq t \leq 1$, let

$$x(t) = x + t\Delta x, \quad z(t) = z + t\Delta z, \quad \text{and} \quad \mu(t) = \mu(x(t), z(t)).$$

Then from part (4) of Theorem 22.2, we have

$$X(t)Z(t)e - \mu(t)e = (1-t)(XZe - \mu e) + t^2 \Delta X \Delta Z e.$$

The right-hand side is the sum of two vectors. Since the length of the sum of two vectors is less than the sum of the lengths (i.e., by the *triangle inequality*), it follows that

$$\|X(t)Z(t)e - \mu(t)e\| \leq (1-t)\|XZe - \mu e\| + t^2\|\Delta X \Delta Z e\| \tag{22.14}$$

(note that we've pulled the scalars out of the norms). Now, since $(x, z) \in \mathcal{N}(1/2)$, we have $\|XZe - \mu e\| \leq \mu/2$. Furthermore, from (22.13) we have that $\|\Delta X \Delta Z e\| = \|PQe\| \leq \mu/4$. Replacing the norms in (22.14) with these upper bounds, we get the following bound:

$$\|X(t)Z(t)e - \mu(t)e\| \leq (1-t)\frac{\mu}{2} + t^2\frac{\mu}{4} \leq \frac{\mu}{2} \tag{22.15}$$

(the second inequality follows from the obvious facts that $t^2 \leq t$ and $\mu/4 \leq \mu/2$).

Now, consider a specific component j. It follows from (22.15) that

$$x_j(t)z_j(t) - \mu(t) \geq -\frac{\mu}{2}.$$

Since $\delta = 1$, part (3) of Theorem 22.2 tells us that $\mu(t) = \mu$ for all t. Therefore the previous inequality can be written as

$$x_j(t)z_j(t) \geq \frac{\mu}{2} > 0. \tag{22.16}$$

This inequality then implies that $x_j(t) > 0$ and $z_j(t) > 0$ for all $0 \leq t \leq 1$ (since they could only become negative by passing through 0, which is ruled out by (22.16)). Putting $t = 1$, we get that $\bar{x}_j > 0$ and $\bar{z}_j > 0$. Since the component j was arbitrary, it follows that $(\bar{x}, \bar{z}) > 0$. Therefore $(\bar{x}, \bar{z}) \in \mathcal{N}(1/4)$. ☐

2.3. Convergence Analysis. The previous theorem showed that the predictor-corrector algorithm is well defined. The next theorem gives us a lower bound on the progress made by each predictor step.

THEOREM 22.5. *In each predictor step, $\theta \geq \frac{1}{2\sqrt{n}}$.*

PROOF. Using the same notation as in the proof of Theorem 22.3, we have the inequality:

$$\begin{aligned}\|X(t)Z(t)e - \mu(t)e\| \leq (1-t)&\|XZe - \mu e\| \\ &+ t^2\|\Delta X \Delta Z e\|.\end{aligned} \tag{22.17}$$

This time, however, $(x, z) \in \mathcal{N}(1/4)$ and $\delta = 0$. Hence,

$$\|XZe - \mu e\| \leq \frac{\mu}{4}$$

and, from parts (1) and (2) of Lemma 22.4,

$$\|\Delta X \Delta Z e\| = \|PQe\| \leq \frac{1}{2}\|r\|^2 = \frac{1}{2}n\mu.$$

Using these two bounds in (22.17), we get the following bound:

$$\|X(t)Z(t)e - \mu(t)e\| \leq (1-t)\frac{\mu}{4} + t^2\frac{n\mu}{2}.$$

Now, fix a $t \leq (2\sqrt{n})^{-1}$. For such a t, we have $t^2 n/2 \leq 1/8$. Therefore, using the fact that $t \leq 1/2$ for $n \geq 2$, we get

$$\begin{aligned} \|X(t)Z(t)e - \mu(t)e\| &\leq (1-t)\frac{\mu}{4} + \frac{\mu}{8} \\ &\leq (1-t)\frac{\mu}{4} + (1-t)\frac{\mu}{4} \\ &= (1-t)\frac{\mu}{2} \\ &= \frac{\mu(t)}{2}. \end{aligned}$$

Hence, as in the previous theorem, $(x(t), z(t)) \in \mathcal{N}(1/2)$. Since t was an arbitrary number less than $(2\sqrt{n})^{-1}$, it follows that $\theta \geq (2\sqrt{n})^{-1}$. □

Let $(x^{(k)}, z^{(k)})$ denote the solution after the kth iteration and let

$$\rho^{(k)} = \rho(x^{(k)}, z^{(k)}) \quad \text{and} \quad \mu^{(k)} = \mu(x^{(k)}, z^{(k)}).$$

The algorithm starts with $x^{(0)} = z^{(0)} = e$. Therefore, $\mu^{(0)} = 1$. Our aim is to show that $\mu^{(k)}$ and $\rho^{(k)}$ tend to zero as k tends to infinity. The previous theorem together with Theorem 22.3 implies that, after an even number of iterations, say $2k$, the following inequality holds:

$$\mu^{(2k)} \leq \left(1 - \frac{1}{2\sqrt{n}}\right)^k.$$

Also, since the corrector steps don't change the value of μ, it follows that

$$\mu^{(2k-1)} = \mu^{(2k)}.$$

From these two statements, we see that

$$\lim_{k\to\infty} \mu^{(k)} = 0.$$

Now, consider $\rho^{(k)}$. It follows from parts (2) and (3) of Theorem 22.2 that the reduction in infeasibility tracks the reduction in noncomplementarity. Hence,

$$\rho^{(k)} = \mu^{(k)}\rho^{(0)}.$$

Therefore, the fact that $\mu^{(k)}$ tends to zero implies the same for $\rho^{(k)}$.

In fact, more can be said:

THEOREM 22.6. *The limits $x^* = \lim_{k\to\infty} x^{(k)}$ and $z^* = \lim_{k\to\infty} z^{(k)}$ exist and (x^*, z^*) is optimal. Furthermore, the vectors x^* and z^* are strictly complementary to each other. That is, for each j, $x_j^* z_j^* = 0$ but either $x_j^* > 0$ or $z_j^* > 0$.*

The proof is fairly technical, and so instead of proving it, we prove the following theorem, which captures the main idea.

THEOREM 22.7. *There exist positive constants* $c_1, c_2, \ldots, c_n$ *such that* $(x, z) \in \mathcal{N}(\beta)$ *implies that* $x_j + z_j \geq c_j > 0$ *for each* $j = 1, 2, \ldots, n$.

PROOF. Put $\mu = \mu(x, z)$ and $\rho = \rho(x, z) = \mu\rho^{(0)}$. Let (x^*, z^*) be a strictly complementary feasible solution (the existence of which is guaranteed by Theorem 10.6). We begin by studying the expression $z^T x^* + x^T z^*$. Since $Ax^* + z^* = 0$, we have that

$$\begin{aligned} z^T x^* + x^T z^* &= z^T x^* - x^T A x^* \\ &= (-A^T x + z)^T x^*. \end{aligned}$$

By the skew-symmetry of A, we see that $-A^T x + z = Ax + z = \rho$. And, since $\rho = \mu\rho^{(0)}$, we get

$$z^T x^* + x^T z^* = \mu {\rho^{(0)}}^T x^*. \tag{22.18}$$

The factor ${\rho^{(0)}}^T x^*$ is a constant (i.e., it does not depend on x or z). Let us denote it by M. Since all the terms in the two products on the left in (22.18) are nonnegative, it follows that each one is bounded by the right-hand side. So if we focus on a particular index j, we get the following bounds:

$$z_j x_j^* \leq \mu M \quad \text{and} \quad x_j z_j^* \leq \mu M. \tag{22.19}$$

Now, we use the assumption that $(x, z) \in \mathcal{N}(\beta)$ to see that

$$x_j z_j \geq (1 - \beta)\mu.$$

In other words, $\mu \leq x_j z_j/(1 - \beta)$, and so the inequalities in (22.19) become

$$z_j x_j^* \leq \frac{M}{1-\beta} z_j x_j \quad \text{and} \quad x_j z_j^* \leq \frac{M}{1-\beta} x_j z_j.$$

Since x_j and z_j are strictly positive, we can divide by them (and the constants) to get

$$\frac{1-\beta}{M} x_j^* \leq x_j \quad \text{and} \quad \frac{1-\beta}{M} z_j^* \leq z_j.$$

Putting

$$c_j = \frac{1-\beta}{M}(x_j^* + z_j^*),$$

we get the desired lower bound on $x_j + z_j$. □

2.4. Complexity of the Predictor-Corrector Algorithm. Of course, in practice we don't run an infinite number of iterations. Instead, we set a priori a threshold and stop when $\mu^{(k)}$ falls below it. The threshold is usually denoted by 2^{-L} where L is some number. Typically, we want the threshold to be about 10^{-8}, which corresponds to $L \approx 26$.

As we saw before, after an even number of iterations, say $2k$, the μ-value is bounded by the following inequality:

$$\mu^{(2k)} \leq \left(1 - \frac{1}{2\sqrt{n}}\right)^k.$$

Hence, it suffices to pick a k big enough to have

$$\left(1 - \frac{1}{2\sqrt{n}}\right)^k \leq 2^{-L}.$$

Taking logarithms of both sides and solving for k, we see that any

$$k \geq \frac{L}{-\log(1 - \frac{1}{2\sqrt{n}})}$$

will do. Since $-\log(1-x) \geq x$, we get

$$2L\sqrt{n} \geq \frac{L}{-\log(1 - \frac{1}{2\sqrt{n}})}.$$

Therefore, any $k \geq 2L\sqrt{n}$ will do. In particular, $k = 2L\sqrt{n}$ rounded up to the nearest integer will suffice. Since k represents half the number of iterations, it follows that it will take at most $4L\sqrt{n}$ iterations for the μ-value to fall below the threshold of 2^{-L}. This bound implies that the method is a *polynomial algorithm*, since it says that any desired precision can be obtained in a number of iterations that is bounded above by a polynomial in n (here, $4L\sqrt{n}$ is not itself a polynomial but is bounded above by say a linear function in n for $n \geq 2$).

2.5. The KKT System. We end this section on homogeneous self-dual problems by briefly discussing the KKT system (22.5)–(22.6). Solving this system of equations is the most time consuming step within each iteration of the predictor-corrector algorithm. There are several ways in which one can organize the computation. The approach that most parallels what we have done before is first to solve (22.6) for Δz,

$$\begin{aligned} \Delta z &= X^{-1}(-Z\Delta x + \delta\mu e - XZe) \\ &= -X^{-1}Z\Delta x + \delta\mu X^{-1}e - z, \end{aligned} \tag{22.20}$$

and then to eliminate it from (22.5) to get the following *reduced KKT system*:

$$(A - X^{-1}Z)\Delta x = -(1-\delta)\rho + z - \delta\mu X^{-1}e.$$

In the next section, we apply the algorithm developed in this section to the homogeneous self-dual problem given by (22.3).

3. Back to Standard Form

We return now to the setup in Section 1. Let z, w, and ψ denote the slack variables for the constraints in problem (22.3):

$$
\begin{array}{lrrrrcl}
\text{maximize} & & & 0 & & & \\
\text{subject to} & & -A^T y & +\, c\phi & +\, z & = & 0, \\
 & Ax & & -\, b\phi & +\, w & = & 0, \\
 & -c^T x & +\, b^T y & & +\, \psi & = & 0, \\
 & & & x,\, y,\, \phi,\, z,\, w,\, \psi & & \geq & 0.
\end{array}
\tag{22.21}
$$

We say that a feasible solution $(\bar{x}, \bar{y}, \bar{\phi}, \bar{z}, \bar{w}, \bar{\psi})$ is *strictly complementary* if $\bar{x}_j + \bar{z}_j > 0$ for all j, $\bar{y}_i + \bar{w}_i > 0$ for all i, and $\bar{\phi} + \bar{\psi} > 0$. Theorem 10.6 ensures the existence of such a solution (why?).

The following theorem summarizes and extends the motivating discussion given in Section 1.

THEOREM 22.8. *Suppose that* $(\bar{x}, \bar{y}, \bar{\phi}, \bar{z}, \bar{w}, \bar{\psi})$ *is a strictly complementary feasible (hence, optimal) solution to* (22.21).

(1) *If* $\bar{\phi} > 0$, *then* $x^* = \bar{x}/\bar{\phi}$ *is optimal for the primal problem* (22.1) *and* $y^* = \bar{y}/\bar{\phi}$ *is optimal for its dual* (22.2).
(2) *If* $\bar{\phi} = 0$, *then either* $c^T\bar{x} > 0$ *or* $b^T\bar{y} < 0$.
   (a) *If* $c^T\bar{x} > 0$, *then the dual problem is infeasible.*
   (b) *If* $b^T\bar{y} < 0$, *then the primal problem is infeasible.*

PROOF. Part (1) was proved in Section 1. For part (2), suppose that $\bar{\phi} = 0$. By strict complementarity, $\bar{\psi} > 0$. Hence, $\bar{x}$ and $\bar{y}$ satisfy

$$
\begin{aligned}
A^T\bar{y} &\geq 0, \\
A\bar{x} &\leq 0, \\
b^T\bar{y} &< c^T\bar{x}.
\end{aligned}
\tag{22.22}
$$

From the last inequality, we see that it is impossible to have $b^T\bar{y} \geq 0$ and $c^T\bar{x} \leq 0$. That is, either $c^T\bar{x} > 0$ or $b^T\bar{y} < 0$ (or both). Suppose, without loss of generality, that $c^T\bar{x} > 0$. We will prove by contradiction that the dual problem is infeasible. To this end, suppose that there exists a vector $y^0 \geq 0$ such that

$$
A^T y^0 \geq c. \tag{22.23}
$$

Since $\bar{x} \geq 0$, we can multiply by it without changing the direction of an inequality. So multiplying (22.23) on the left by $\bar{x}^T$, we get

$$
\bar{x}^T A^T y^0 \geq \bar{x}^T c.
$$

Now, the right-hand side is strictly positive. But inequality (22.22) together with the nonnegativity of y^0 implies that the left-hand side is nonpositive:

$$
\bar{x}^T A^T y^0 = (A\bar{x})^T y^0 \leq 0.
$$

This is a contradiction and therefore the dual must be infeasible. □

3.1. The Reduced KKT System. The right-hand side in the reduced KKT system involves the vector of infeasibilities. We partition this vector into three parts as follows:

$$\begin{bmatrix} \sigma \\ \rho \\ \gamma \end{bmatrix} = \begin{bmatrix} & -A^T & c \\ A & & -b \\ -c^T & b^T & \end{bmatrix} \begin{bmatrix} x \\ y \\ \phi \end{bmatrix} + \begin{bmatrix} z \\ w \\ \psi \end{bmatrix} = \begin{bmatrix} -A^T y + c\phi + z \\ Ax - b\phi + w \\ -c^T x + b^T y + \phi \end{bmatrix}.$$

The reduced KKT system for (22.3) is given by

$$\begin{bmatrix} -X^{-1}Z & -A^T & c \\ A & -Y^{-1}W & -b \\ -c^T & b^T & -\psi/\phi \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta \phi \end{bmatrix} = \begin{bmatrix} \hat{\sigma} \\ \hat{\rho} \\ \hat{\gamma} \end{bmatrix}, \tag{22.24}$$

where

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\rho} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} -(1-\delta)\sigma + z - \delta\mu X^{-1}e \\ -(1-\delta)\rho + w - \delta\mu Y^{-1}e \\ -(1-\delta)\gamma + \psi - \delta\mu/\phi \end{bmatrix}.$$

This system is not symmetric. One could use a general purpose equation solver to solve it, but its special structure would be mostly ignored by such a solver. To exploit the structure, we solve this system in two stages. We start by using the first two equations to solve simultaneously for Δx and Δy in terms of $\Delta\phi$:

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -X^{-1}Z & -A^T \\ A & -Y^{-1}W \end{bmatrix}^{-1} \left(\begin{bmatrix} \hat{\sigma} \\ \hat{\rho} \end{bmatrix} - \begin{bmatrix} c \\ -b \end{bmatrix} \Delta\phi \right).$$

Introducing abbreviating notations, we can write

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} - \begin{bmatrix} g_x \\ g_y \end{bmatrix} \Delta\phi, \tag{22.25}$$

where the vectors

$$f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} g_x \\ g_y \end{bmatrix}$$

are found by solving the following two systems of equations:

$$\begin{bmatrix} -X^{-1}Z & -A^T \\ A & -Y^{-1}W \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} \hat{\sigma} \\ \hat{\rho} \end{bmatrix}$$

and

$$\begin{bmatrix} -X^{-1}Z & -A^T \\ A & -Y^{-1}W \end{bmatrix} \begin{bmatrix} g_x \\ g_y \end{bmatrix} = \begin{bmatrix} c \\ -b \end{bmatrix}.$$

Then we use (22.25) to eliminate Δx and Δy from the last equation in (22.24):

$$\begin{bmatrix} -c^T & b^T \end{bmatrix} \left(\begin{bmatrix} f_x \\ f_y \end{bmatrix} - \begin{bmatrix} g_x \\ g_y \end{bmatrix} \Delta\phi \right) - \frac{\psi}{\phi}\Delta\phi = \hat{\gamma}.$$

We then solve for $\Delta\phi$:

$$\Delta\phi = \frac{c^T f_x - b^T f_y + \hat{\gamma}}{c^T g_x - b^T g_y - \psi/\phi}.$$

Given $\Delta\phi$, (22.25) determines Δx and Δy. Once these vectors are known, (22.20) is used to compute the step directions for the slack variables:

$$\begin{aligned}
\Delta z &= -X^{-1}Z\Delta x + \delta\mu X^{-1}e - z \\
\Delta w &= -Y^{-1}W\Delta y + \delta\mu Y^{-1}e - w \\
\Delta\psi &= -\frac{\psi}{\phi}\Delta\phi + \delta\mu/\phi - \psi.
\end{aligned}$$

We now see that the reduced KKT system can be solved by solving two systems of equations for f and g. These two systems both involve the same matrix. Furthermore, these systems can be formulated as quasidefinite systems by negating the first equation and then reordering the equations appropriately. For example, the quasidefinite system for g is

$$\begin{bmatrix} -Y^{-1}W & A \\ A^T & X^{-1}Z \end{bmatrix} \begin{bmatrix} g_y \\ g_x \end{bmatrix} = \begin{bmatrix} -b \\ -c \end{bmatrix}.$$

Therefore, the techniques developed in Chapter 20 can be used to solve these systems of equations. In particular, to solve the two systems, the quasidefinite matrix only needs to be factored once. Then the two systems can be solved by doing two forward and two backward substitutions. Since factorization requires more computation than forward and backward substitutions, one would expect to be able to solve these two systems in much less time than if they were each being solved from scratch. In fact, it is often the case that one can solve two systems involving the same quasidefinite matrix in little more time than is required to solve just one such system.

The full homogeneous self-dual method is summarized in Figure 22.1.

4. Simplex Method vs. Interior-Point Methods

Finally, we compare the performance of interior-point methods with the simplex method. For this comparison, we have chosen the homogeneous self-dual method described in this chapter and the self-dual simplex method (see Figure 7.1). In the interest of efficiency certain liberties have been taken with the implementations. For example, in the homogeneous self-dual method, (18.6) is used to compute "long" step lengths instead of the more conservative "short" step lengths in (22.10). The code fragments implementing each of these two algorithms are shown in Appendix A.

A standard collection of test problems, the so-called NETLIB suite, were used in the comparison. Problems in this collection are formulated with bounds and ranges:

$$\begin{array}{lrcccl}
\text{minimize} & & & c^T x & & \\
\text{subject to} & b & \le & Ax & \le & b + r \\
 & l & \le & x & \le & u.
\end{array}$$

However, to keep the algorithms as simple as possible, they were implemented only for problems in our standard inequality form. Therefore, the problems from the NETLIB suite were converted to standard form as follows:

initialize

$$(x, y, \phi, z, w, \psi) = (e, e, 1, e, e, 1)$$

while (not optimal) {

$$\mu = (z^T x + w^T y + \psi\phi)/(n + m + 1)$$

$$\delta = \begin{cases} 0, & \text{on odd iterations} \\ 1, & \text{on even iterations} \end{cases}$$

$$\hat{\rho} = -(1-\delta)(Ax - b\phi + w) + w - \delta\mu Y^{-1} e$$

$$\hat{\sigma} = -(1-\delta)(-A^T y + c\phi + z) + z - \delta\mu X^{-1} e$$

$$\hat{\gamma} = -(1-\delta)(b^T y - c^T x + \psi) + \psi - \delta\mu/\phi$$

solve the two $(n+m) \times (n+m)$ quasidefinite systems:

$$\begin{bmatrix} -Y^{-1}W & A \\ A^T & X^{-1}Z \end{bmatrix} \begin{bmatrix} f_y \\ f_x \end{bmatrix} = \begin{bmatrix} \hat{\rho} \\ -\hat{\sigma} \end{bmatrix}$$

and

$$\begin{bmatrix} -Y^{-1}W & A \\ A^T & X^{-1}Z \end{bmatrix} \begin{bmatrix} g_y \\ g_x \end{bmatrix} = \begin{bmatrix} -b \\ -c \end{bmatrix}$$

$$\Delta\phi = \frac{c^T f_x - b^T f_y + \hat{\gamma}}{c^T g_x - b^T g_y - \psi/\phi}$$

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} - \begin{bmatrix} g_x \\ g_y \end{bmatrix} \Delta\phi$$

$$\Delta z = -X^{-1} Z \Delta x + \delta\mu X^{-1} e - z$$

$$\Delta w = -Y^{-1} W \Delta y + \delta\mu Y^{-1} e - w$$

$$\Delta\psi = -\frac{\psi}{\phi}\Delta\phi + \delta\mu/\phi - \psi$$

$$\theta = \begin{cases} \max\{t : (x(t), \ldots, \psi(t)) \in \mathcal{N}(1/2)\}, & \text{on odd iterations} \\ 1, & \text{on even iterations} \end{cases}$$

$$\begin{aligned} x &\leftarrow x + \theta\Delta x, & z &\leftarrow z + \theta\Delta z \\ y &\leftarrow y + \theta\Delta y, & w &\leftarrow w + \theta\Delta w \\ \phi &\leftarrow \phi + \theta\Delta\phi, & \psi &\leftarrow \psi + \theta\Delta\psi \end{aligned}$$

}

FIGURE 22.1. The homogeneous self-dual method.

$$\begin{array}{rrcl} -\ \text{maximize} & -c^T x - c^T l & & \\ \text{subject to} & -Ax & \leq & -b + Al \\ & Ax & \leq & b + r - Al \\ & x & \leq & u - l \\ & x & \geq & 0. \end{array}$$

Of course, this transformation is invalid when any of the lower bounds are infinite. Therefore, such problems have been dropped in our experiment. Also, this transformation introduces significant computational inefficiency but, since it was applied equally to the problems presented to both methods, the comparison remains valid.

The results of our experiment are shown in Table 22.1. The most obvious observation is that the simplex method is generally faster and that, for many problems, the slower method is not more than 3 or 4 times slower. For problems in this suite, these results are consistent with results reported in the literature. However, it must be noted that the problems in this suite range only from small to medium in size. The largest problem, fit2p, has about 3,000 constraints and about 14,000 variables. By today's standards, this problem is considered of medium size. For larger problems, reports in the literature indicate that interior point methods tend to be superior although the results are very much dependent on the specific class of problems. In the remaining chapters of this book we shall consider various extensions of the linear programming model. We shall see that the simplex method is particularly well suited for solving integer programming problems studied in Chapter 23 whereas interior point methods are more appropriate for extensions into the quadratic and convex programming problems studied in Chapters 24 and 25. These considerations are often more important than speed. There are, of course, exceptions. For example, the interior-point method is about 900 times faster than the simplex method on problem fit2p. Such a difference cannot be ignored.

When comparing algorithms it is always tempting to look for ways to improve the slower method. There are obvious enhancements to the interior-point method used in this implementation. For example, one could use the same LDL^T factorization to compute both the predictor and the corrector directions. When implemented properly, this enhancement alone can almost halve the times for this method.

Of course, the winning algorithm can also be improved (but, significant overall improvements such as the one just mentioned for the interior-point method are not at all obvious). Looking at the table, we note that the interior-point method solved both fit2p and fit2d in roughly the same amount of time. These two problems are duals of each other and hence any algorithm that treats the primal and the dual symmetrically should take about the same time to solve them. Now, look at the simplex method's performance on these two problems. There is a factor of 36 difference between them. The reason is that, even though we have religiously adhered to primal-dual symmetry in our development of the simplex method, an asymmetry did creep in. To see it, note that the basic matrix is always a square submatrix of $\begin{bmatrix} A & I \end{bmatrix}$. That is, it is an $m \times m$ matrix. If we apply the algorithm to the dual problem, then the basis matrix is $n \times n$. Hence, even though the sequence of iterates generated should be identical with the two problems, the computations involved in each iteration can be very different if m and n are not about the same. This is the case for the fit2p/fit2d pair. Of course, one can easily think up schemes to overcome this difficulty. But even if the performance of the simplex method on fit2p can be brought in line with its performance on fit2d, it will still be about 25 times slower than the interior-point on this problem—a difference that remains significant.

Name	Time		Name	Time	
	Simplex method	Interior point		Simplex method	Interior point
25fv47	2 min 55.70 s	3 min 14.82 s	maros	1 min 0.87 s	3 min 19.43 s
80bau3b	7 min 59.57 s	2 min 34.84 s	nesm	1 min 40.78 s	6 min 21.28 s
adlittle	0 min 0.26 s	0 min 0.47 s	pilot87	*	*
afiro	0 min 0.03 s	0 min 0.11 s	pilotnov	*	4 min 15.31 s
agg	0 min 1.09 s	0 min 4.59 s	pilots	*	32 min 48.15 s
agg2	0 min 1.64 s	0 min 21.42 s	recipe	0 min 0.21 s	0 min 1.04 s
agg3	0 min 1.72 s	0 min 26.52 s	sc105	0 min 0.28 s	0 min 0.37 s
bandm	0 min 15.87 s	0 min 9.01 s	sc205	0 min 1.30 s	0 min 0.84 s
beaconfd	0 min 0.67 s	0 min 6.42 s	sc50a	0 min 0.09 s	0 min 0.17 s
blend	0 min 0.40 s	0 min 0.56 s	sc50b	0 min 0.12 s	0 min 0.15 s
bnl1	0 min 38.38 s	0 min 46.09 s	scagr25	0 min 12.93 s	0 min 4.44 s
bnl2	3 min 54.52 s	10 min 19.04 s	scagr7	0 min 1.16 s	0 min 1.05 s
boeing1	0 min 5.56 s	0 min 9.14 s	scfxm1	0 min 4.44 s	0 min 7.80 s
boeing2	0 min 0.80 s	0 min 1.72 s	scfxm2	0 min 14.33 s	0 min 18.84 s
bore3d	0 min 1.17 s	0 min 3.97 s	scfxm3	0 min 28.92 s	0 min 28.92 s
brandy	0 min 5.33 s	0 min 8.44 s	scorpion	0 min 3.38 s	0 min 2.64 s
czprob	0 min 50.14 s	0 min 41.77 s	scrs8	0 min 7.15 s	0 min 9.53 s
d2q06c	*	1 h 11 min 1.93 s	scsd1	0 min 0.86 s	0 min 3.88 s
d6cube	2 min 46.71 s	13 min 44.52 s	scsd6	0 min 2.89 s	0 min 9.31 s
degen2	0 min 17.28 s	0 min 17.02 s	scsd8	0 min 28.87 s	0 min 16.82 s
degen3	5 min 55.52 s	3 min 36.73 s	sctap1	0 min 2.98 s	0 min 3.08 s
dfl001	8 h 55 min 33.05 s	**	sctap2	0 min 7.41 s	0 min 12.03 s
e226	0 min 4.76 s	0 min 6.65 s	sctap3	0 min 11.70 s	0 min 17.18 s
etamacro	0 min 17.94 s	0 min 43.40 s	seba	0 min 27.25 s	0 min 11.90 s
fffff800	0 min 10.07 s	1 min 9.15 s	share1b	0 min 2.07 s	0 min 10.90 s
finnis	0 min 4.76 s	0 min 6.17 s	share2b	0 min 0.47 s	0 min 0.71 s
fit1d	0 min 18.15 s	0 min 11.63 s	shell	0 min 16.12 s	0 min 29.45 s
fit1p	7 min 10.86 s	0 min 16.47 s	ship04l	0 min 3.82 s	0 min 13.60 s
fit2d	1 h 3 min 14.37 s	4 min 27.66 s	ship04s	0 min 3.48 s	0 min 10.81 s
fit2p	36 h 31 min 31.80 s	2 min 35.67 s	ship08l	0 min 17.83 s	0 min 39.06 s
forplan	0 min 3.99 s	*	ship08s	0 min 8.85 s	0 min 19.64 s
ganges	0 min 44.27 s	0 min 34.89 s	ship12l	0 min 26.55 s	1 min 8.62 s
gfrdpnc	0 min 11.51 s	0 min 8.46 s	ship12s	0 min 16.75 s	0 min 30.33 s
greenbea	22 min 45.49 s	43 min 4.32 s	sierra	0 min 10.88 s	0 min 42.89 s
grow15	0 min 8.55 s	0 min 58.26 s	standata	0 min 0.57 s	0 min 6.60 s
grow22	0 min 11.79 s	2 min 0.53 s	standmps	0 min 2.41 s	0 min 13.44 s
grow7	0 min 3.61 s	0 min 13.57 s	stocfor1	0 min 0.22 s	0 min 0.92 s
israel	0 min 1.83 s	0 min 2.66 s	stocfor2	0 min 45.15 s	0 min 40.43 s
kb2	0 min 0.15 s	0 min 0.34 s	wood1p	0 min 14.15 s	7 min 18.47 s
lotfi	0 min 0.81 s	0 min 3.36 s	woodw	1 min 48.14 s	8 min 53.92 s
maros-r7	*	1 h 31m12.06 s			

(*) Denotes numerical difficulties
(**) Denotes insufficient memory

TABLE 22.1. Comparison between the self-dual simplex method and the homogeneous self-dual interior-point method.

Exercises

22.1 When $n = 1$, the set $\mathcal{N}(\beta)$ is a subset of $\mathbb{R}^2$. Graph it.

22.2 Suppose there is an algorithm for which one can prove that

$$\mu^{(k)} \leq \left(1 - \frac{a}{f(n)}\right)^k,$$

for every $k \geq 1$, where $f(n)$ denotes a specific function of n, such as $f(n) = n^2$, and a is a constant. In terms of a and f and the "precision" L, give a (tight) upper bound on the number of iterations that would be sufficient to guarantee that

$$\mu^{(k)} \leq 2^{-L}.$$

22.3 In Section 3 of Chapter 20, we extended the primal-dual path-following method to treat problems in general form. Extend the homogeneous self-dual method in the same way.

22.4 *Long-step variant.* Let

$$\mathcal{M}(\beta) = \{(x, z) : \min XZe \geq (1 - \beta)\mu(x, z)\}.$$

(The notation $\min XZe$ denotes the scalar that is the minimum of all the components of the vector XZe. Throughout this problem, given any vector v, the notation $\min v$ ($\max v$) will denote the minimum (maximum) of the components of v.) Fix $\frac{1}{2} < \beta < 1$ (say $\beta = 0.95$). A *long-step* variant of the homogeneous self-dual method starts with an initial $(x, z) \in \mathcal{M}(\beta)$ and in every iteration uses

$$\delta = 2(1 - \beta)$$

and

$$\theta = \max\{t : (x + t\Delta x, z + t\Delta z) \in \mathcal{M}(\beta)\}.$$

The goal of this exercise is to analyze the complexity of this algorithm by completing the following steps.

(a) Show that $\mathcal{N}(\beta) \subset \mathcal{M}(\beta) \subset \mathcal{M}(1) = \{(x, z) > 0\}$.

(b) Show that $\max(-PQe) \leq \|r\|^2/4$. *Hint: Start by writing*

$$p_j q_j \geq \sum_{i: p_i q_i < 0} p_i q_i$$

and then use the facts that $p^T q = 0$, $p_i + q_i = r_i$, *and that for any two real numbers* a *and* b, $(a + b)^2 \geq 4ab$ *(prove this).*

(c) Show that if $(x, z) \in \mathcal{M}(\beta)$, then $\|r\|^2 \leq n\mu$. *Hint: Use* (22.11) *to write* $\|r\|^2 = \sum_j (x_j z_j - \delta\mu)^2 / x_j z_j$. *Expand the numerator, use the definitions of* μ *and* δ *to simplify, and then use the assumption that* $(x, z) \in \mathcal{M}(\beta)$ *to take care of the remaining denominator.*

(d) Show that if $(x, z) \in \mathcal{M}(\beta)$, then

$$\theta \geq \min\{1, -\frac{\beta\delta\mu}{\min PQe}\} \geq \frac{4\beta\delta}{n}.$$

Hint: Using the same notation as in the proof of Theorem 22.3, fix $t \leq \min\{1, -\beta\delta\mu / \min PQe\}$, *write*

$$x_j(t) z_j(t) - \mu(t) = (1 - t)(x_j z_j - \mu) + t^2 \Delta x_j \Delta z_j,$$

and then replace the right-hand side by a lower bound that is independent of j. *From there, follow your nose until you get the first inequality. The second inequality follows from parts* (b) *and* (c).

(e) As usual letting $\mu^{(k)}$ denote the μ value associated with the solution on the kth iteration of the algorithm, show that

$$\mu^{(k)} \leq \left(1 - \frac{4\beta\delta}{n}(1 - \delta)\right)^k.$$

(f) Give an upper bound on the number of iterations required to get $\mu^{(k)} \leq 2^{-L}$.

(g) Show that θ can be computed by solving n (univariate) quadratic equations.

(h) A robust implementation of a quadratic equation solver uses the formula

$$x = \begin{cases} \frac{-b-\sqrt{b^2-4ac}}{2a}, & b \geq 0, \\ \frac{2c}{-b+\sqrt{b^2-4ac}}, & b < 0, \end{cases}$$

for one of the two roots to $ax^2 + bx + c = 0$ (a similar formula is used for the other one). Show that the two expressions on the right are mathematically equal and suggest a reason to prefer one over the other in the particular cases indicated.

Notes

The first study of homogeneous self-dual problems appeared in Tucker (1956). This chapter is based on the papers Mizuno et al. (1993), Ye et al. (1994), and Xu et al. (1993). The step length formula (22.10) forces the algorithm studied in this chapter to take much shorter steps than those in Chapter 18. In general, algorithms that are based on steps that confine the iterates to $\mathcal{N}(\beta)$ are called *short-step methods*. A *long-step* variant of the algorithm can be obtained by enlarging the set $\mathcal{N}(\beta)$. Such a variant is the subject of Exercise 22.4. For this method, a worst case analysis shows that it takes on the order of n steps to achieve a given level of precision. Xu et al. (1993) describes an efficient implementation of the long-step variant.

The predictor–corrector method is a standard technique used in the numerical solution of ordinary differential equations. Mehrotra (1992) (see also Mehrotra 1989) was the first to apply this technique in the context of interior-point methods, although the related notion of forming power series approximations was suggested earlier by N.K. Karmarkar and is described in Adler et al. (1989).