Chapter 26
Elastic Stresses and Deformations

An important class of soil mechanics problems is the determination of the stresses and deformations in a soil body, by the application of a certain load. The load may be the result of construction of a road, a dyke, or the foundation of a building. The actual load may be the weight of the structure, but it may also consist of the forces due to traffic, wave loads, or the weight of the goods stored in a building. The stresses in the soils must be calculated in order to verify whether these stresses can be withstood by the soil (i.e. whether the stresses remain below the failure criterion), or in order to determine the deformations of the soil, which must remain limited.

26.1 Stresses and Deformations

A three dimensional computation of stresses and deformations in general involves three types of equations : equilibrium, constitutive relations, and compatibility. For soils the main difficulty is that the constitutive relations are rather complicated, and that their accurate description and formulation requires a large number of parameters, which are not so easy to determine, and which must be determined for every soil anew. In principle this should include the non-linear behavior of soils, both in compression and in shear, and possible effects such as time dependence (creep), dilatancy, contractancy and anisotropy. The calculation of the real stresses and deformations in a soil is a well nigh impossible task, for which advanced numerical models are being developed. Such models, usually based upon the finite element method, are applied very often in engineering practice, and it can be expected that their use will be further expanded.

As an introduction into the methods of analysis the problem will be severely schematized here, and will be kept as simple as possible, by assuming that the material is isotropic linear elastic. This means that it is assumed that the relation between stresses and strains is described by Hooke's law. This is a severe approximation,

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but it may still be useful, as it contains all the necessary elements of a continuum analysis. Also, it will appear that in many cases some of the results, in particular the calculation of the vertical stresses, may be reasonably accurate. The idea then is that for the stresses in a soil body caused by the application of a certain load, it is perhaps not so important what the precise properties of the materials are. A complete linear elastic computation at least ensures that equilibrium is satisfied, whereas the field of deformations and displacements satisfies the compatibility equations. For the vertical stresses in a layered soil it probably does not matter so much what the precise stiffness of al the layers is, as long as equilibrium, the geometry, and the distribution of the load have been taken into account. For other quantities, such as the vertical deformations, the stiffness of the layers may be very important, and the results of an elastic analysis may not be so relevant. It will be shown, however, that an analysis on the basis of linear elasticity may still be used as the first step in a reliable computation for the deformations as well.

It may also be mentioned that the applicability of a linear elastic analysis has been verified for several problems by comparison with more complex computations. For instance, the results of computations for anisotropic materials, or layered materials, have been compared with the solutions for the linear elastic approximation. This confirms that the errors in the vertical normal stresses often are very small. On the other hand, the horizontal stresses, and the displacements, are very sensitive to the description of the material properties. It is fortunate that the vertical normal stresses often are the most interesting quantities, and these appear to be least sensitive for the material properties.

A useful procedure is to determine the stresses from a linear elastic analysis, and then, in a next step, to calculate the deformations from these stresses, using the best known relations between stresses and strains. From a theoretical or scientific viewpoint this is not justified, as the compatibility relations are ignored in the second step, and the coupling between stresses and the real deformations is also disregarded, but for engineering it appears to be a powerful and useful method. For instance, for a layered soil the stresses may be calculated assuming that the soil is homogeneous and linearly elastic, completely ignoring the difference in properties of the various layers, and then in a second step the deformations of each layer are calculated using Terzaghi's logarithmic compression formula, or some other realistic formula. The vertical deformations of the layers are finally added to determine the settlement of the soil surface. This procedure will be elaborated in Chap. 30.

26.2 Elasticity

For the analysis of stresses and strains in a homogenous, isotropic linear elastic material various methods have been developed. The general theory can be found in many textbooks on the theory of elasticity. Here, only the basic equations will be given, without giving the details of the derivations. Some of these details, and some derivations of solutions are given in Appendix B.

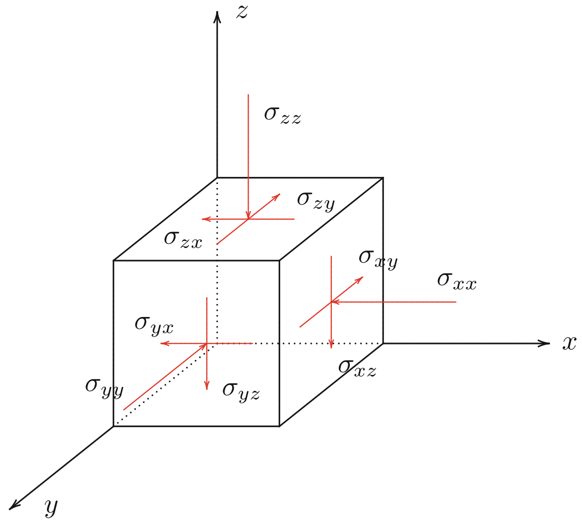

Fig. 26.1 Stresses on element

In this chapter, and in the next chapters, the analysis always concerns the calculation of stresses and deformations caused by some applied load. This means that the stresses and deformations in each case are incremental quantities. The initial stresses should be added in order to determine the actual stresses. It is assumed that these initial stresses also account for the weight of the soil, so that for the analysis of incremental stresses the weight of the soil itself may be disregarded. Thus there will be no body forces due to gravity. For a small element the stresses on the three visible faces are shown in Fig. 26.1.

The equations of equilibrium in the three coordinate directions are

$$\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}&=0,\\ \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}&=0,\\ \frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}&=0.\end{aligned} \tag{26.1}$$

Because of equilibrium of moments the stress tensor must be symmetric,

$$\begin{aligned}\sigma_{xy}&=\sigma_{yx},\\ \sigma_{yz}&=\sigma_{zy},\\ \sigma_{zx}&=\sigma_{xz}.\end{aligned} \tag{26.2}$$

The equations of equilibrium constitute a set of six equations involving nine stress components. In itself this can never be sufficient for a mathematical solution. The deformations must also be considered before a solution can be contemplated.

For a linear elastic material the relation between stresses and strains is given by Hooke's law,

$$\begin{aligned} \varepsilon_{xx} &= -\frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})], \\ \varepsilon_{yy} &= -\frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})], \\ \varepsilon_{zz} &= -\frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})], \end{aligned} \tag{26.3}$$

$$\begin{aligned} \varepsilon_{xy} &= -\frac{1+\nu}{E}\sigma_{xy}, \\ \varepsilon_{yz} &= -\frac{1+\nu}{E}\sigma_{yz}, \\ \varepsilon_{zx} &= -\frac{1+\nu}{E}\sigma_{zx}, \end{aligned} \tag{26.4}$$

where E is the modulus of elasticity (Young's modulus), and ν is Poisson's ratio. The minus sign in the equations has been introduced to account for the unusual combination of sign conventions: stresses are considered positive for compression, and strains are considered positive for extension. The Eqs. (26.3) and (26.4) add six equations to the system, at the same time introducing six additional variables.

The six strains can be related to the three components of the displacement vector,

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x}, \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y}, \\ \varepsilon_{zz} &= \frac{\partial u_z}{\partial z}, \end{aligned} \tag{26.5}$$

$$\begin{aligned} \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \\ \varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \\ \varepsilon_{zx} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right). \end{aligned} \tag{26.6}$$

These are the *compatibility equations*. In total there are now just as many equations as there are variables, so that the system may be solvable, at least if there are a sufficient number of boundary conditions.

For a number of problems solutions of the system of equations can be found in the literature on the theory of elasticity, see for instance the books by Timoshenko and Goodier (1951), Sneddon (1951), Poulos and Davis (1974), and Sadd (2005).

In soil mechanics the solutions for a half space or a half plane, with a horizontal upper surface, are of special interest. The solutions for some of these problems are given in Appendix B. In order to conform to the sign conventions used in the extensive literature the stresses in that appendix are denoted by τ, and the sign convention is that tensile stresses are considered positive. In the next chapters some of the most important solutions for soil mechanics are further discussed, without detailed derivations.

References

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