Chapter 33
Coulomb

Long before the analysis of Rankine the French scientist Coulomb presented a theory on limiting states of stress in soils (in 1776), which is still of great value. The theory enables to determine the stresses on a retaining wall for the cases of active and passive earth pressure. The method is based upon the assumption that the soil fails along straight slip planes.

33.1 Active Earth Pressure

For the active case (a retreating wall) the procedure is illustrated in Fig. 33.1. It is assumed that in case of a displacement of the wall towards the left, a triangular wedge of soil will slide down, along a straight slip plane. The angle of the slope with the vertical direction is denoted by θ. It is also assumed that at the moment of sliding, the weight of the soil wedge is just in equilibrium with the forces on the slip surface and the forces on the wall. For reasons of simplicity it is assumed, at least initially, that the force between the soil and the wall (Q) is directed normal to the surface of the wall, i.e. shear stresses along the wall are initially neglected. In later chapters such shear stresses will be taken into account as well. The purpose of the analysis is to determine the magnitude of the force Q. The principle of Coulomb's method is that it is stated that the wall must be capable of withstanding the force Q for all possible slip planes. Therefore the slip plane that leads to the largest value of Q is to be determined. The various slip planes are characterized by the angle θ, and this angle will be determined such that the maximum value of Q is obtained.

The starting point of the analysis is the weight of the soil wedge (W), per unit width perpendicular to the plane shown in the figure,

$$W = \frac{1}{2}\gamma h^2 \tan\theta. \tag{33.1}$$

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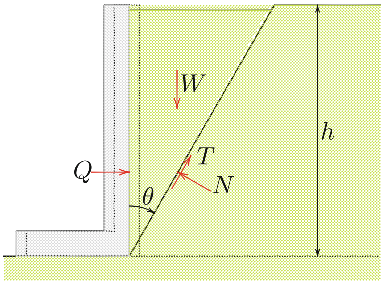

Fig. 33.1 Active earth pressure

This weight must be balanced by the horizontal force Q (horizontal because the wall has been assumed to be perfectly smooth), and the forces N and T on the slip plane. The direction of the shear force T is determined by the assumed sliding direction, with the soil body moving down, in order to follow the motion of the wall to the left. Furthermore, because the length of the slip plane is $h/\cos\theta$,

$$T = \frac{ch}{\cos\theta} + N\tan\phi. \tag{33.2}$$

The equations of equilibrium of the soil body, in horizontal and vertical direction, are

$$Q + T\sin\theta - N\cos\theta = 0, \tag{33.3}$$

$$W - N\sin\theta - T\cos\theta = 0. \tag{33.4}$$

With Eq. (33.2) the shear force T can be eliminated. This gives

$$Q = \frac{N}{\cos\phi}\cos(\theta+\phi) - ch\tan\theta, \tag{33.5}$$

$$W = \frac{N}{\cos\phi}\sin(\theta+\phi) + ch. \tag{33.6}$$

From these two equations the normal force N can be eliminated,

$$Q = W\frac{\cos(\theta+\phi)}{\sin(\theta+\phi)} - ch\frac{\cos\phi}{\cos\theta\sin(\theta+\phi)}. \tag{33.7}$$

With Eq. (33.1) this gives

$$Q = \frac{1}{2}\gamma h^2\frac{\sin\theta\cos(\theta+\phi)}{\cos\theta\sin(\theta+\phi)} - ch\frac{\cos\phi}{\cos\theta\sin(\theta+\phi)}. \tag{33.8}$$

This equation expresses the force Q as a function of the angle θ. The relation is rather complex (the angle θ appears in 6 places), so that it does not seem to be very simple to determine the maximum value. However, the expression can be simplified by using various trigonometric relations, such as $\sin\theta\cos(\theta+\phi) = \cos\theta\sin(\theta+\phi) - \sin\phi$. This gives

$$Q = \frac{1}{2}\gamma h^2 - \frac{\frac{1}{2}\gamma h^2 \sin\phi + ch\cos\phi}{\cos\theta\sin(\theta+\phi)}. \tag{33.9}$$

Now the angle θ appears in 2 places only, in the denominator of the second term. The maximum value of Q can be determined by the maximum value of the function

$$f(\theta) = \cos\theta\sin(\theta+\phi).$$

The maximum of this function occurs when its derivative with respect to θ is zero. Differentiation gives

$$\frac{df}{d\theta} = \cos(2\theta+\phi),$$

and a second differentiation gives

$$\frac{d^2 f}{d\theta^2} = -2\sin(2\theta+\phi).$$

It now follows that $df/d\theta = 0$ if $2\theta + \phi = \frac{1}{2}\pi$, or

$$\frac{df}{d\theta} = 0 : \quad \theta = \frac{1}{4}\pi - \frac{1}{2}\phi. \tag{33.10}$$

Then $d^2 f/d\theta^2 = -2$, so that the function f indeed has a maximum for this value of θ. This means that the horizontal force Q also has a maximum for $\theta = \frac{1}{4}\pi - \frac{1}{2}\phi$. This maximum value is, after some elaboration,

$$\theta = \frac{1}{4}\pi - \frac{1}{2}\phi : \quad Q = \frac{1}{2}\gamma h^2 K_a - 2ch\sqrt{K_a}, \tag{33.11}$$

in which K_a is the coefficient of active earth pressure, defined before,

$$K_a = \frac{1-\sin\phi}{1+\sin\phi}. \tag{33.12}$$

These results are in full agreement with the results obtained in the previous chapter on active earth pressure, see Eq. (32.11). The value for the horizontal force Q corresponds to the sliding of a wedge of soil along a slip plane making an angle $\frac{1}{4}\pi - \frac{1}{2}\phi$ with the vertical direction. These are just the planes shown in Fig. 32.3. In the previous chapter it was found that along these planes the stresses first reach the Mohr–Coulomb envelope. It might be noted that in this analysis possible tension cracks in the soil have been ignored.

Coulomb's method contains a possibly confusing step, in the procedure of *maximizing* the force Q to determine the appropriate value of the angle θ. This might suggest that the procedure gives a high value for Q, whereas in reality the value of Q indicates the smallest possible value of the horizontal force against a retaining wall, as can be seen from Rankine's analysis. The confusion is caused by the assumption in Coulomb's analysis that the soil slides along a slope defined by an angle θ with the vertical, and not along any other plane. For a value of θ other than the critical value $\theta = \frac{1}{4}\pi - \frac{1}{2}\phi$, the force Q may be smaller, but in that case there will be other planes on which the maximum shear stress exceeds Coulomb's maximum $\tau_f = c + \sigma_n \tan\phi$. In the analysis it ought to be investigated whether the assumed slip plane, at an angle θ, is indeed the most critical plane. This is the case only if $\theta = \frac{1}{4}\pi - \frac{1}{2}\phi$, as can be seen from Rankine's analysis. In that analysis the stresses on all planes are considered, by using Mohr's circle. In Coulomb's analysis the stresses on planes other than the assumed sliding plane are not considered at all.

In engineering practice, the horizontal stress against a retaining wall, or a sheet pile wall is often calculated using the active stress coefficient K_a. This may seem on the unsafe side, because it gives the smallest possible value of the horizontal stress, and will occur only in case of failure of the soil. The application is based upon the following argument. It is admitted that the analysis following Rankine or Coulomb, for the active stress state, yields the smallest possible value for the lateral force. In reality the lateral force may be higher, especially if the foundation of the retaining wall is stiff and strong. If the lateral force is so large that the wall's foundation can not withstand that force, it will deform, away from the soil. During that deformation the lateral force will decrease. Eventually this deformation may be so large that the active state of stress is attained. If the foundation and the structure are strong enough to withstand the active state of stress, the deformations will stop as soon as this active state is reached. These deformations may be large, but the structure will not fail. Thus, the structure will be safe if it can withstand active earth pressure, provided that there is no objection to a considerable deformation. For instance, the pile foundation of a quay wall in a harbor can be designed on the basis of active earth pressure against the quay wall, if it is accepted that considerable lateral deformations (say 1% or 2% of the height of the wall) of the quay wall may occur. If this is undesirable, for esthetic reasons or because other structures (the cranes) might be damaged by such large deformations, the foundation must be designed for larger lateral forces. This will mean that many more piles are needed.

33.2 Passive Earth Pressure

For the case of passive earth pressure (i.e. the case of a wall that is being pushed towards the soil mass, by some external cause) Coulomb's procedure is as follows, see Fig. 33.2. Because the wedge of soil in this case is being pushed upwards, the shear force T will be acting in downward direction. The weight of the wedge is, as in the active case,

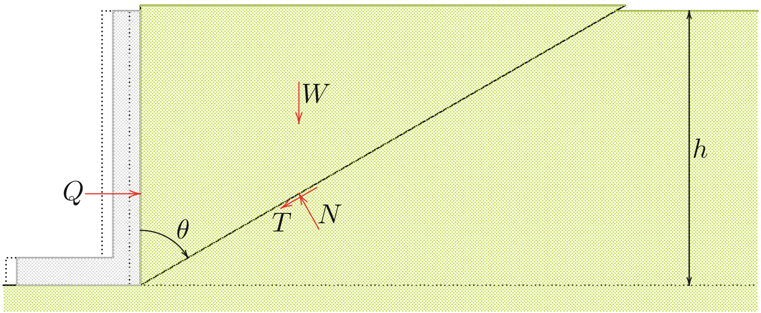

Fig. 33.2 Passive earth pressure

$$W = \frac{1}{2}\gamma h^2 \tan\theta. \tag{33.13}$$

The equations of equilibrium in x- and z-direction now are

$$Q - T\sin\theta - N\cos\theta = 0, \tag{33.14}$$

$$W - N\sin\theta + T\cos\theta = 0. \tag{33.15}$$

After elimination of T and N from the equations, and some trigonometric manipulations, the force Q is found to be

$$Q = \frac{1}{2}\gamma h^2 + \frac{\frac{1}{2}\gamma h^2 \sin\phi + ch\cos\phi}{\cos\theta\sin(\theta-\phi)}. \tag{33.16}$$

Again this force appears to be dependent on the angle θ. The minimum value of Q occurs if the function

$$f(\theta) = \cos\theta\sin(\theta-\phi),$$

has its largest value. Differentiation gives

$$\frac{df}{d\theta} = \cos(2\theta-\phi),$$

and

$$\frac{d^2f}{d\theta^2} = -2\sin(2\theta-\phi).$$

It now follows that $df/d\theta = 0$ if $2\theta - \phi = \frac{1}{2}\pi$, or

$$\frac{df}{d\theta} = 0 : \quad \theta = \frac{1}{4}\pi + \frac{1}{2}\phi. \tag{33.17}$$

Then $d^2 f/d\theta^2 = -2$, and the function f indeed has a maximum for that value of θ. That means that the horizontal force Q has a minimum. This minimum is

$$\theta = \frac{1}{4}\pi + \frac{1}{2}\phi \ : \quad Q = \frac{1}{2}\gamma h^2 K_p + 2ch\sqrt{K_p}, \tag{33.18}$$

where K_p is the passive earth pressure coefficient, defined before,

$$K_p = \frac{1 + \sin\phi}{1 - \sin\phi}. \tag{33.19}$$

Again, the result is in complete agreement with the value obtained in Rankine's analysis. Coulomb's procedure appears to lead to the maximum (passive) earth pressure.

Coulomb's method can easily be extended to more general cases. It is possible, for instance, that the surface of the wall is inclined with respect to the vertical direction, and the soil surface may also be sloping. Also, the soil may carry a given surface load. For all these cases the method can easily be extended. The general procedure is to assume a straight slip plane, consider equilibrium of the sliding wedge, and then maximizing or minimizing the force against the wall. Analytical, graphical and numerical methods have been developed. In the next chapter a number of tables is presented.

Example 33.1 A vertical wall retains a mass of dry sand, of 4 m height. The friction angle of the sand is 30°, and the volumetric weight is 17 kN/m^3. What is the design value of the horizontal force (per meter width) on the wall, if eventual lateral deformations are acceptable?

Solution

It can be assumed that for dry sand $c = 0$. Equation (33.11) then gives $Q = \frac{1}{2}\gamma h^2 K_a$. In this case, with $\phi = 30°$: $K_a = 0.333$. It follows that $Q = 45.3$ kN/m. If the angle of internal friction is 10% larger, $\phi = 33°$ and the value of $K_a = 0.2948$. Then it follows that $Q = 40.1$ kN/m, which is 11.4% smaller.

If $\phi = 30°$ and the client wishes that the wall should not deform under any circumstances, one may consider the passive case, so that $Q = \frac{1}{2}\gamma h^2 K_p = 405$ kN/m. This is the force needed to force the wall into the soil, and it may be assumed that this is larger than the force in the absence of any displacement.

Problem 33.1

In some textbooks the coefficients K_a and K_p are defined as

$$K_a = \tan^2(\frac{1}{4}\pi - \frac{1}{2}\phi), \quad K_p = \tan^2(\frac{1}{4}\pi + \frac{1}{2}\phi).$$

Is that an error?