Chapter 27
Boussinesq

In this chapter some useful solutions for the stresses in an elastic half space are given. These solutions were first obtained by the French scientist Joseph Boussinesq in 1885, and can be found in many books on the theory of elasticity.

27.1 Boussinesq's Problem

The problem to be considered is to obtain a solution for the stresses and strains in a homogeneous isotropic linear elastic half space, loaded by a vertical point force on the surface, see Fig. 27.1. A derivation of this solution is given in Appendix B.

The stresses are found to be

$$\sigma_{zz} = \frac{3P}{2\pi}\frac{z^3}{R^5}, \tag{27.1}$$

$$\sigma_{rr} = \frac{P}{2\pi}\left[\frac{3r^2z}{R^5} - (1-2\nu)\frac{1}{R(R+z)}\right], \tag{27.2}$$

$$\sigma_{\theta\theta} = \frac{P}{2\pi}\frac{1-2\nu}{R^2}\left(\frac{R}{R+z} - \frac{z}{R}\right), \tag{27.3}$$

$$\sigma_{rz} = \frac{3P}{2\pi}\frac{rz^2}{R^5}. \tag{27.4}$$

In these equations r is the cylindrical coordinate,

$$r = \sqrt{x^2+y^2}, \tag{27.5}$$

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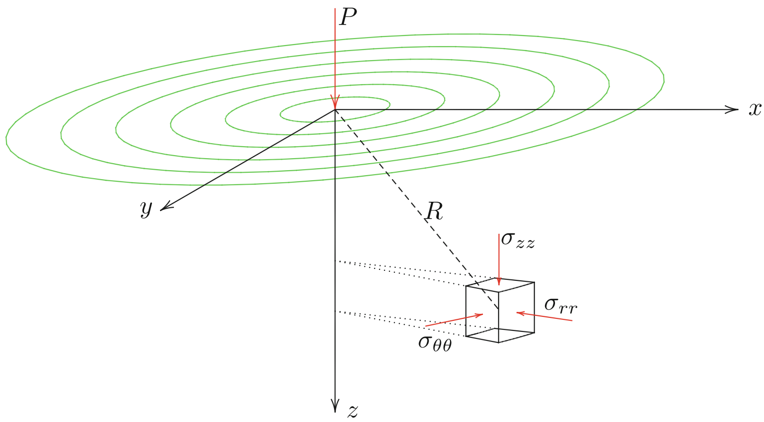

Fig. 27.1 Point load on half space

and R is the spherical coordinate,

$$R = \sqrt{x^2 + y^2 + z^2}. \tag{27.6}$$

The solution for the displacements is

$$u_r = \frac{P(1+\nu)}{2\pi E R}\left[\frac{r^2 z}{R^3} - (1-2\nu)\left(1 - \frac{z}{R}\right)\right], \tag{27.7}$$

$$u_\theta = 0, \tag{27.8}$$

$$u_z = \frac{P(1+\nu)}{2\pi E R}\left[2(1-\nu) + \frac{z^2}{R^2}\right]. \tag{27.9}$$

The vertical displacement of the surface is particularly interesting. This is

$$z = 0 : \quad u_z = \frac{P(1-\nu^2)}{\pi E r}. \tag{27.10}$$

For $r \to 0$ this tends to infinity, indicating that at the point of application of the point load the displacement is infinitely large. This singular behavior is a consequence of the singularity in the surface load, as in the origin the stress is infinitely large. That the displacement in that point is also infinitely large may not be so surprising.

Another interesting quantity is the distribution of the stresses as a function of depth, just below the point load, i.e. for $r = 0$. This is found to be

$$r = 0 : \quad \sigma_{zz} = \frac{3P}{2\pi z^2}, \tag{27.11}$$

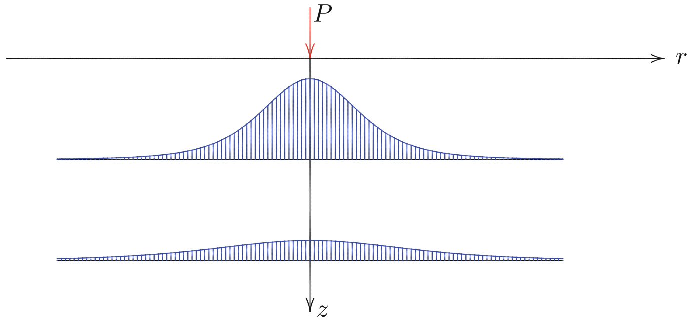

Fig. 27.2 Vertical normal stress σ_{zz}

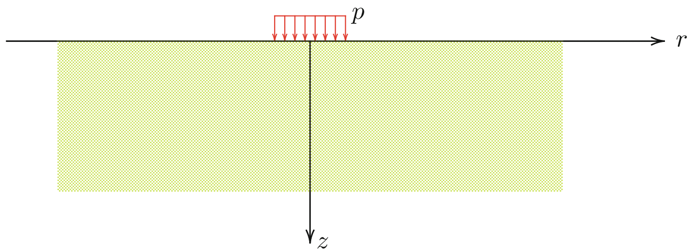

Fig. 27.3 Uniform load over circular area

$$r = 0 : \quad \sigma_{rr} = \sigma_{\theta\theta} = -(1 - 2\nu)\frac{P}{4\pi z^2}. \tag{27.12}$$

These stresses decrease with depth, of course.

In engineering practice it is sometimes assumed, as a first approximation, that at a certain depth the stresses are distributed uniformly over an area obtained by drawing a line from the load at an angle of 45°. That would mean that the vertical normal stress at a depth z would be $P/\pi z^2$, homogeneously over a circle of radius z. Comparing this approximation with the analytical solution (27.1) it appears to be incorrect (the error is 50% if $r = 0$), but the trend is correct, as the stresses indeed decrease with $1/z^2$. In Fig. 27.2 the distribution of the vertical normal stress σ_{zz} is represented as a function of the cylindrical coordinate r, for two values of the depth z.

The assumption of a linear elastic material behavior means that the entire problem is linear, as the equations of equilibrium and compatibility are also linear. This implies that the principle of superposition of solutions can be applied. Boussinesq's solution can be used as the starting point of more general types of loading, such as a system of point loads, or a uniform load over a certain given area.

As an example consider the case of a uniform load of magnitude p over a circular area, of radius a. The solution for this case can be found by integration over a circular area, see Fig. 27.3. The stresses along the axis $r = 0$, i.e. below the center of the load, are found to be

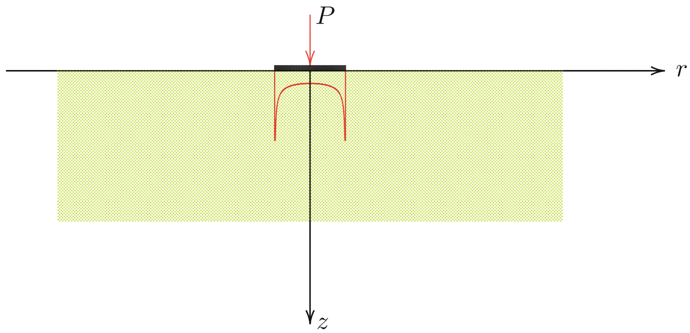

Fig. 27.4 Rigid plate on half space

$$r = 0 \; : \quad \sigma_{zz} = p\left(1 - \frac{z^3}{b^3}\right), \tag{27.13}$$

$$r = 0 : \quad \sigma_{rr} = p\left[(1+\nu)\frac{z}{b} - \frac{1}{2}\left(1 - \frac{z^3}{b^3}\right)\right], \tag{27.14}$$

where $b = \sqrt{z^2 + a^2}$.

The displacement of the origin is

$$r = 0, \; z = 0 \; : \quad u_z = 2(1-\nu^2)\frac{pa}{E}. \tag{27.15}$$

This solution will be used as the basis of a more general case in the next chapter.

Another important problem, which was already solved by Boussinesq, is the problem of a half space loaded by a vertical force on a rigid plate. The force is represented by $P = \pi a^2 \overline{p}$, see Fig. 27.4. The distribution of the normal stresses below the plate is found to be

$$z = 0, \; 0 < r < a \; : \quad \sigma_{zz} = \frac{\frac{1}{2}\overline{p}}{\sqrt{1 - r^2/a^2}}. \tag{27.16}$$

This stress distribution is shown in Fig. 27.4. At the edge of the plate the stresses are infinitely large, as a consequence of the constant displacement of the rigid plate. In reality the material near the edge of the plate will probably deform plastically. It can be expected, however, that the real distribution of the stresses below the plate will be of the form shown in the figure, with the largest stresses near the edge. The center of the plate will subside without much load.

The displacement of the plate is

$$z = 0, \; 0 < r < a \; : \quad u_z = \frac{\pi}{2}(1-\nu^2)\frac{\overline{p}a}{E}. \tag{27.17}$$

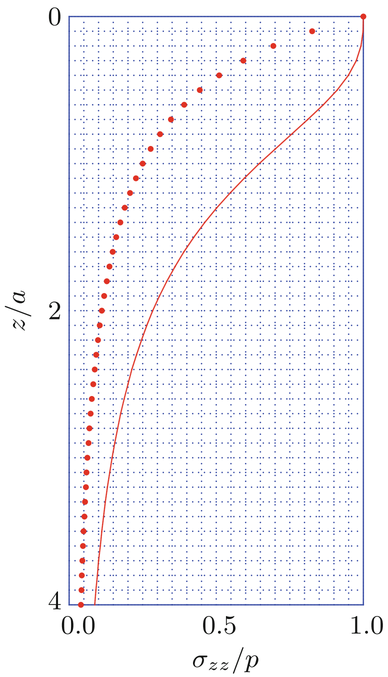

Fig. 27.5 Exact and approximate solution

When this is compared with the displacement below a uniform load, see (27.15), it appears that the displacement of the rigid plate is somewhat smaller, as could be expected.

Example 27.1 The distribution of the vertical stresses at a depth z below a uniform load p on a circular area of radius a can be approximated by a uniform distribution of stress over an area of radius $z + a$. The stress below the center of the load then would be $\sigma_{zz} = pa^2/(z + a)^2$. In Fig. 27.5 this approximation is compared with the exact solution given in Eq. (27.13). The exact solution is shown by the fully drawn line, the approximate solution by the dots. The approximation appears not to be very good.

An approximation of the surface settlement can be obtained by integrating the vertical deformations. This vertical deformation can be approximated by $\varepsilon_{zz} = \sigma_{zz}/E$, where E is Young's modulus, and the effect of horizontal stresses has been neglected. Thus the approximate vertical deformation is $\varepsilon_{zz} = pa^2/(E(z + a)^2)$. Integration from $z = 0$ to $z = \infty$ gives $u_z = pa/E$, which is about a factor 2 smaller than the exact result given in Eq. (27.15). The structure of the approximation agrees very well with the exact solution, however.

Actually this agreement is less surprising than it might seem. Because of the linearity of the system the displacement must be proportional to the load p, and the only other parameters in the problem are the radius a and Young's modulus E. The only combination of these parameters that will lead to a displacement (which has the dimension of a length) is pa/E.

In engineering practice the displacement due to a loaded plate is often expressed by a subgrade constant c, by writing $u_z = p/c$. In this case it appears that $c = E/a$, which relates the subgrade constant to Young's modulus and the plate size a. In reality a dimensionless multiplication factor may have to be included, of course.