Chapter 10
Flow Net

Two dimensional groundwater flow through a homogeneous soil can often be described approximately in a relatively simple way by a *flow net*, that is a net of potential lines and stream lines. The principles will be discussed briefly in this chapter.

10.1 Potential and Stream Function

The groundwater potential, or just simply the potential, Φ is defined as

$$\Phi = kh, \tag{10.1}$$

where k is the permeability coefficient (or hydraulic conductivity), and h is the groundwater head. It is assumed that the hydraulic conductivity k is a constant throughout the field. If this is not the case the concept of a potential can not be used. Darcy's law, see (8.1), can now be written as

$$\begin{aligned} q_x &= -\frac{\partial \Phi}{\partial x}, \\ q_z &= -\frac{\partial \Phi}{\partial z}, \end{aligned} \tag{10.2}$$

or, using vector notation,

$$\mathbf{q} = -\nabla \Phi. \tag{10.3}$$

In mathematical physics any quantity whose gradient is a vector field (for example forces or velocities), is often denoted as a *potential*. For that reason in groundwater theory Φ is also called the potential. In some publications the groundwater head h

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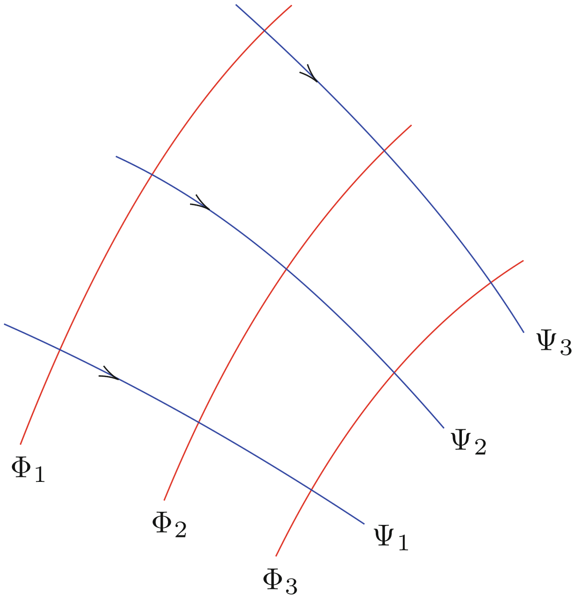


Fig. 10.1 Potential lines and Stream lines

itself is sometimes called the potential, but strictly speaking that is not correct, even though the difference is merely the constant k.

The Eq. (10.2) indicate that no groundwater flow will flow in a direction in which the potential Φ is not changing. This means that in a figure with lines of constant potential (these are denoted as *potential lines*) the flow is everywhere perpendicular to these potential lines, see Fig. 10.1.

The flow can also be described in terms of a *stream function*. This can best be introduced by noting that the flow must always satisfy the equation of continuity, see (8.2), i.e.

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z} = 0. \tag{10.4}$$

This means that a function Ψ must exist such that

$$\begin{aligned} q_x &= -\frac{\partial \Psi}{\partial z}, \\ q_z &= +\frac{\partial \Psi}{\partial x}. \end{aligned} \tag{10.5}$$

By the definition of the components of the specific discharge in this way, as being derived from this function Ψ, the *stream function*, the continuity equation (10.4) is automatically satisfied, as can be verified by substitution of Eqs. (10.5) into (10.4).

It follows from (10.5) that the flow is precisely in x-direction if the value of Ψ is constant in x-direction. This can be checked by noting that the condition $q_z = 0$

can only be satisfied if $\partial\Psi/\partial x = 0$. Similarly, the flow is in z-direction only if Ψ is constant in z-direction, because it follows that $q_x = 0$ if $\partial\Psi/\partial z = 0$. This suggests that in general the stream function Ψ is constant in the direction of flow. Along the stream lines in Fig. 10.1 the value of Ψ is constant. Formally this property can be proved on the basis of the total differential

$$d\Psi = \frac{\partial\Psi}{\partial x}dx + \frac{\partial\Psi}{\partial z}dz = q_z dx - q_x dz. \tag{10.6}$$

This will be zero if $dz/dx = q_z/q_x$, and that means that the direction in which $d\Psi = 0$ is given by $dz/dx = q_z/q_x$, which is precisely the direction of flow. It can be concluded that in a mesh of potential lines and stream lines the value of Ψ is constant along the stream lines.

If the x-direction coincides with the direction of flow, the value of q_z is 0. It then follows from (10.2) to (10.5) that in that case Φ is constant in z-direction, and that Ψ is constant in x-direction. Furthermore, in that case one may write, approximately

$$\frac{\Delta\Phi}{\Delta x} = \frac{\Delta\Psi}{\Delta z}. \tag{10.7}$$

It now follows that if the intervals $\Delta\Phi$ and $\Delta\Psi$ are chosen to be equal, then $\Delta x = \Delta z$, i.e. the potential line and the stream line locally form a small square. That is a general property of the system of potential lines and streamlines (the *flow net*): potential lines and stream lines form a system of "squares".

The physical meaning of $\Delta\Phi$ can be derived immediately from its definition, see Eq. 10.2. If the difference in head between two potential lines, along a stream line, is Δh, then $\Delta\Phi = k\Delta h$. The physical meaning of $\Delta\Psi$ can best be understood by considering a point in which the flow is in x-direction only. In such a point $q = q_x = -\Delta\Psi/\Delta z$, or $\Delta\Psi = -q\,\Delta z$. In general one may write

$$\Delta\Psi = -q\,\Delta n, \tag{10.8}$$

where n denotes the direction perpendicular to the flow direction, with the relative orientation of n and s being the same as for z and x. If the thickness of the plane of flow is denoted by B, the area of the cross section between two stream lines is ΔnB. It now follows that

$$\Delta\Psi = -\Delta Q/B. \tag{10.9}$$

The quantity $\Delta\Psi$ appears to be equal to the discharge per unit thickness being transported between two stream lines. It will appear that this will enable to determine the total discharge through a system.

10.2 Flow Under a Structure

As an example the flow under a structure will be considered, see Fig. 10.2. In this case a sluice has been constructed into the soil. It is assumed that the water level on the left side of the sluice is a distance H higher than the water on the right side. At a certain depth the permeable soil rests on an impermeable layer. To restrict the flow under the sluice a sheet pile wall has been installed on the upstream side of the sluice bottom.

The flow net for a case like this can be determined iteratively. The best procedure is by sketching a small number of stream lines, say 2 or 3, following an imaginary water particle from the upstream boundary to the downstream boundary. These stream lines of course must follow the direction of the constraining boundaries at the top and the bottom of the flow field. The knowledge that the stream lines must everywhere be perpendicular to the potential lines can be used by drawing the stream lines perpendicular to the horizontal potential lines to the left and to the right of the sluice. After sketching a tentative set of stream lines, the potential lines can be sketched, taking care that they must be perpendicular to the stream lines. In this stage the distance between the potential lines should be tried to be taken equal to the distance between the stream lines. In the first trial this will not be successful, at least not everywhere, which means that the original set of stream lines must be modified. This then must be done, perhaps using a new sheet of transparent paper superimposed onto the first sketch. A better set of stream lines can then be sketched such that a better approximation of a net of squares is obtained.

The entire process must be repeated a few times, until finally a satisfactory system of squares is obtained, see Fig. 10.2. Near the corners in the boundaries some special "squares" may be obtained, sometimes having 5 sides. This must be accepted, because the boundary imposes the bend in the boundary. In the case of Fig. 10.2 at the right end of the net one half of a square is left. It turns out that there are 12.5 intervals between potential lines, which means that the interval between two potential lines is

$$\Delta\Phi = \frac{kH}{12.5}. \tag{10.10}$$

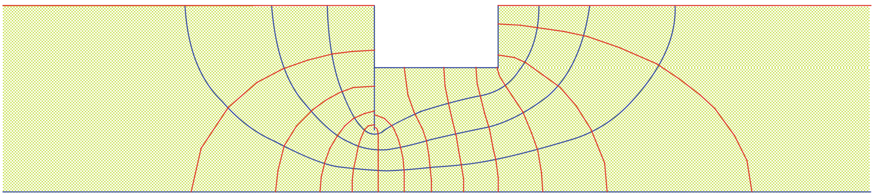


Fig. 10.2 Flow net

Because the flow net consists of squares it follows that $\Delta\Psi = \Delta\Phi$, so that

$$\Delta\Psi = \frac{kH}{12.5}. \tag{10.11}$$

Because there appear to be 4 stream bands, the total discharge now is

$$Q = \frac{4}{12.5}\,kHB = 0.32\,kHB, \tag{10.12}$$

in which B is the width perpendicular to the plane of the figure. The value of the discharge Q must be independent of the number of stream lines that has been chosen, of course. This is indeed the case, as can be verified by repeating the process with 4 interior stream lines rather than 3. It will then be found that the number of potential intervals will be larger, about in the ratio 5 to 4. The ratio of the number of squares in the direction of flow to the number of squares in the direction perpendicular to the flow remains (approximately) constant.

From the completed flow net the groundwater head in every point of the field can be determined. For instance, it can be observed that between the point at the extreme left below the bottom of the sluice and the exit point at the right, about 6 squares can be counted (5 squares and two halves). This means that the groundwater head in that point is

$$h = \frac{6}{12.5}\,H = 0.48\,H, \tag{10.13}$$

if the head is measured with respect to the water level on the right side.

The pore water pressure can be derived if the head is known, as well as the elevation, because $h = z + p/\gamma_w$. The evaluation of the water pressure may be of importance for the structural engineer designing the concrete floor, and for the geotechnical engineer who wishes to know the effective stresses, so that the deformations of the soil can be calculated.

From the flow net the force on the particles can also be determined (the seepage force). According to Eq. (6.16) the seepage force is

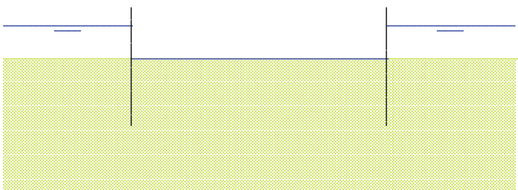
$$\begin{aligned} j_x &= -\gamma_w \frac{\partial h}{\partial x}, \\ j_z &= -\gamma_w \frac{\partial h}{\partial z}. \end{aligned} \tag{10.14}$$

In the case illustrated in Fig. 10.2 it can be observed that at the right hand exit, next to the structure, in the last (half) square $\Delta h = -H/(2 \times 12.5)$ and $\Delta z = 0.3\,d$, if d is the depth of the structure into the ground. Then, approximately, $\partial h/\partial z = -0.133\,H/d$, so that $j_z = 0.133\gamma_w\,H/d$. This is a positive quantity, indicating that the force acts in upward direction, as might be expected. The particles at the soil surface are also acted upon by gravity, which leads to a volume force of magnitude $-(\gamma_s - \gamma_w)$, negative because it is acting in downward direction. It seems tempting to conclude that there is no danger of erosion of the soil particles if the upward force is smaller than the downward force. This would mean, assuming that $\gamma_s/\gamma_w = 2$, so that $(\gamma_s - \gamma_w)/\gamma_w = 1$, that the critical value of H/d would be about 7.5. Only if the value of H/d would be larger than 7.5 erosion of the soil would occur, with the possible loss of stability of the floor foundation at the right hand side.

In reality the danger may be much greater. If the soil is not completely homogeneous, the gradient $\partial h/\partial z$ at the downstream exit may be much larger than the value calculated here. This will be the case if the soil at the downstream side is less permeable than the average. In that case a pressure may build up below the impermeable layer, and the situation may be much more dangerous. On the basis of continuity one might say, very roughly, that the local gradient will vary inversely proportional to the value of the hydraulic conductivity, because $k_1 i_1 = k_2 i_2$. This means that locally the gradient may be much larger than the average value that will be calculated on the basis of a homogeneous average value of the permeability. Locally soil may be eroded, which will then attract more water, and this may lead to further erosion. The phenomenon is called *piping*, because a pipe may be formed, just below the structure. Piping is especially dangerous if a structure is built directly on the soil surface. If the structure of Fig. 10.2 were built on the soil surface, and not into it, the velocities at the downstream side would be even larger (the squares would be very small), with a greater risk of piping.

Prescribing a safe value for the gradient is not so simple. For that reason large safety factors are often used. In the case of vertical outflow, as in Fig. 10.2, a safety factor 2, or even larger, is recommended. In cases with horizontal outflow the safety factor must be taken much larger, because in that case there is no gravity to oppose erosion. In many cases piping has been observed, even though the maximum gradient was only about 0.1, assuming homogeneous conditions. Technical solutions are reasonably simple, although they may be costly. A possible solution is that on the upstream side, or near the upstream side, the resistance to flow is enlarged, for instance by putting a blanket of clay on top of the soil, or into it. Another class of solutions is to apply a drainage at the downstream side, for instance by the installation of a gravel pack near the expected outflow boundary. In the case of Fig. 10.2 a perfect solution would be to make the sheet pile wall longer, so that it reaches into the impermeable layer. A large dam built upon a permeable soil should be protected by an impermeable core or sheet pile wall, *and* a drain at the downstream side. The large costs of these measures are easily justified when compared to the cost of loosing the dam.

Problem 10.1 Sketch a flow net for the situation shown in Fig. 8.7, and calculate the total discharge. Compare the result with the estimate made at the end of that chapter.



Problem 10.2 A building pit in a lake is being constructed, using a sheet pile wall surrounding the building pit. Inside the wall the water level is lowered (by pumping) to the level of the ground surface. Outside the sheet pile wall the water level is 5 m higher. It has been installed to a depth of 10 m below ground surface. The thickness of the soil layer is 20 m. Sketch a flow net, and determine the maximum gradient inside the sheet pile wall.

Problem 10.3 Suppose that in a case as considered in the previous problem the soil consists of 1 m clay on top of a thick layer of homogeneous sand. In that case the capacity of the pumps will be much smaller, which is very favorable. Are there any risks involved?