Chapter 15
Consolidation

In the previous chapters it has been assumed that the deformation of a soil is uniquely determined by the stress. This means that a time dependent response has been excluded. In reality the behavior is strongly dependent on time, however, especially for clay soils. In compression of a soil the porosity decreases, and as a result there is less space available for the pore water. This pore water can be expelled from the soil, but in clays this may take a certain time, due to the small permeability. This process is called *consolidation*. Its basic equations are considered in this chapter.

15.1 Differential Equation

The analysis will be restricted to one dimensional deformation, assuming that the soil does not deform in lateral direction. It is also assumed that the water can only flow in vertical direction. This will be the case during an oedometer test, or in the field, in case of a surcharge load over a large area, see Fig. 15.1.

To simplify the analysis it will be assumed that the change in stress is small compared to the initial stress. In that case the stress-strain relation may be linearized, using an elastic coefficient $D = K + \frac{4}{3}G$, see (14.19). The precise value of that coefficient depends upon the initial stress. The relation between the increment of effective stress $\Delta\sigma'$ and the increment of strain $\Delta\varepsilon$ can now be written as

$$\Delta\sigma' = -\left(K + \frac{4}{3}G\right)\,\Delta\varepsilon. \tag{15.1}$$

In the remainder of this chapter the notation Δ will be omitted. Thus the increment of the effective stress will be denoted simply by σ', and the increment of the strain by ε, so that

A. Verruijt, *An Introduction to Soil Mechanics*, Theory and Applications of Transport in Porous Media 30, DOI 10.1007/978-3-319-61185-3_15

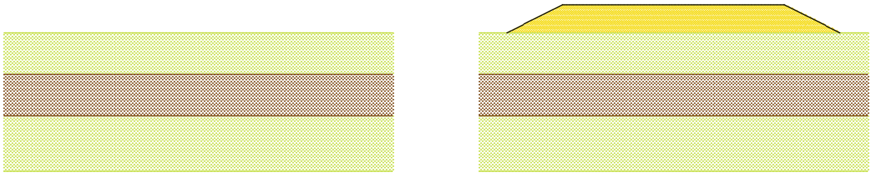

Fig. 15.1 Uniform load

$$\sigma' = -\left(K + \frac{4}{3}G\right)\varepsilon. \tag{15.2}$$

Using stresses and strains with respect to some initial state is very common in soil mechanics. For the strains there is actually no other possibility. Strains can only be measured with respect to some initial state, and in this initial state the soil is not stress free. Gravity is always acting, and the stresses due to gravity have been developed gradually during geological history. The logical procedure is to regard the state of stress including the influence of the weight of the soil layers as a given initial state, and to regard all effects of engineering activity with respect to that initial state. It should be noted that to obtain the true stresses in the field the initial stresses should be added to the incremental stresses.

In the analysis of consolidation it is customary to write Eq. (15.2) in its inverse form,

$$\varepsilon = -m_v\,\sigma', \tag{15.3}$$

where m_v is denoted as the *compressibility coefficient*. If the incremental vertical total stress is denoted by σ, and the incremental pore pressure by p, then Terzaghi's principle of effective stress is

$$\sigma' = \sigma - p. \tag{15.4}$$

It follows from (15.3) that

$$\varepsilon = -m_v(\sigma - p). \tag{15.5}$$

The total stress σ is often known, as a function of time. Its value is determined by the load. Let it be assumed that initially $\sigma = 0$, indicating no additional load.

During the application of the load the total stress σ is supposed to be increased by a given amount, in a very short time interval, after which the total stress remains constant. The pore pressure may vary during that period. To describe its generation and dissipation the continuity of the water must be considered.

Consider an elementary volume V in the soil, see Fig. 15.2. The volume of water is $V_w = nV$, where n is the porosity. The remaining volume, $V_p = (1-n)V$ is the total volume of the particles. As usual, the particles are considered as incompressible. This means that the volume V can change only if the porosity changes. This is possible only if the water in the pores is compressed, or if water flows out of the element.

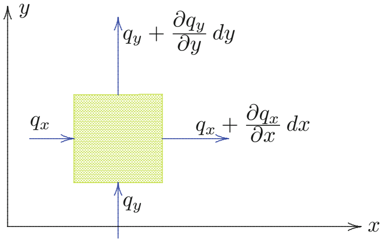

Fig. 15.2 Outflow

The first possibility, a volume change by compression of the pore water, can be caused by a change of the pore pressure p. It can be expected that the change of volume is proportional to the change of the pressure, and to the original volume, i.e.

$$\Delta V_1 = -\beta V_w \Delta p = -n\beta V \Delta p, \tag{15.6}$$

where β represents the *compressibility* of the water. For pure water handbooks of physics give $\beta = 0.5 \times 10^{-9}$ m^2/N, which is very small. Water is practically incompressible. However, when the water contains some small bubbles of gas (air or natural gas), the value of β may be much larger, approximately

$$\beta = S\beta_0 + \frac{(1-S)}{p_0}, \tag{15.7}$$

where β_0 is the compressibility of pure water, S is the degree of saturation, and p_0 is the absolute pressure in the water, considered with respect to vacuum (this means that under atmospheric conditions $p_0 =$ 100 kPa). If $S = 0.99$ and the pressure is $p_0 =$ 100 kPa, then $\beta = 10^{-7}$ m^2/N. That is still a small value, but about 200 times larger than the compressibility of pure water. The apparent compressibility of the water is now caused by the compression of the small air bubbles. The formula (15.7) can be derived on the basis of Boyle's gas law. Taking into account the compressibility of the fluid, even though the effect is small, makes the analysis more generally applicable.

The second possibility of a volume change, as a result of a net outflow of water, is described by the divergence of the specific discharge, see Fig. 15.2. There is a net loss of water when the outflow from the element is larger than the inflow into it. In a small time Δt the volume change is

$$\Delta V_2 = -(\nabla \cdot \mathbf{q}) V \Delta t = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) V \Delta t. \tag{15.8}$$

The minus sign expresses that a positive value of $\nabla \cdot \mathbf{q}$ indicates that there is a net outflow, which means that the volume will decrease. The volume increase ΔV_2 then is negative.

The total volume change in a small time Δt now is

$$\Delta\varepsilon_{\text{vol}} = \frac{\Delta V}{V} = \frac{\Delta V_1 + \Delta V_2}{V} = -n\beta\Delta p - \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)\Delta t. \tag{15.9}$$

After division by Δt, and passing into the limit $\Delta t \to 0$, the resulting equation is

$$\frac{\partial\varepsilon_{\text{vol}}}{\partial t} = -n\beta\frac{\partial p}{\partial t} - \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right). \tag{15.10}$$

This is an important basic equation of the theory of consolidation, the *storage equation*. It expresses that a volume change ($\partial e/\partial t$) can be caused by either a pressure change (the factor n indicating how much water is present, and the factor β indicating its compressibility), or by a net outflow of water from the pores.

In the one dimensional case of vertical flow only, the storage equation reduces to

$$\frac{\partial\varepsilon_{\text{vol}}}{\partial t} = -n\beta\frac{\partial p}{\partial t} - \frac{\partial q_z}{\partial z}. \tag{15.11}$$

The value of the specific discharge q_z depends upon the pressure gradient, through Darcy's law,

$$q_z = -\frac{k}{\gamma_w}\frac{\partial p}{\partial z}. \tag{15.12}$$

It should be noted that it is not necessary to take into account a term for the pressure gradient due to gravity, because p indicates the increment with respect to the initial state, in which gravity is taken into account.

It follows from (15.11) and (15.12), assuming that the hydraulic conductivity k is constant,

$$\frac{\partial\varepsilon_{\text{vol}}}{\partial t} = -n\beta\frac{\partial p}{\partial t} + \frac{k}{\gamma_w}\frac{\partial^2 p}{\partial z^2}. \tag{15.13}$$

This equation contains two variables, the volume strain ε_{vol} and the fluid pressure p. Another equation is needed for a full description of the problem. This second equation is provided by the relation of the deformation of the soil to the stresses.

In the one dimensional case considered here the lateral strains are zero, so that the volume strain ε_{vol} is equal to the vertical strain ε,

$$\varepsilon_{\text{vol}} = \varepsilon. \tag{15.14}$$

It now follows from (15.5), (15.13) and (15.14), if it is assumed that the compressibility m_v is constant in time,

$$\frac{\partial p}{\partial t} = \frac{m_v}{m_v + n\beta}\frac{\partial\sigma}{\partial t} + c_v\frac{\partial^2 p}{\partial z^2}, \tag{15.15}$$

where c_v is the *consolidation coefficient*,

$$c_v = \frac{k}{\gamma_w(m_v + n\beta)}. \tag{15.16}$$

Equation (15.15) is the basic differential equation for the one dimensional consolidation process. From this equation the pore pressure p must be determined. The variation of the total stress σ with time, $\partial\sigma/\partial t$, is supposed to be given by the loading conditions.

The simplest type of loading occurs when the total stress σ is constant during the entire process. This will be the case if the load does not change after its initial application. Then

$$\frac{\partial p}{\partial t} = c_v \frac{\partial^2 p}{\partial z^2}, \tag{15.17}$$

In mathematical physics an equation of this type is denoted as a *diffusion equation*. The same equation describes the process of heating or cooling of a strip of metal. The variable then is the temperature.

It may be noted that the differential equation does not become simpler when the water is assumed to be incompressible ($\beta = 0$). Only the coefficient c_v is affected. The compressibility of the water does not complicate the mathematics.

15.2 Boundary Conditions and Initial Condition

To complete the formulation of the problem, the boundary conditions and initial conditions must be added to the differential equation (15.17). In the case of an oedometer test, see Fig. 15.3, the sample is usually drained at the top, using a thin sheet of filter paper and a steel porous plate, or a porous stone. In the container in which the sample and its surrounding ring are placed, the water level is kept constant. This means that at the top of the sample the excess pore pressure is zero,

$$z = h \;:\quad p = 0. \tag{15.18}$$

The soil sample may also be drained at its bottom, but alternatively, it may be supported by an impermeable plate. In that case the boundary condition at the bottom of the sample is

$$z = 0 \;:\quad \frac{\partial p}{\partial z} = 0, \tag{15.19}$$

indicating no outflow at the bottom of the sample. These two boundary conditions are physically sufficient. In general a second order differential equation requires two boundary conditions.

The initial condition is determined by the way of loading. A common testing procedure is that a load is applied in a very short time (by placing a weight on the loading plate). After this loading the load is kept constant. At the time of loading an

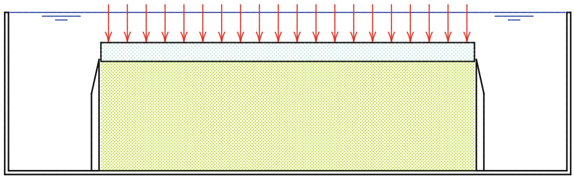

Fig. 15.3 Oedometer test

immediate increase of the pore pressure is generated, that can be determined in the following way. The storage equation (15.10) is integrated over a short time interval Δt, giving

$$\varepsilon_{\text{vol}} = -n\beta p - \int_0^{\Delta t} \frac{\partial q_z}{\partial z}\, dt.$$

The integral represents the amount of water that has flowed out of the soil in the time interval Δt. If $\Delta t \to 0$ this must be zero, so that

$$t = 0 : \quad \varepsilon_{\text{vol}} = -n\beta p. \tag{15.20}$$

On the other hand, it follows from (15.5), taking into account that in this case $\varepsilon_{\text{vol}} = \varepsilon$,

$$\varepsilon_{\text{vol}} = -m_v(\sigma - p). \tag{15.21}$$

From Eqs. (15.20) and (15.21) it now follows that

$$t = 0 : \quad p = \frac{\sigma}{1 + n\beta/m_v}. \tag{15.22}$$

This is the initial condition. It means that at the time of loading, $t = 0$, the pore water pressure p is given.

If the water is considered as completely incompressible (which is a reasonable assumption when the soil is completely saturated with water) Eq. (15.22) reduces to

$$t = 0,\ \beta = 0 : \quad p = \sigma. \tag{15.23}$$

In that case the initial pore pressure equals the given load. That can be understood by noting that in case of an incompressible pore fluid there can be no immediate volume change. This means that there can be no vertical strain, as the volume change equals the vertical strain in this case of a sample that is laterally confined by the stiff steel ring. Hence there can be no vertical strain at the moment of loading, and therefore the effective stress can not increase at that instant. In this case, of lateral confinement and incompressible water, the entire load is initially carried by the water in the pores, and the effective stress remains equal to its initial value. This type of response is called *undrained*. It is characteristic of soft soils under rapid loading.

It should be noted that throughout this chapter the deformation and the flow have been one dimensional. In a more general three dimensional case, there may be lateral deformations, and an immediate deformation is very well possible, although the volume must remain constant if the fluid is incompressible. There can then be an immediate change of the effective stresses. The water will then carry only part of the load. The three dimensional theory of consolidation is an interesting topic for further study.