Chapter 38
Limit Analysis

Coulomb's method for the analysis of soil pressures considers extreme conditions, in which the soil is on the verge of failure. This type of analysis can be given a firm theoretical basis by the *theory of plasticity*. This also enables to generalize the method, and to investigate the possible limitations and the validity of the method.

38.1 Plasticity

In the analysis of stresses and strains in continuum mechanics three types of equations are needed: equilibrium conditions, constitutive relations, and compatibility equations. The general purpose is to determine the stresses and strains in a certain body, under the influence of given stresses and displacements on the surface of that body. Even for the simplest type of material, a linear elastic body, for which the constitutive relations are linear relations between stresses and strains (Hooke's law), this is a formidable task, which can be solved only for simple cases, such as a half space, a perfect sphere or a cylindrical body. Approximate solutions may be found for various materials, including linear elastic materials, using advanced numerical methods, such as the finite element method. Such numerical methods will not be considered in this book, however. An alternative may be formed by *limit analysis*, on the basis of plasticity theory, which aims not to give the complete field of actual stresses and deformations, but is restricted to give a possible upper or lower limit of the stresses or the deformations.

A. Verruijt, *An Introduction to Soil Mechanics*, Theory and Applications of Transport in Porous Media 30, DOI 10.1007/978-3-319-61185-3_38

38.2 Basic Theorems of Plasticity Theory

In considerations of limit analysis not all the details of the constitutive relations are taken into account, but one aspect is given priority, namely the failure criterion of the material. For soils this may be the Mohr–Coulomb criterion, described by a cohesion c and a friction angle ϕ. Also, not all the conditions of equilibrium and compatibility equations are taken into account, but only a subset of these equations. The purpose of limit analysis is not to determine the complete field of actual stresses and strains, but only to determine certain limiting values. The problem may be to determine a *lower bound* for the maximum allowable load on a soil body, or to determine an *upper bound* for this maximum load. If a lower bound for the failure load can be found, it is certain that no failure will occur as long as the real load remains below this lower bound. If an upper bound can be found it is certain that failure will occur if the real load is greater than this upper bound.

In its simplest form the theory of plasticity uses a single constant failure condition, which is a function of the stresses only. This condition expresses that for certain combinations of stresses in a point of the material the deformations increase without bounds (this is called *plastic yielding*), and that for smaller stresses no plastic deformations will occur. A material with such a simple yield condition is called a *perfectly plastic* material. For soils a suitable yield condition is the Mohr–Coulomb criterion, although more complex yield conditions have also been studied.

In formulating the basic theorems of the theory of plasticity two types of fields are being used, which can be defined as follows.

1. An *equilibrium system*, or a *statically admissible field* of stresses is a distribution of stresses that satisfies the following conditions:

   a. it satisfies the conditions of equilibrium in each point of the body,
   b. it satisfies the boundary conditions for the stresses,
   c. the yield condition is not exceeded in any point of the body.

2. A *mechanism*, or a *kinematically admissible field* of displacements is a distribution of displacements and deformations that satisfies the following conditions:

   a. the displacement field is compatible, i.e. no gaps or overlaps are produced in the body (sliding of one part along another part is allowed),
   b. it satisfies the boundary conditions for the displacements,
   c. wherever deformations occur the stresses satisfy the yield condition.

The basic theorems of the plasticity theory are,

1. *Lower bound theorem*. The true failure load is larger than the load corresponding to an equilibrium system.
2. *Upper bound theorem*. The true failure load is smaller than the load corresponding to a mechanism, if that load is determined using the virtual work principle.

The first theorem states that if for a certain load an equilibrium system can be found (ignoring compatibility), then that load can certainly be carried. The second theorem states that if a mechanism can be found corresponding to a certain load (where equilibrium is taken into account only insofar as it corresponds to the chosen deformation), then this load can certainly *not* be carried.

It may be noted that in these theorems and in the definition of the statically or kinematically admissible fields, the constitutive relations are not mentioned, and therefore they play no role, except for the statement that the material will yield if the stresses satisfy the yield condition.

A proof of the two theorems is given in Appendix C. When studying these proofs it will appear that they have only a limited validity. The most important restriction is that for a material with friction, such as a soil, for which the yield condition is the Mohr–Coulomb criterion, with a cohesion c and a friction angle ϕ, the theorems are valid only if during plastic deformation a continuing volume expansion occurs, of magnitude $\sin\phi$ times the rate of shear deformation. That seems to be an unrealistic behavior, as it can be expected that in the case of continuing plastic deformations the volume will remain practically constant. This is also what has often been confirmed in experimental studies. An ever continuing plastic volume expansion would mean that the material expands without bounds, and that seems to be improbable. This means that the basic theorems of plasticity are not valid for soils, except for $\phi = 0$, i.e. for purely cohesive materials. For such a material the theory predicts that the volume is constant during plastic deformations, and that is in agreement with experimental evidence.

Because for $\phi = 0$ the theorems are valid, it follows that for such a material *safe* and *unsafe* predictions of the behavior of a soil body can be made. For rapid loadings of saturated clays it can indeed be assumed that $\phi = 0$ and $c = s_u$, the undrained shear strength, see Chap. 24. For sands, for which it is essential that the friction angle $\phi > 0$, the theorems are not valid, at least in principle. In engineering practice they may nevertheless be used, often in a somewhat modified form. Great care should be taken in formulating conclusions from limit analysis for sands.

Actually, the limit theorems have already been used in the Chaps. 32 and 33. Rankine's considerations, see Chap. 32, are based upon equilibrium systems, choosing the horizontal stress such that the limit of yielding is reached. This means that the failure load is approached from below. In the analysis following Coulomb, see Chap. 33, the basis is a kinematic system, with sliding along a straight slip plane. Then the failure load is approached from above.

In the next chapters limiting states will be considered for a variety of structures, using limit analysis. These include the bearing capacity of a shallow footing, and the stability of slopes.