Chapter 16
Analytical Solution

In this chapter an analytical solution of the one dimensional consolidation problem is given. In soil mechanics this solution was first given by Terzaghi (1923). In mathematics the solution had been known since the beginning of the 19th century. Fourier developed the solution to determine the heating and cooling of a metal strip, which is governed by the same differential equation. Terzaghi knew that solution, and adapted the parameters to the case of consolidation.

16.1 The Problem

The mathematical problem of one dimensional consolidation has been presented in the previous chapter. The differential equation is

$$\frac{\partial p}{\partial t} = c_v \frac{\partial^2 p}{\partial z^2}, \tag{16.1}$$

with the initial condition

$$t = 0: \quad p = p_0 = \frac{q}{1 + n\beta/m_v}, \tag{16.2}$$

in which q is the load applied at time $t = 0$. It is assumed that this load remains constant for $t > 0$.

The boundary conditions are, for the case of a sample of height $2h$, drained at its top and its bottom,

$$z = -h: \quad p = 0, \tag{16.3}$$

$$z = h: \quad p = 0. \tag{16.4}$$

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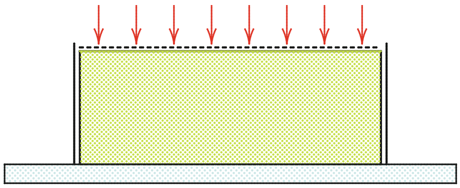

Fig. 16.1 Consolidation of a soil sample

These equations describe the consolidation of a soil sample in an oedometer test, or a confined compression test, with a constant load, and drained at the top and bottom of the sample. A variant of the problem is that of a sample of thickness h, drained at its top and with an impermeable bottom, so that the boundary condition (16.3) is replaced by

$$z = 0 : \quad \partial p / \partial z = 0. \tag{16.5}$$

Such a sample drains to the top only, see Fig. 16.1. The problem considered in this chapter assumes drainage to both the top and the bottom of a sample of height $2h$. Taking the upper half of the solution of this problem only leads to the same solution as the problem with the boundary conditions (16.5) and (16.4).

16.2 Solution

The problem defined by the Eqs. (16.1)–(16.4) can be solved, for instance, by separation of variables, or, even better, by the Laplace transform method, which is given in many books on advanced mathematics, see e.g. Churchill (1972) or Carslaw and Jaeger (1948).

The Laplace transform $\overline{p}$ of the pressure p is defined as

$$\overline{p} = \int_0^\infty \exp(-st) dt. \tag{16.6}$$

The basic principle of the Laplace transform method is that the differential equation (16.1) is multiplied by $\exp(-st)dt$, and then integrated from $t = 0$ to $t = \infty$. This gives, using partial integration and the initial condition (16.2),

$$s\overline{p} - p_0 = c_v \frac{d^2\overline{p}}{dz^2}. \tag{16.7}$$

The partial differential equation (16.1) has now been transformed into an ordinary differential equation. Its solution is

$$\overline{p} = \frac{p_0}{s} + A\exp(z\sqrt{s/c_v}) + B\exp(-z\sqrt{s/c_v}). \tag{16.8}$$

Here A and B are integration constants, that do not depend upon z, but may depend upon the transform parameter s. These constants may be determined from the boundary conditions (16.3) and (16.4),

$$A = -\frac{p_0}{2s\cosh(h\sqrt{s/c_v})}, \tag{16.9}$$

$$B = -\frac{p_0}{2s\cosh(h\sqrt{s/c_v})}. \tag{16.10}$$

The transform of the pore pressure now is

$$\frac{\overline{p}}{p_0} = \frac{1}{s} - \frac{\cosh(z\sqrt{s/c_v})}{s\cosh(h\sqrt{s/c_v})}. \tag{16.11}$$

The remaining problem now is the inverse transformation of the expression (16.11). This is a mathematical problem, that requires some experience with the Laplace transform method, including Heaviside's inversion theorem. This theorem states that the inverse transform of a function of the form $f(s) = P(s)/Q(s)$ consists of a series of terms, one for each of the zeros of the denominator $Q(s)$. Each of these terms gives a contribution of the form

$$\frac{p}{p_0} = \frac{P(s_j}{Q'(s_j)}\exp(-s_j t). \tag{16.12}$$

Without giving all the details it is stated here that the final result is

$$\frac{p}{p_0} = \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{2j-1}\cos\left[(2j-1)\frac{\pi}{2}\frac{z}{h}\right]\exp\left[-(2j-1)^2\frac{\pi^2}{4}\frac{c_v t}{h^2}\right]. \tag{16.13}$$

This is the analytical solution of the problem. It is shown in Fig. 16.2. The data in this solution may be obtained from a simple computer program, for instance a program in Turbo Pascal, see Program CONSOL1D.PAS, listed below. The program gives the values of the pore water pressure as a function of depth, for a certain value of time. In the program the terms of the infinite series are taken into account until the argument of the exponential function reaches the value 20. This is based upon the notion that al terms containing a factor exp(−20), or smaller, can be disregarded.

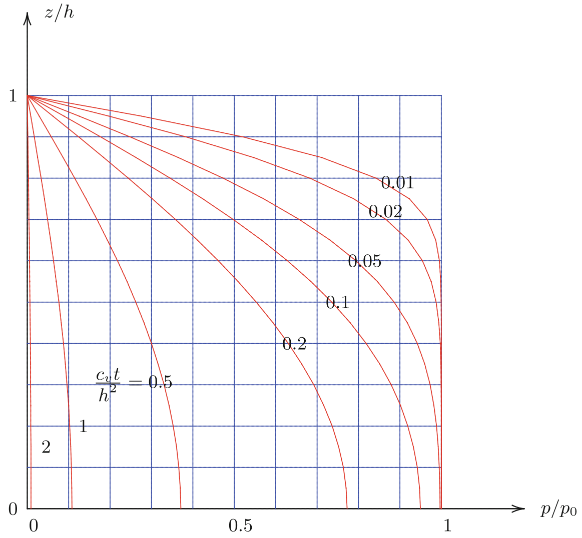

Fig. 16.2 Analytical solution of Terzaghi's problem

```
program CONSOL1D;
uses crt;
const
 nn=10;
var
 i,j,jj,k:integer;h,cv,tt,jt,pi,f,a,pa,pp:real;
 z,p,t:array[0..nn] of real;
procedure title;
 begin
  clrscr;gotoxy(35,1);textbackground(7);textcolor(0);write(' CONSOL1D ');
  textbackground(0);textcolor(7);writeln;
 end;
procedure next;
 var
  c:char;
 begin
  gotoxy(25,25);textbackground(7);textcolor(0);
  write(' Touch any key to continue ');write(chr(8));
  c:=readkey;textbackground(0);textcolor(7)
 end;
begin
 h:=1;cv:=1;t[0]:=0;t[1]:=0.01;t[2]:=0.02;t[3]:=0.05;t[4]:=0.1;
 t[5]:=0.2;t[6]:=0.5;t[7]:=1;t[8]:=2;
 for k:=1 to 8 do
  begin
   tt:=cv*t[k]/(h*h);pi:=3.1415926;a:=4/pi;pa:=pi/2;pp:=pa*pa;
   p[0]:=0;p[nn]:=0;z[0]:=0;z[nn]:=h;
   for i:=0 to nn-1 do
    begin
     z[i]:=i*h/nn;p[i]:=0;
     f:=1;j:=1;jj:=2*j-1;jt:=jj*jj*pp*tt;
```

```
      while (jt<20) do
       begin
        p[i]:=p[i]+(a*f/jj)*cos(jj*pa*z[i])*exp(-jt);
        j:=j+1;f:=-f;jj:=2*j-1;jt:=jj*jj*pp*tt;
       end;
    end;
   title;writeln(' cv*t/h^2 = ',t[k]:6:3);writeln;
   for j:=0 to nn do
    begin
     writeln(' z/h = ',z[nn-j]:6:3,', p/p0 = ',p[nn-j]:6:3);
    end;
   next;
  end;
end.
```

Program CONSOL1D.PAS

At a first glance the solution (16.13) may not seem to give much insight, but after some closer inspection many properties of the solution can be obtained from it. It is for instance easy to see that for $z = h$ the pressure $p = 0$, which shows that the solution satisfies the boundary condition (16.4). The cosine of each term of the series (16.13) is zero if $z = h$, because $\cos(\pi/2) = 0$, $\cos(3\pi/2) = 0$, $\cos(5\pi/2) = 0$, etc. It can also be verified easily that the solution (16.13) satisfies the differential equation (16.1), because each individual term satisfies that equation. That the boundary condition (16.5) is satisfied can most easily be checked by noting that after differentiation with respect to z each term will contain a factor $\sin(\ldots z)$, and these are all zero if $z = 0$. To check the initial condition is not so easy, because for $t = 0$ the series converges very slowly. The verification can best be performed from the computer program, taking the value of t very small.

A good impression of the solution can be obtained by investigating its behavior for large values of time. Because the exponential functions contain a factor $(2j-1)^2$, i.e. factors 1, 9, 16, …, all later terms can be disregarded if the first term is small. This means that for large values of time the series can be approximated by its first term,

$$\frac{c_v t}{h^2} \gg 0.1 : \quad \frac{p}{p_0} \approx \frac{4}{\pi} \cos\left(\frac{\pi}{2}\frac{z}{h}\right) \exp\left(-\frac{\pi^2}{4}\frac{c_v t}{h^2}\right). \tag{16.14}$$

After a sufficiently long time only one term of the series remains, which is a cosine function in z-direction. Its values tend to zero if $t \to \infty$. The approximation (16.14) can be used if time t is not too small. In practice it can be applied for all values for which $c_v t/h^2 > 0.2$.

16.3 The Deformation

Once that the pore pressures are known, the deformations can easily be calculated. The vertical strain is given by

$$\varepsilon = -m_v(\sigma - p). \tag{16.15}$$

This means that the total deformation of the sample is

$$\Delta h = \int_0^h \varepsilon\, dz = -m_v h q + m_v \int_0^h p\, dz. \tag{16.16}$$

The first term on the right hand side is the final deformation, which will be reached when all pore pressures have been reduced to zero. That value will be denoted by Δh_∞,

$$\Delta h_\infty = -m_v h q. \tag{16.17}$$

Immediately after the application of the load the pore pressure $p = p_0$, see Eq. (16.2). The deformation then is, with (16.16),

$$\Delta h_0 = -m_v h q \frac{n\beta/m_v}{1 + n\beta/m_v}. \tag{16.18}$$

If the water is incompressible ($\beta = 0$), this is zero, as expected. The expressions (16.17) and (16.18) are negative if $q > 0$, which indicates that the sample will become shorter when loaded.

To describe the deformation as a function of time, a useful quantity is the *degree of consolidation*, defined as

$$U = \frac{\Delta h - \Delta h_0}{\Delta h_\infty - \Delta h_0}. \tag{16.19}$$

This is a dimensionless quantity, varying between 0 (for $t = 0$) and 1 (for $t \to \infty$). The degree of consolidation indicates how far the consolidation process has been progressed.

With (16.16), (16.17) and (16.18) one obtains

$$U = \frac{1}{h} \int_0^h \frac{p_0 - p}{p_0} dz. \tag{16.20}$$

And with (16.13) this gives

$$U = 1 - \frac{8}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} \exp\left[-(2j-1)^2 \frac{\pi^2}{4} \frac{c_v t}{h^2}\right]. \tag{16.21}$$

For $t \to \infty$ this is indeed equal to 1. The value $U = 0$ for $t = 0$ can be verified from the series

$$\sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}. \tag{16.22}$$

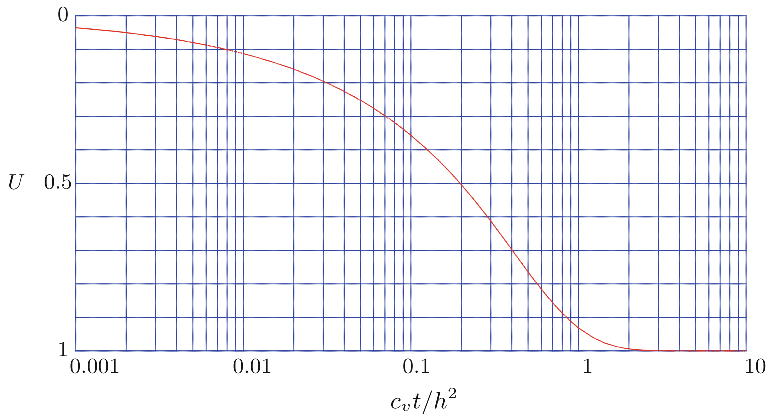

Fig. 16.3 Degree of consolidation

The degree of consolidation, which is a function of the dimensionless time parameter $c_v t/h^2$ only, is shown in Fig. 16.3.

The degree of consolidation, which is a function of the dimensionless time $c_v t/h^2$ is shown in Fig. 16.3. The data may be calculated by the Program DEGREE.PAS, listed below.

```
program DEGREE;
uses crt;
const
 nn=20;
var
 j,jj,k:integer;
 h,cv,tt,pi,pp,a,u:real;
 t:array[0..nn] of real;
procedure title;
 begin
  clrscr;gotoxy(35,1);textbackground(0);textcolor(7);write(' DEGREE ');
  textbackground(7);textcolor(0);writeln;
 end;
procedure next;
 var
  c:char;
 begin
  gotoxy(25,25);textbackground(0);textcolor(7);
  write(' Touch any key to continue ');write(chr(8));
  c:=readkey;textbackground(7);textcolor(0)
 end;
begin
 title;
 h:=1;cv:=1;pi:=3.1415926;pp:=pi*pi/4;a:=2/pp;
 t[0]:=0;t[1]:=0.001;t[2]:=0.002;t[3]:=0.005;t[4]:=0.01;t[5]:=0.02;
 t[6]:=0.05;t[7]:=0.1;t[8]:=0.2;t[9]:=0.5;t[10]:=1;t[11]:=2;
 t[12]:=5;t[13]:=10;
 for k:=1 to 13 do
  begin
```

```
    tt:=cv*t[k]/(h*h);j:=1;jj:=(2*j-1)*(2*j-1);jt:=jj*pp*tt;u:=1;
    while (jt<20) do
     begin
      u:=u-a*exp(-jt)/jj;
      j:=j+1;jj:=(2*j-1)*(2*j-1);jt:=jj*pp*tt;
     end;
    writeln(' t = ',t[k]:6:3,', u = ',u:6:3;)
  end;
 next;
end.
```

Program DEGREE.PAS

Theoretically speaking the consolidation process takes infinitely long to be completed. For engineering practice, however, it is sufficient if the first (and largest) term in (16.21), the infinite series, is about 0.01. Then 99 % of the final deformation has been reached. It can be seen that this is the case if $c_v t/h^2 = 1.784$, or roughly speaking $c_v t/h^2 = 2$. This means that

$$t_{99\,\%} = \frac{2h^2}{c_v} = \frac{2h^2(m_v + n\beta)\gamma_w}{k}. \tag{16.23}$$

This very useful formula is a summary of the process of consolidation. Because the coefficient of consolidation c_v is the quotient of the permeability k and the compressibility m_v, it can be seen from Eq. (16.23) that the consolidation process takes longer if the permeability is smaller, or if the compressibility is larger. This is understandable if one realizes that the consolidation process consists of compression of the soil, retarded by the outflow of water. If the permeability is smaller the outflow is slower, and the consolidation therefore takes longer. And if the compressibility is large much water must be expelled, and that takes a long time.

For engineering practice it is also very important that the time t appears in the formula (16.21) in the combination $c_v t/h^2$. This means that the process will take 4 times as long if the layer is a factor 2 thicker. It also means that if in a laboratory test on a sample of 2 cm thickness, the consolidation process has been found to be finished after 1 h (this can best be measured by measuring the pore pressures, and then waiting until they are practically zero), the consolidation of that soil in the field for a layer of 2 m thickness, will take 10,000 times as long, that is more than 1 year.

Another important consequence of the fact that the consolidation process is governed by the factor $c_v t/h^2$ is that the duration of the consolidation process can be shortened considerably by reducing the drainage length h. As an example one may consider the consolidation process of a clay layer of 10 m thickness. Suppose that the permeability k is about 10^{-9} m/s. Let it furthermore be expected that the final deformation of the clay layer by a load of 50 kPa (the weight of 3 m dry sand) is 20 cm. This means that the value of the compressibility m_v is, with (15.3) : $m_v = 0.0004\ \text{m}^2/\text{kN}$. The coefficient of consolidation then is, with (15.16), $c_v = 0.25 \times 10^{-6}\ \text{m}^2/\text{s}$. The consolidation time is, with (16.23), $t_{99\,\%} = 2 \times 10^8$ s. That is about 6 years, which means that it will take many years before the deformation reaches its final value of

20 cm. To speed up the consolidation process a large number of vertical drains may be installed, in the form of plastic filter material. If these drains are installed in a pattern with mutual distances of about 1.60 m, the drainage length becomes about a factor 6 smaller (0.80 m rather than 5 m). If it is assumed that the horizontal permeability is equal to the vertical permeability, the duration of the consolidation process will be a factor 36 shorter, that is about 2 months. For a new road, or a new town extension this means that the settlements are concentrated in a much shorter time span.

16.4 Approximation for Small Values of Time

If the time parameter $c_v t/h^2$ is very small, many terms are needed in the analytical solutions to obtain accurate results. That may not be a great disadvantage if the computations are performed by a computer program, but it does not give much insight into the solution. A more convenient approximation can be obtained using a theorem from Laplace transform theory saying that an approximation for small values of t can be obtained by assuming the value of s in the transformed solution as very large. Again, the details are omitted here. The result for the degree of consolidation is found to be

$$U = \frac{\Delta h - \Delta h_0}{\Delta h_\infty - \Delta h_0} \approx \frac{2}{\sqrt{\pi}} \sqrt{\frac{c_v t}{h^2}}. \tag{16.24}$$

It appears that in the beginning of the consolidation process its advance increases with the square root of time.

The approximate formula (16.24) also enables to estimate how short the loading time of a load must be to be considered as instantaneous. It can be seen that only 1% of the consolidation process has been completed if $c_v t/h^2 = 10^{-4}\pi/4$, or about $t = t_{1\%}$, with

$$t_{1\%} = 10^{-4} \frac{h^2}{c_v}. \tag{16.25}$$

A load that is applied faster than this value of time can be considered as an instantaneous load.

Example 16.1 A clay sample of 2 cm thickness is being tested in an oedometer. The sample is drained on both sides. The coefficient of consolidation is known to be $c_v = 10^{-7}\text{m}^2/\text{s}$. At a certain moment of time the sample is loaded. Calculate the time for the pore water pressure in the center of the sample to be reduced to 50% of its initial value.

Solution

In this case the reduction of the pore pressure is large enough to justify the use of only one term of the analytical solution, as given in Eq. (16.14). In this case only the top half of the symmetric sample may be considered, with $h = 1$ cm. The bottom of this half, $z = 0$, corresponds to the center of the actual sample. Equation 16.14 now gives

$$\frac{4}{\pi} \exp\left(-\frac{\pi^2}{4}\frac{c_v t}{h^2}\right) = \frac{p}{p_0} = 0.5.$$

It follows that $c_v t/h^2 = 0.3788$, which is certainly large enough for the approximation (16.14) to be applicable. Using the given values of c_v and h it finally follows that $t = 379$ s.

Example 16.2 Determine the error in the approximation (16.14) for $c_v t/h^2 = 0.2$, by calculating the second term in the series (16.13), for $z = 0$.

Solution

The second term in the series (16.13) can be calculated by taking $j = 2$ and $z = 0$. This gives for the contribution of that term

$$\frac{\Delta p}{p_0} = -\frac{4}{3\pi} \exp\left[-\frac{9\pi^2}{4}\frac{c_v t}{h^2}\right].$$

With $c_v t/h^2 = 0.2$ it follows that $\Delta p/p_0 = 0.005$, which means that the error in the approximate solution is about 0.5%.

Problem 16.1 In a test on a clay sample of 2 cm thickness it has been measured that after 15 min the pore pressures are practically zero. What will be the duration of the consolidation process for a layer of the same clay, of 5 m thickness?

Problem 16.2 In a laboratory test on a clay sample it has been forgotten to measure the deformation immediately after the application of the load. The measurement after 1 min was a deformation of 0.06 mm, and after 4 min a deformation of 0.08 mm. Estimate the initial deformation.

Problem 16.3 The computer program in this chapter can not be used if $t = 0$, because then the loop will continue forever. The series solutions do converge, however. Formulate a better criterion for terminating the series, and install this improvement in the programs.

Problem 16.4 Extend the computer program of this chapter with facilities for graphical output, or output on a printer.

References

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