Chapter 17
Numerical Solution

The dissipation of the pore water pressures during the consolidation process can be calculated very simply by a numerical solution procedure, using the finite difference method. This is presented in this chapter, keeping the method as simple as possible. Many more advanced, and more powerful numerical methods have been developed. They can be found on the internet.

17.1 Finite Differences

The differential equation for one dimensional consolidation is Eq. (15.17),

$$\frac{\partial p}{\partial t} = c_v \frac{\partial^2 p}{\partial z^2}. \tag{17.1}$$

The time derivative can be approximated by

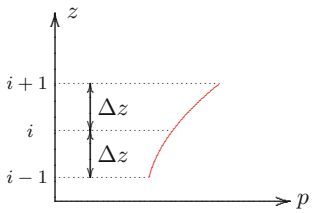
$$\frac{\partial p}{\partial t} \approx \frac{p_i(t + \Delta t) - p_i(t)}{\Delta t}, \tag{17.2}$$

where the index i indicates that the values refer to the pressures in the point $z = z_i$. Equation (17.2) can be considered as the definition of the partial derivative $\partial p/\partial t$, except that the limit $t \to 0$ has been omitted. Finite differences will also be used in the z-direction. For this purpose the thickness h of the sample is subdivided into n small elements of thickness Δz,

$$\Delta z = \frac{h}{n}. \tag{17.3}$$

A. Verruijt, *An Introduction to Soil Mechanics*, Theory and Applications of Transport in Porous Media 30, DOI 10.1007/978-3-319-61185-3_17

Fig. 17.1 Second derivative



The second derivative with respect to z can be approximated by

$$\frac{\partial^2 p}{\partial z^2} \approx \frac{p_{i+1}(t) - 2p_i(t) + p_{i-1}(t)}{(\Delta z)^2}. \tag{17.4}$$

This relation is illustrated in Fig. 17.1. The formula can most simply be obtained by noting that the second derivative is the derivative of the first derivative. This means that the second derivative is the difference of the slope in the upper part of the figure and the slope in the lower part of the figure, divided by the distance Δz. It can also be verified from the figure that for a straight line the expression (17.4) indeed gives a value zero, because then the value in the center is just the average of the values at the two values above it and below it.

Substitution of (17.2) and (17.4) into (17.1) gives

$$p_i(t + \Delta t) = p_i(t) + \alpha\big\{p_{i+1}(t) - 2p_i(t) + p_{i-1}(t)\big\}, \tag{17.5}$$

where

$$\alpha = c_v \frac{\Delta t}{(\Delta z)^2}. \tag{17.6}$$

The expression (17.5) is an explicit formula for the new value of the pore pressure in the point i, if the old values (at time t) in that point and in the two points just above and just below it are known.

The boundary conditions must also be represented in a numerical way. For the boundary condition at the upper boundary, where the pressure p must be zero, see (15.18), this is very simple,

$$p_n = 0. \tag{17.7}$$

The boundary condition at the bottom of the sample is that for $z = 0$ the derivative $\partial p/\partial z = 0$, see (15.19). That can best be approximated by continuing the numerical subdivision by one more interval below $z = 0$, so that in a point at a distance Δz below the lower boundary a value of the pore pressure is defined, say p_{-1}. By requiring that $p_{-1} = p_1$, whatever the value of p_0 is, the condition $\partial p/\partial z = 0$ is satisfied at the symmetry axis $z = 0$. This means that the numerical equivalent of the boundary condition at $z = 0$ is

$$p_{-1} = p_1. \tag{17.8}$$

The general algorithm (17.5) for the point $i = 0$ can now be written as

$$p_0(t + \Delta t) = p_0(t) + \alpha\big\{2p_1(t) - 2p_0(t)\big\}. \tag{17.9}$$

The two boundary conditions (17.7) and (17.9), which are valid at all values of time, complete the algorithm (17.5), together with the initial conditions

$$t = 0 \ : \quad p_i = p_0, \ i = 0, 1, 2, \ldots n - 1, \qquad p_n = 0. \tag{17.10}$$

At the initial time $t = 0$ all values are known : all values of the pressure are p_0, except the one at the top, where the pressure is zero. The new values, after a time step Δt, can be calculated using the algorithm (17.5). This can be applied for all values of i in the interval $0, 1, 2, \ldots n - 1$. At the top, for $i = n$, the value of the pressure remains zero.

A simple computer program, in Turbo Pascal, is shown below as the Program NUMCONS.PAS.

```
program numcons;
uses crt;
const nn=20;
var
 z,p,pa:array[0..nn] of real;tt:array[0..8] of real;
 i,j,k,nt:integer;a,alpha,t,dt,step,h,dz,cv,tc:real;d:char;
procedure title;
 begin
  clrscr;gotoxy(36,1);textbackground(7);textcolor(0);write(' NUMCONS ');
  textbackground(0);textcolor(7);writeln;
 end;
procedure next;
 begin
  gotoxy(25,25);textbackground(7);textcolor(0);
  write(' Touch any key to continue ');write(chr(8));
  d:=readkey;textbackground(0);textcolor(7)
 end;
begin
 title;writeln;writeln;
 writeln('This is a program for the analysis of one-dimensional');
 writeln('consolidation of a homogeneous soil layer by the finite');
 writeln('difference method.');next;
 h:=10;cv:=0.1;alpha:=0.2;t:=0.0;
 dz:=h/nn;for i:=0 to nn do z[i]:=i*dz/h;
 p[0]:=0;p[nn]:=0;for i:=1 to nn-1 do p[i]:=1.0;
 tt[0]:=0.0;tt[1]:=0.01;tt[2]:=0.02;tt[3]:=0.05;tt[4]:=0.1;
 tt[5]:=0.2;tt[6]:=0.5;tt[7]:=1.0;tt[8]:=2.0;
 for k:=1 to 8 do
  begin
   dt:=tt[k]-tt[k-1];
   nt:=round(int(h*h*dt/(alpha*dz*dz)));if nt<1 then nt:=1;
   a:=h*h*dt/(nt*dz*dz);
   for j:=1 to nt do
    begin
     t:=t+dt/nt;tc:=cv*t/(h*h);
     for i:=1 to nn-1 do pa[i]:=p[i]+a*(p[i-1]-2*p[i]+p[i+1]);
```

```
      for i:=1 to nn-1 do p[i]:=pa[i];
     end;
    title;writeln(' cv*t/h^2 = ',t:6:3,', steps = ',nt);writeln;
    for i:=0 to nn do writeln(' z/h = ',z[i]:6:3,', p/p0 = ',p[i]:6:3);
    next;
   end;
  clrscr;
end.
```

Program NUMCONS.PAS

The numerical results are compared with the analytical results in Fig. 17.2. The values of the dimensionless time $c_v t/h^2$ for which the pore pressures are shown, are the same as those used in Fig. 16.2. It appears that the numerical data agree very well with the analytical results. The accuracy of the numerical solution, and its simplicity, may serve to explain the popularity of the numerical method.

The numerical data have been calculated by subdividing the height h in 20 equal parts, $\Delta z = h/20$. The value of α has been chosen as $\alpha = 0.2$. This means that $\Delta t = 0.0005\, h^2/c_v$. It turns out that in that case about 2/0.0005 = 10000 time steps are needed to complete the entire consolidation process, until the pore pressures have been reduced to practically zero (for $c_v t/h^2 = 2$), but even this many time steps are executed very quickly on a computer.

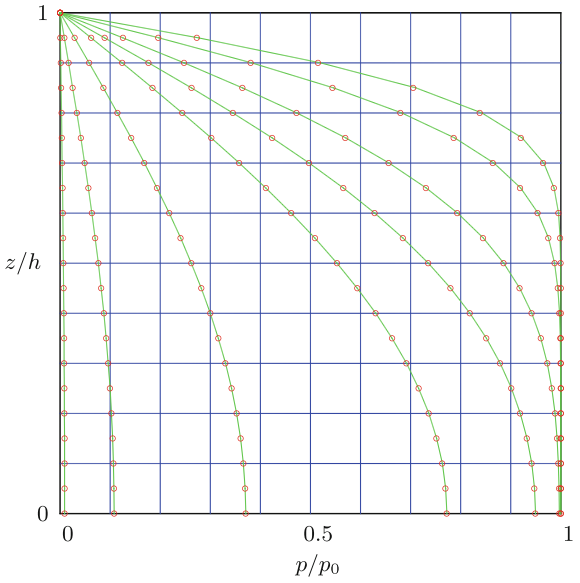


Fig. 17.2 Numerical—analytical

17.2 Numerical Stability

The stability of the numerical process can be investigated by calculating the development of a small error in the numerical process. For this purpose it may be assumed that near the end of the consolidation process, when all pore pressures should be zero, some errors remain, with $p_i(t) = \epsilon$ and $p_{i+1}(t) = p_{i-1}(t) = -\epsilon$. The algorithm (17.5) then gives $p_i(t + \Delta t) = (1 - 4\alpha)\epsilon$. The error will decrease if the new value is smaller than the old one, in absolute value. This will be the case if

$$| 1 - 4\alpha | < 1. \tag{17.11}$$

This means that

$$0 < \alpha < \frac{1}{2}. \tag{17.12}$$

Of course, all distributions of errors should gradually be reduced to zero, and it is not certain that the requirement (17.12) is sufficient for stability. However, more fundamental investigations show that the criterion (17.12) is sufficient to guarantee that all possible distributions of errors will eventually be reduced to zero.

The criterion (17.12) means that the algorithm used in this chapter is stable only if the time step is positive (that seems to be self-evident), and not too large,

$$\Delta t < \frac{1}{2} \frac{(\Delta z)^2}{c_v}. \tag{17.13}$$

To satisfy this criterion the value of the factor α in the program NUMCONS.PAS has been taken as 0.20. It is a simple matter to modify the program, and take a somewhat larger value, larger than $\frac{1}{2}$. It will then appear immediately that the process is unstable. The pore pressures will become larger and larger, alternating between negative and positive values. If the time step is chosen such that the criterion (17.13) is satisfied, the numerical process is always stable, as can be verified by running the program with different values of the time step. The numerical results are always very accurate as well, provided that he stability criterion (17.13) is satisfied.

17.3 Numerical Versus Analytical Solution

As may be evident from this chapter and the previous one, the numerical solution method is simpler than the analytical solution, and much easier to use. It may be added that the numerical solution method can easier be generalized than the analytical method. It is, for instance, rather simple to develop a numerical solution for the consolidation of a layered soil, with different values for the permeability and the compressibility in the various layers. The analytical solution for such a layered

system can also be constructed, at least in principle, but this is a reasonably complex mathematical exercise.

In general an analytical solution has the advantage that it may give a good insight in the character of the solution. For instance, the analytical solution of the consolidation problem indicates that its progress is governed by the parameter $c_v t/h^2$, which enables to compare a field situation with a laboratory test on the same material. Such insight can also be obtained directly from the differential equation and the boundary and initial conditions, however, even in the absence of a solution of the problem. This can be illustrated as follows.

The basic equations of the consolidation problem can be made dimensionless by introducing a dimensionless vertical coordinate $Z = z/h$ and a dimensionless pore pressure $P = p/p_0$. As the time dimension only appears in the consolidation coefficient c_v, this means that the time t can only be made dimensionless by the introduction of a parameter $T = c_v t/h^2$. The problem then is, in dimensionless form,

$$\frac{\partial P}{\partial T} = \frac{\partial^2 P}{\partial Z^2}, \tag{17.14}$$

with the initial condition

$$T = 0 : \quad P = 1, \tag{17.15}$$

and the boundary conditions

$$Z = 0 : \quad \frac{\partial P}{\partial Z} = 0. \tag{17.16}$$

$$Z = 1 : \quad P = 0, \tag{17.17}$$

The material property c_v and the size h have now been eliminated from the mathematical problem, and the only numerical values in the problem are the numbers 0 and 1. Both Z and P are of the order of magnitude of 1. This will then probably also hold for T, and it can be expected that the process will be finished when $T \gg 1$. This means that it can be stated that the process will be governed by the factor $T = c_v t/h^2$, as was indeed found in the analytical solution in the previous chapter. The additional information from the analytical solution is that it indicates that the consolidation process will be practically finished when $T \approx 2$, and that can not be concluded from the basic equations only.

The fact that the behavior in time of the consolidation process is determined by the parameter $c_v t/h^2$ means that it can also be predicted that any loading in a time span Δt can be considered as *rapid* when the value of $c_v \Delta t/h^2$ is small compared to 1, say about 0.0001 or smaller. That was concluded also in the previous chapter from the analytical solution, see Eq. (16.25), but it can also be concluded from the formulation of the problem in dimensionless form, without knowing the solution.

The numerical solution presented in this chapter appears to be stable only if a certain stability criterion is satisfied. It may be mentioned that there exist other

numerical procedures that are unconditionally stable. By using a different type of finite differences, such as a backward finite difference or a central finite difference for the time derivative, a stable process is obtained. The numerical procedures then are somewhat more complicated, however. Another effective method is to use a formulation by finite elements. This also makes it very easy to include variable soil properties, which is rather complicated when using a finite difference method.

It may finally be mentioned that many universities and consulting companies have developed numerical methods for two-dimensional or three-dimensional problems of consolidation. There are many possibilities for a further study of consolidation theory, or of numerical methods.

Example 17.1 The consolidation process of a clay layer of 4 m thickness is solved by a numerical procedure. The consolidation coefficient of the clay is $c_v = 10^{-6}$ m^2/s. The layer is subdivided into 20 small layers. What is the maximum allowable magnitude of the time step?

Solution

In this case of a subdivision of the layer into 20 parts the thickness of each sub-layer is $\Delta z = h/n = 0.20$ m. The stability criterion (17.13) then gives $\Delta t < 20 \times 10^3$ s.

It may be noted that the duration of the total consolidation process is given by Eq. (16.23), which expresses that $t_{99\,\%} = 2h^2/c_v$. With Eq. (17.13) it then follows that $t_{99\,\%} = 4n^2\,\Delta t$, indicating that about $4n^2$ time steps must be calculated for the numerical process to be finished.

To make a more accurate analysis of the example the number of sub-layers may be taken twice as large, say $n = 40$. The time step then must be 4 times as small, because of the factor $(\Delta z)^2$ in the stability criterion. The price to be paid for this increased accuracy is that 4 times as many time steps must be calculated. As in each time step the number of calculations is also twice as large, the total computation time will be 8 times as large. This may still be very small, however.

Problem 17.1 Execute the calculations described above, using the program NUMCONS.PAS or some other program, for instance using Excel, and investigate the influence of the value of the parameter α, say $\alpha = 0.25$ and $\alpha = 1.00$.