Chapter 31
Lateral Stresses

In the previous chapters some elastic solutions of soil mechanics problems have been given. It was argued that elastic solutions may provide a reasonable approximation of the vertical stresses in a soil body loaded at its surface by a vertical load. Also, an approximate procedure for the prediction of settlements has been presented. In the next chapters, the analysis of the horizontal stresses will be discussed. This is of particular interest for the forces on a retaining structure, such as a retaining wall or a sheet pile wall.

31.1 Coefficient of Lateral Earth Pressure

As stated before, see Chap. 5, even in the simplest case of a semi-infinite soil body, without surface loading, see Fig. 31.1, it is impossible to determine all stresses caused by the weight of the soil. It seems reasonable to assume that in a homogeneous soil body with a horizontal top surface the shear stresses σ_{zx}, σ_{zy} and σ_{xy} are zero, and it also seems natural to assume that the vertical normal stress σ_{zz} increases linearly with depth, $\sigma_{zz} = \gamma z$. These assumptions ensure that the condition of vertical equilibrium is satisfied. The horizontal stresses σ_{xx} and σ_{yy}, however, can not be determined unequivocally from the equilibrium conditions.

Actually, it can be stated that the stresses must satisfy the equations of equilibrium,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0, \tag{31.1}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0, \tag{31.2}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} - \gamma = 0, \tag{31.3}$$

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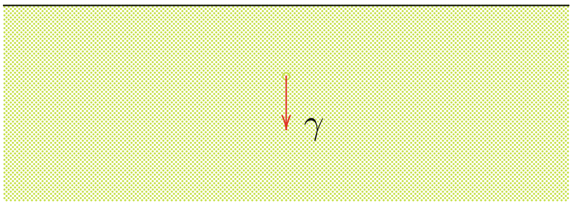

Fig. 31.1 Half space

$$\sigma_{yz} = \sigma_{zy}, \tag{31.4}$$

$$\sigma_{zx} = \sigma_{xz}, \tag{31.5}$$

$$\sigma_{xy} = \sigma_{yx}. \tag{31.6}$$

These equations constitute a set of six conditions for the nine stress components, at every point of the soil body. It seems probable that many solutions of these equations are possible, and it can not be decided, without further analysis, what the best solution is. It seems natural to assume, at least for a homogeneous material, or a material consisting of horizontal layers, that the stress state may be such that the vertical normal stress increases linearly with depth, in proportion to the unit weight of the soil. More precisely, it is assumed that the stresses can be written as

$$\sigma_{zz} = \gamma z, \tag{31.7}$$

$$\sigma_{xx} = \sigma_{yy} = f(z), \tag{31.8}$$

$$\sigma_{yz} = \sigma_{zy} = 0, \tag{31.9}$$

$$\sigma_{zx} = \sigma_{xz} = 0, \tag{31.10}$$

$$\sigma_{xy} = \sigma_{yx} = 0. \tag{31.11}$$

This field of stresses satisfies all the equilibrium conditions, and the boundary conditions on the upper surface of the soil body, i.e. for $z = 0$ the stresses on a horizontal plane are zero, $\sigma_{zz} = 0$, $\sigma_{zx} = 0$, and $\sigma_{zy} = 0$. That all shear stresses in the soil body are zero seems a realistic assumption if all vertical columns of the soil have the same properties. There will probably be no shear stress transfer between these columns.

The function $f(z)$ in Eq. (31.8) remains arbitrary, and in principle the stresses σ_{xx} and σ_{yy} need not be equal. It has been assumed that the horizontal stress in any horizontal plane is the same in all directions, so that there are no preferential directions in the horizontal plane. Theoretically speaking, the function $f(z)$, which describes the horizontal stresses, need not be continuous. Discontinuities in this function are allowed, and may occur especially if there are discontinuities in the soil properties. It may be remarked that even the expressions for the vertical normal stress σ_{zz} and for the shear stresses do not follow necessarily from the equilibrium conditions. It may well be that these stresses depend upon x and y, if the soil stiffness is not constant in horizontal planes. In case of a very soft inclusion in a rather stiff soil body, the stresses may be concentrated in a region around the soft inclusion. This is called *arching*, as the stiffer soil may form a certain arch to transmit the load from upper layers to the subsoil. In homogeneous soil, however, or in soils without large differences in stiffness, the stress distribution given above can be considered as a reasonable approximation. Such a soil body has often been created, in its geological history, by gradual sedimentation, often under water. In such conditions the gradual increase of the thickness of the soil body will normally lead to a stress state of the form given above.

The stress state described by Eqs. (31.7)–(31.11) can be made somewhat more practical by writing $f(z) = K\sigma_{zz}$, where K is an unknown coefficient, that may depend upon the vertical coordinate z. The horizontal stresses then are

$$\sigma_{xx} = \sigma_{yy} = K\sigma_{zz} = K\gamma z, \tag{31.12}$$

where K is the *coefficient of lateral earth pressure*. It gives the ratio of the lateral normal (effective) stress to the vertical (effective) stress. Theoretically speaking, the problem has not be cleared, because the value of K is still unknown, but it seems to make sense to assume that the horizontal stresses will also increase with depth, if the vertical stresses do so. Thus, it can be assumed that the coefficient K will not vary too much, at least compared to the original function $f(z)$.

It may be mentioned that for historical reasons the coefficient K is denoted as a coefficient of *earth* pressure, in agreement with most soil mechanics literature. This is one of the few instances where the word earth is used in soil mechanics, rather than the word soil, or ground. No special meaning should be attached to this terminology. In this book the coefficient will sometimes also be denoted as the horizontal (or lateral) stress coefficient.

The value of the lateral earth pressure coefficient K depends upon the material, and also on the geological history of the soil. In this chapter some examples of possible values, or the possible range of values, will be given, for certain simple materials. It will appear to be illustrative to describe the relations between vertical and horizontal stresses in a stress path. In Chap. 26 the quantities σ and τ have been introduced for that purpose, being the location of the center and the radius of Mohr's circle. In this case these quantities are

$$\sigma = \frac{1}{2}(\sigma_{zz} + \sigma_{xx}), \tag{31.13}$$

$$\tau = \frac{1}{2}|\sigma_{zz} - \sigma_{xx}|. \tag{31.14}$$

It now follows, with (31.12), and assuming that $K \leq 1$,

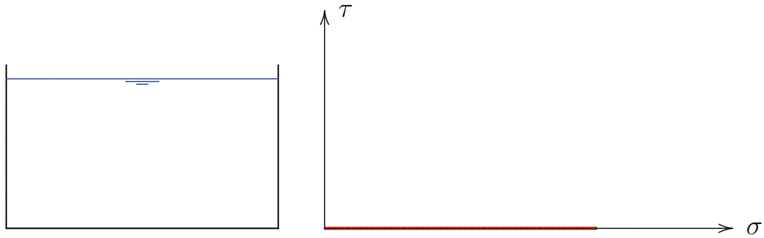

Fig. 31.2 Stress path for a fluid

$$\frac{\tau}{\sigma} = \frac{1-K}{1+K}. \tag{31.15}$$

Often the horizontal stress will indeed be smaller than the vertical stress, so that $K < 1$, but this is not absolutely necessary.

31.2 Fluid

In a fluid the shear stresses can be neglected, compared to the pressure. This means that the normal stress is equal in all directions. This means that

$$K = 1. \tag{31.16}$$

If $K = 1$ the horizontal stress is equal to the vertical stress. With (31.15) this gives

$$\frac{\tau}{\sigma} = 0. \tag{31.17}$$

The stress path is shown in Fig. 31.2. This stress path refers to the case that a container is gradually filled with water. It would also apply if gravity would gradually develop in a fluid.

Soil is not a fluid, but certain very soft soils come close: the mud collected by dredging often is similar to a thick fluid. Very soft clay, with a high water content, also behaves similar to a fluid. When spread out it will flow until an almost horizontal surface has been formed. For such soils the value of K will be close to 1, and the stress path of Fig. 31.2 is realistic.

31.3 Elastic Material

A possible approach to the behavior of soils is to consider it as an elastic material. In such a material the stresses and strains satisfy Hooke's law. In a situation in which

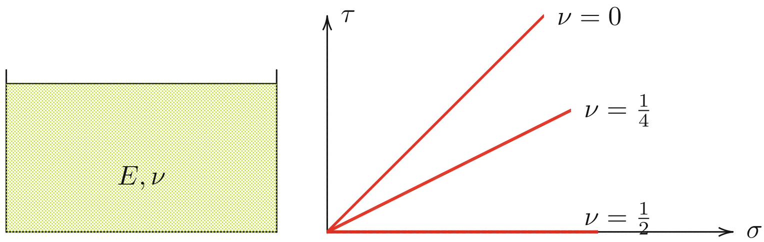

Fig. 31.3 Stress path for elastic material

there can be no lateral deformation, the stresses must satisfy the condition

$$\varepsilon_{xx} = -\frac{1}{E}\big[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\big] = 0,$$

$$\varepsilon_{yy} = -\frac{1}{E}\big[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})\big] = 0,$$

if the z-direction is vertical. In a medium of large horizontal extent it can be expected that $\sigma_{xx} = \sigma_{yy}$. Then

$$\varepsilon_{xx} = \varepsilon_{yy} = 0 \; : \quad \sigma_{xx} = \sigma_{yy} = \frac{\nu}{1-\nu}\sigma_{zz}, \tag{31.18}$$

or

$$K = \frac{\nu}{1-\nu}. \tag{31.19}$$

If Poisson's ratio varies between 0 and 0.5, the value of K varies from 0 to 1.

It follows from (31.15) and (31.19) that in this case

$$\frac{\tau}{\sigma} = 1 - 2\nu. \tag{31.20}$$

For a number of values of Poisson's ratio ν, between 0 and $\frac{1}{2}$, the stress path is shown in Fig. 31.3. If $\nu = \frac{1}{2}$ the horizontal stresses are equal to the vertical stresses. In that case there are no volume changes, just as in a fluid. The stress path then is equal to the stress path in a fluid. If $\nu = 0$ the stress path has a slope of 45°.

If the horizontal strains are not zero, but it is still assumed that the two horizontal stresses, σ_{xx} and σ_{yy}, are equal, these stresses are

$$\sigma_{xx} = \sigma_{yy} = \frac{\nu}{1-\nu}\sigma_{zz} - \frac{E}{1-\nu}\varepsilon_{xx}. \tag{31.21}$$

The first term is the value obtained in Eq. (31.18), when there are no lateral strains, and the second term describes the influence of the lateral strain. In case of a positive horizontal strain, the horizontal stress decreases, and then K is getting smaller. A negative horizontal strain, for instance due to some lateral compression, will result in a larger horizontal stress. The value of K then will seem to increase. These are general tendencies, with a validity beyond elasticity.

In some older publications Eq. (31.19) has been proposed as a generally applicable relation for soil and rock. That is not true. In the elastic analysis given above it is assumed that the stresses are being developed gradually, by gravity being applied gradually, on an existing soil in an unstressed state. And during this entire process the relation between stress and strain should be linear, and no horizontal deformations should occur. Geological history usually is much more complex, and the material behavior is non-linear. This means that the value of the lateral stress coefficient K in general can not be predicted with any accuracy. It can be expected that in a region between two deep rivers the value of K will be relatively small, whereas in a valley between two mountain ridges that are moving towards each other due to tectonic motion, the stress coefficient K will be relatively large.

31.4 Elastic Material Under Water

In order to take groundwater into account, the soil may be schematized as a linear elastic material, that is being deposited under water, see Fig. 31.4. If the weight of the material is again carried by the vertical stresses, the vertical total stress will increase linearly with depth,

$$\sigma_{zz} = \gamma z, \tag{31.22}$$

in which γ is the total volumetric weight of the soil, including the water in the pores. The pore pressures are assumed to be hydrostatic,

$$p = \gamma_w z, \tag{31.23}$$

so that the vertical effective stresses are

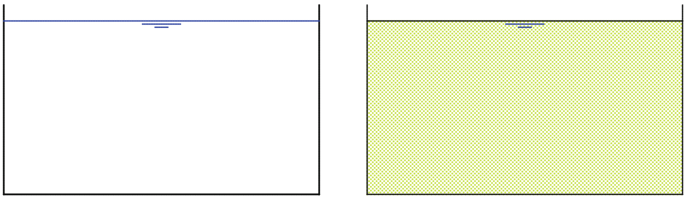

Fig. 31.4 Elastic material under water

$$\sigma'_{zz} = \sigma_{zz} - p = (\gamma - \gamma_w)z. \tag{31.24}$$

It is again assumed that in the process of the development of these stresses during deposition of the soil no horizontal deformations of the soil skeleton occur. The deformation of this soil skeleton is determined by the effective stresses, and in this case, for a linear elastic material, it follows that

$$\sigma'_{xx} = \sigma'_{yy} = \frac{\nu}{1-\nu}\sigma'_{zz} = \frac{\nu}{1-\nu}\big(\gamma - \gamma_w\big)z. \tag{31.25}$$

This means that

$$K' = \frac{\nu}{1-\nu}, \tag{31.26}$$

where the symbol K' indicates the lateral stress coefficient for the effective stresses.

The horizontal total stress now is

$$\sigma_{xx} = \sigma'_{xx} + p = K'(\gamma - \gamma_w)z + \gamma_w z. \tag{31.27}$$

This could be written as

$$\sigma_{xx} = K\sigma_{zz}, \tag{31.28}$$

where then

$$K = K' - \big(1 - K'\big)\frac{\gamma_w}{\gamma}. \tag{31.29}$$

It should be noted that this relation is valid only under very special conditions. The derivation assumes that the groundwater table coincides with the soil surface, and that the soil is homogeneous in depth. Actually, it seems that a lateral stress coefficient should be used preferably for the effective stresses only. The horizontal total stresses should be determined afterwards by adding the pore pressure to the horizontal effective stress.

Example 31.1 The effective stress path (ESP) for the case of an elastic material under water can be constructed by noting that in this case

$$\sigma' = \frac{1}{2}\big(\sigma'_{zz} + \sigma'_{xx}\big) = \frac{1}{2}\big(1 + K'\big)\big(\gamma - \gamma_w\big)z,$$

$$\tau' = \frac{1}{2}\big(\sigma'_{zz} - \sigma'_{xx}\big) = \frac{1}{2}\big(1 - K'\big)\big(\gamma - \gamma_w\big)z,$$

so that

$$\frac{\tau'}{\sigma'} = \frac{1-K'}{1+K'} = 1 - 2\nu.$$

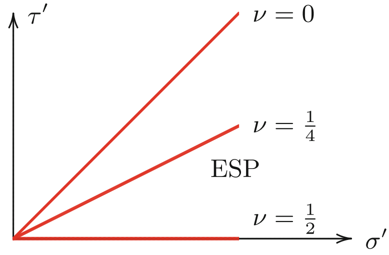

Fig. 31.5 Elastic material under water

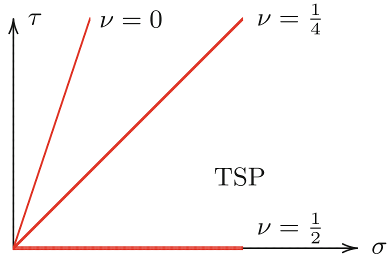

Fig. 31.6 Elastic material under water

For three values of ν the effective stress path is shown in Fig. 31.5. It may be noted that the shape of these paths is the same as in the case of a dry elastic material, see Fig. 31.3. There is a difference in scale, however, because the effective weight under water is smaller.

The total stress path (TSP) for the case of an elastic material under water can be constructed by using the expressions (31.22) and (31.27). The parameters for the total stress path are

$$\sigma = \frac{1}{2}(\sigma_{zz} + \sigma_{xx}) = \frac{1}{2}(1 + K)\gamma z,$$

$$\tau = \frac{1}{2}(\sigma_{zz} - \sigma_{xx}) = \frac{1}{2}(1 - K)\gamma z,$$

so that

$$\frac{\tau}{\sigma} = \frac{1 - K}{1 + K}.$$

Using the expressions (31.29) and (31.26) for the parameters K and K', and assuming that $\gamma_w/\gamma = \frac{1}{2}$, this can be written as

$$\frac{\tau}{\sigma} = 3\,\frac{1 - 2\nu}{1 + 2\nu}. \tag{31.30}$$

For three values of ν the total stress path is shown in Fig. 31.6.