Chapter 40
Prandtl

In this section a solution is given for the problem of a strip load on a half plane that is both statically admissible and kinematically admissible. This solution must therefore give the true failure load. The solution was found by the German scientist Ludwig Prandtl in (1920).

40.1 Prandtl's Solution

The lower bound part of Prandtl's solution, with an equilibrium system of stresses, will be presented in this chapter. The proof that this solution is also kinematically admissible, which is much more difficult, will be omitted here. A complete proof can be found in textbooks on the theory of plasticity. As in the previous chapter, the material is considered to be weightless ($\gamma = 0$), and frictionless ($\phi = 0$), so that its only relevant property is the cohesive strength c. That is a great restriction, but it will be relaxed in later chapters.

The stresses will be formulated using polar coordinates. In order to verify the equilibrium conditions, these must be expressed into polar coordinates first.

40.2 Equilibrium Equations in Polar Coordinates

Figure 40.1 shows a small element of a two-dimensional region, in polar coordinates r and θ, with all the stresses acting upon it. Equilibrium in r-direction requires that

$$\sigma^*_{rr}(r + \Delta r)\Delta\theta - \sigma_{rr} r\Delta\theta + \sigma^*_{\theta r}\Delta r - \sigma_{\theta r}\Delta r - \sigma_{\theta\theta}\Delta r\Delta\theta = 0.$$

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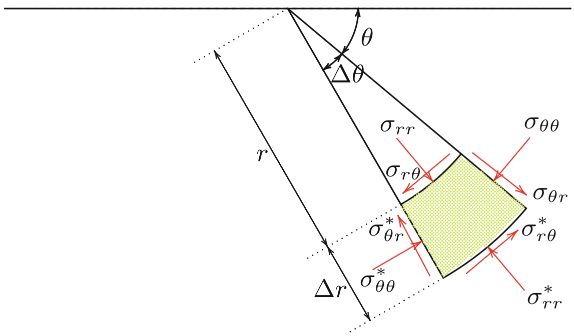
Fig. 40.1 Polar coordinates



The last term in this equation is needed because the forces $\sigma_{\theta\theta}\Delta r$ and $\sigma^*_{\theta\theta}\Delta r$, which differ only infinitesimally, do not have precisely the same direction. Their directions differ by an amount $\Delta\theta$. Together they give a contribution to the forces in r-direction. By writing

$$\sigma^*_{rr} - \sigma_{rr} = \frac{\partial\sigma_{rr}}{\partial r}\Delta r,$$

$$\sigma^*_{\theta r} - \sigma_{\theta r} = \frac{\partial\sigma_{\theta r}}{\partial\theta}\Delta\theta,$$

the equilibrium equation becomes, after division by $r\Delta r\Delta\theta$,

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0. \tag{40.1}$$

This is the equation of equilibrium in radial direction.

Equilibrium in tangential direction requires that

$$\sigma^*_{\theta\theta}\Delta r - \sigma_{\theta\theta}\Delta r + \sigma^*_{r\theta}(r + \Delta r)\Delta\theta - \sigma_{r\theta} r\Delta\theta + \sigma_{\theta r}\Delta r\Delta\theta = 0.$$

In this case the last term may deserve some explanation. This term is the result of the angle $\Delta\theta$ between the forces $\sigma_{\theta r}\Delta r$ and $\sigma^*_{\theta r}\Delta r$. Using the equality $\sigma_{\theta r} = \sigma_{r\theta}$ the following equation is obtained,

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2}{r}\sigma_{r\theta} = 0. \tag{40.2}$$

This is the equation of equilibrium in tangential direction.

40.3 Prandtl's Schematization

The basic principle of Prandtl's solution of the problem of the determination of the failure load of a half plane carrying a strip load on its surface is a subdivision of the region below the load into three zones, see Fig. 40.2, two triangles and a wedge. In each of these three zones the stress state is assumed to be critical. The load can most simply be derived from a consideration of equilibrium.

In zone I the stresses are assumed to be

$$\text{I}: \quad \sigma_{xx} = 2c, \quad \sigma_{zz} = 0, \quad \sigma_{xz} = 0. \tag{40.3}$$

This stress state satisfies the equilibrium conditions and the boundary conditions on the upper surface (zero shear stress and zero normal stress), and it does not violate the yield condition in any point. Actually, in every point of this zone the yield condition is just reached. On a plane inclined at an angle of 45° the stresses are, see also Fig. 40.3, $\sigma_{\theta\theta} = c$, and $\sigma_{\theta r} = -c$. The sign of these stresses can best be verified by comparison with the definitions of positive stress components, as illustrated in Fig. 40.1, and the stress distribution shown in Fig. 40.3. On the interface between zones I and II the normal stress in radial direction is $\sigma_{rr} = c$.

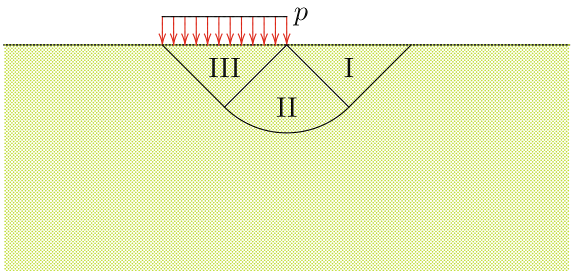


Fig. 40.2 Prandtl's schematization

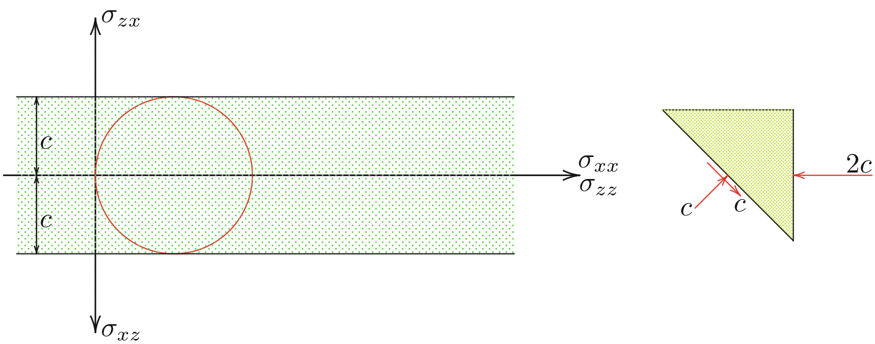


Fig. 40.3 Stresses in zone I

For zone II, the wedge, it is assumed that everywhere in this zone $\sigma_{rr} = \sigma_{\theta\theta}$ and $\sigma_{\theta r} = \sigma_{r\theta} = -c$. Throughout this zone the Mohr circle then just touches the envelope. The two equations of equilibrium, (40.1) and (40.2) now reduce to

$$\frac{\partial \sigma_{rr}}{\partial r} = 0, \tag{40.4}$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 2c. \tag{40.5}$$

These equations can be satisfied by the stress field

$$\text{II} : \quad \sigma_{rr} = \sigma_{\theta\theta} = c + 2c(\theta - \tfrac{1}{4}\pi), \quad \sigma_{\theta r} = \sigma_{r\theta} = -c, \tag{40.6}$$

where the integration constant has been chosen such that $\sigma_{\theta\theta}$ is continuous on the interface between zone I and zone II, where $\theta = \frac{1}{4}\pi$. On the interface between zone II and zone III the angle $\theta = \frac{3}{4}\pi$. Then

$$\theta = \tfrac{3}{4}\pi : \quad \sigma_{rr} = \sigma_{\theta\theta} = c(\pi + 1), \quad \sigma_{\theta r} = \sigma_{r\theta} = -c. \tag{40.7}$$

In zone III the stresses are again be assumed to be constant. A possible stress field is, see also Fig. 40.4,

$$\text{III} : \quad \sigma_{xx} = \pi c, \quad \sigma_{zz} = (\pi + 2)c, \quad \sigma_{xz} = 0. \tag{40.8}$$

The boundary conditions for the stresses are satisfied if $p = (\pi + 2)c$. This solution satisfies all conditions for an equilibrium system. This means that the load in this solution is a lower bound. The failure load is at least equal to that lower bound,

$$p_c \geq (\pi + 2)c = 5.14c. \tag{40.9}$$

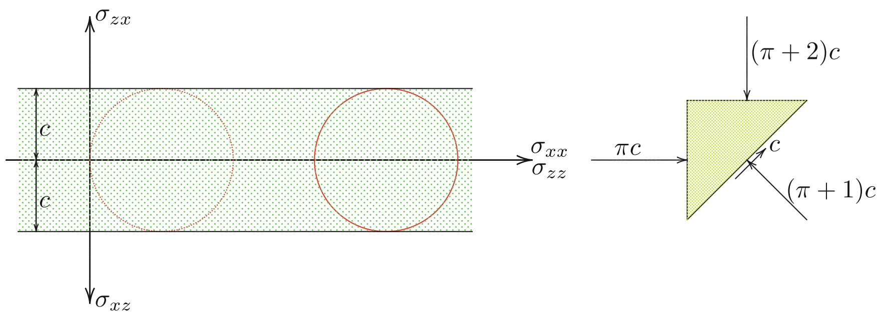


Fig. 40.4 Stresses in zone III

It can be shown that Prandtl's solution is also an upper bound, by considering a convenient deformation field, with the wedge being subdivided into a large number of small triangular wedges. In that case a complication arises in the top corner of the wedge, where the displacements are singular. The derivation for this case, which also happens to yield a load of $(\pi + 2)c$, will not be considered here, see any textbook on the theory of plasticity, e.g. Hill (1960).

Assuming that the proof that the value $(\pi + 2)c$ is an upper bound can indeed be given, it follows that the true failure load in the case of a strip load is

$$p_c = (\pi + 2)c = 5.14c. \qquad (40.10)$$

This value is indeed higher than the lower bounds obtained in the previous chapter, and lower than the upper bounds obtained in that chapter. This confirms the validity of the upper and lower bound theorems.

References

R. Hill, *The Mathematical Theory of Plasticity* (Clarendon Press, Oxford, 1960)

L. Prandtl, Über die Härte plastischer Körper. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse **1920**, 74–85 (1920)