Chapter 32
Rankine

The possible stresses in a soil are limited by the Mohr-Coulomb failure criterion. In 1857 the Scottish engineer W.J.M. Rankine used this criterion to establish limiting values for the stresses in the interior of a soil mass. This will be shown to lead to limits for the lateral stress coefficient K in this chapter.

For simplicity the considerations will be restricted to dry soils at first. The influence of pore water will be investigated later.

32.1 Mohr–Coulomb

As seen before, see Chap. 20, the stress states in a soil can be limited, with good approximation, by the Mohr–Coulomb failure criterion. This criterion is that the shear stresses on any plane are limited by the condition

$$\tau < \tau_f = c + \sigma \tan \phi, \tag{32.1}$$

where c is the cohesion, and ϕ is the angle of internal friction. The criterion can be illustrated using Mohr's circle, see Fig. 32.1.

If it is assumed that σ_{zz} and σ_{xx} are principal stresses, and that σ_{zz} is known (by the weight of the load and the soil), it follows that the value of the horizontal stress σ_{xx} can not be smaller than indicated by the small circle, and not larger than defined by the large circle. The ratio between the minor and the major principal stress can be determined by noting, see Fig. 32.2, that the radius of Mohr's circle is $\frac{1}{2}(\sigma_1 - \sigma_3)$, and that the location of the center is at a distance $\frac{1}{2}(\sigma_1 + \sigma_3)$ from the origin. It follows that for a circle touching the envelope,

$$\sin \phi = \frac{\frac{1}{2}(\sigma_1 - \sigma_3)}{\frac{1}{2}(\sigma_1 + \sigma_3) + c \cot \phi}.$$

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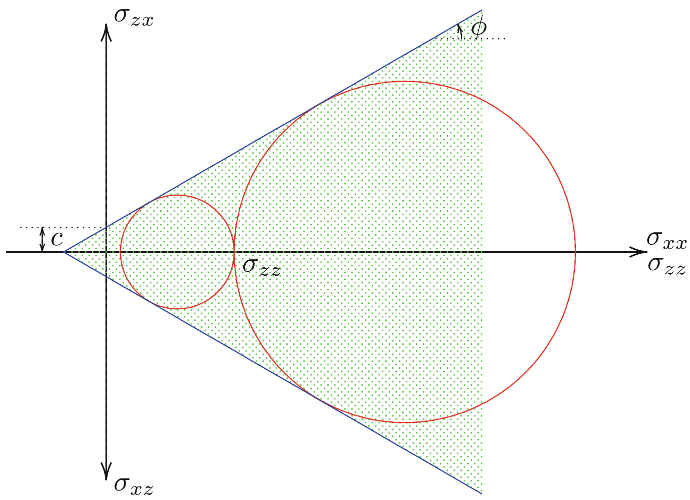

Fig. 32.1 Mohr–Coulomb

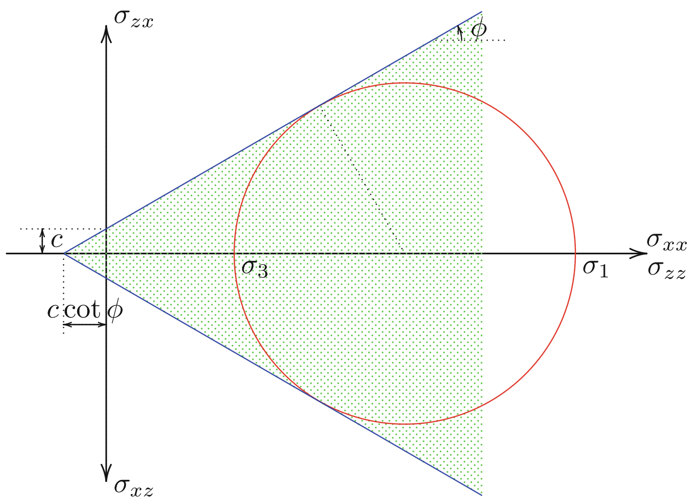

Fig. 32.2 Ratio of principal stresses

so that

$$\sigma_3 = \frac{1-\sin\phi}{1+\sin\phi}\sigma_1 - 2c\frac{\cos\phi}{1+\sin\phi}. \tag{32.2}$$

This relation has been derived before, in Chap. 20. The two coefficients in this equation can be related by noting that

$$\frac{\cos\phi}{1+\sin\phi} = \frac{\sqrt{1-\sin^2\phi}}{1+\sin\phi} = \frac{\sqrt{(1+\sin\phi)(1-\sin\phi)}}{1+\sin\phi} = \sqrt{\frac{1-\sin\phi}{1+\sin\phi}}.$$

This means that Eq. (32.2) can be written as

$$\sigma_3 = K_a\sigma_1 - 2c\sqrt{K_a}, \tag{32.3}$$

where

$$K_a = \frac{1-\sin\phi}{1+\sin\phi}. \tag{32.4}$$

Apart from the constant term $2c\sqrt{K_a}$ there appears to be a given ratio of the minor and the major principal stress.

Formula (32.3) can be written in inverse form as

$$\sigma_1 = K_p\sigma_3 + 2c\sqrt{K_p}, \tag{32.5}$$

where

$$K_p = \frac{1+\sin\phi}{1-\sin\phi}. \tag{32.6}$$

The coefficients K_a and K_p, which give the smallest and the largest ratio of the two principal stresses (apart from a constant term), are denoted as the coefficients of *active earth pressure* (K_a) and *passive earth pressure* (K_p), respectively.

It can be seen that

$$K_a < K < K_p. \tag{32.7}$$

If $\phi = 30°$ (this is a common value for sand, on the small side), it follows that

$$\phi = 30° \;:\quad \frac{1}{3} < K < 3. \tag{32.8}$$

The lateral stress coefficient K appears to be limited by values about $\frac{1}{3}$, and about 3. The precise limits depend upon the angle of internal friction ϕ.

As seen in the previous chapter for the elastic case, lateral extension of the soil leads to a smaller value of the lateral stress coefficient K, whereas lateral compression leads to a larger value of the coefficient K. The extreme situations are denoted as *active earth pressure* and *passive earth pressure*, respectively. The case of active earth pressure is supposed to occur when a retaining structure is being pushed away by the soil stresses. Passive earth pressure denotes that the structure is being pushed into the ground, in which a reaction is being developed.

The large difference between the minimum and the maximum lateral stress is characteristic for frictional materials such as soils.

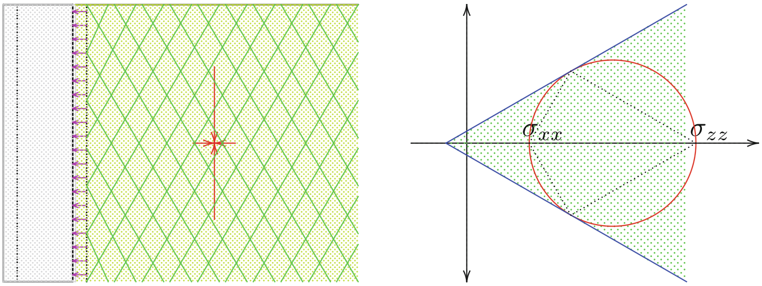

Fig. 32.3 Active earth pressure

32.2 Active Earth Pressure

It can be expected that the smallest value of the horizontal stress occurs in the case of a retaining wall that is moving away from the soil, see Fig. 32.3. The Mohr circle for that case is also shown in the figure. The pole for the vectors normal to the planes is the rightmost point of the circle. This means that the critical shear stress acts on planes making an angle $\frac{1}{4}\pi + \frac{1}{2}\phi$ with the horizontal direction, that is an angle of $\frac{1}{4}\pi - \frac{1}{2}\phi$ with the vertical direction. These planes have been indicated in the left half of Fig. 32.3. It is sometimes imagined that the soil indeed slides along these planes in case of failure.

The vertical stresses along the wall are

$$\sigma_{zz} = \gamma z, \tag{32.9}$$

in which γ is the volumetric weight of the soil, and z is the depth below soil surface. The horizontal stresses now are, with (32.3),

$$\sigma_{xx} = K_a \gamma z - 2c\sqrt{K_a}. \tag{32.10}$$

The total horizontal force on a wall of height h is obtained by integration from $z = 0$ to $z = h$. This gives

$$Q = \frac{1}{2} K_a \gamma h^2 - 2ch\sqrt{K_a}. \tag{32.11}$$

The distribution of the horizontal normal stress σ_{xx} against the wall is shown in Fig. 32.4. It appears that at the top of the wall tensile stresses are generated, over a depth of $2c/\gamma\sqrt{K_a}$. That may be possible in the soil for a short while, in undrained conditions, with negative stresses in the water. In fully drained conditions this is not possible, because then there should be tensile stresses between the particles, or between the particles and the wall. Therefore, it is usually assumed that in a top layer of the soil, of height $2c/\gamma\sqrt{K_a}$, cracks will appear in the soil, and between the

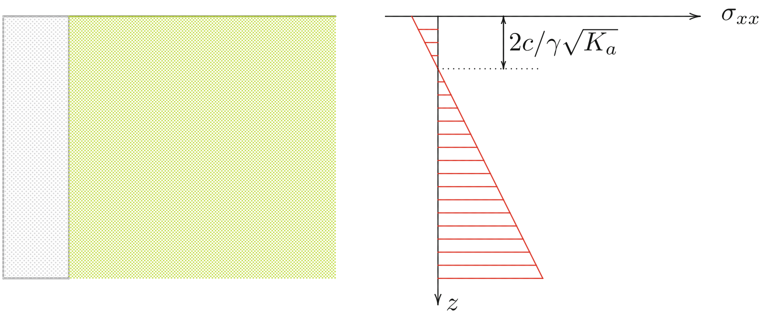

Fig. 32.4 Horizontal stress in case of active earth pressure

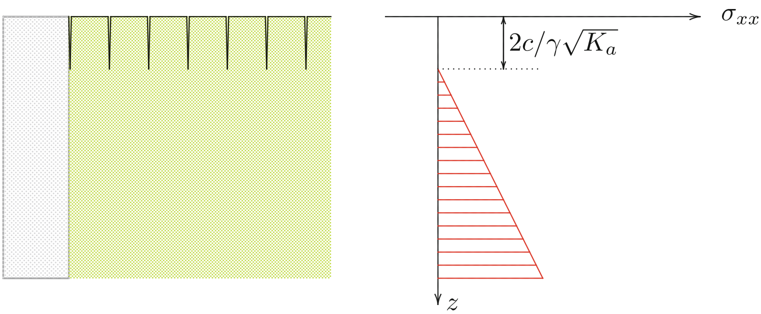

Fig. 32.5 Horizontal stresses, with cracks in the soil

soil and the wall. For that case the stress distribution is shown in Fig. 32.5. For the vertical stresses the cracked zone acts as a surcharge.

In the case with cracks at the surface, shown in Fig. 32.5, the total horizontal force now is

$$Q = \frac{1}{2} K_a \gamma h_r^2, \tag{32.12}$$

in which h_r is the reduced height of the wall,

$$h_r = h - 2c/\gamma\sqrt{Ka}. \tag{32.13}$$

32.3 Passive Earth Pressure

The case of passive earth pressure, in which the horizontal soil stress has its maximum value, can be considered to correspond to a smooth vertical wall that is being pushed in horizontal direction into the soil, see Fig. 32.6. Again the Mohr circle has been shown in the figure as well, with the pole in this case being located in the leftmost point of the circle. The critical shear stress $\tau = \tau_f = c + \sigma \tan\phi$ occurs on planes

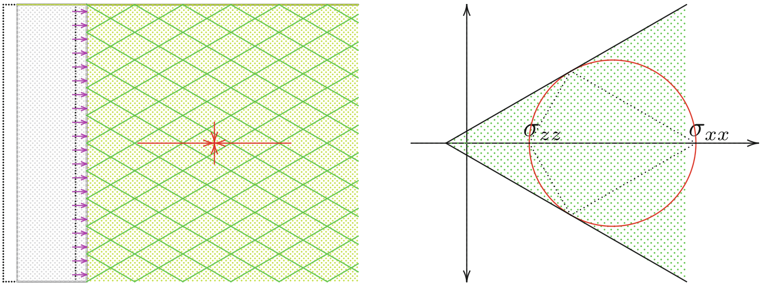

Fig. 32.6 Passive earth pressure

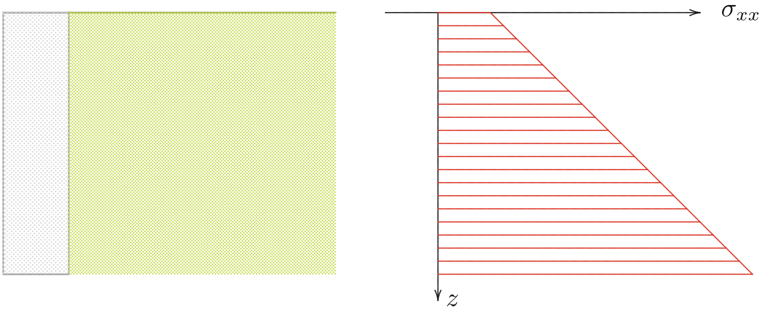

Fig. 32.7 Horizontal stresses in case of passive earth pressure

making an angle $\frac{1}{4}\pi - \frac{1}{2}\phi$ with the horizontal direction. These planes have also been indicated in the left half of the figure. In this case the potential sliding planes are shallower than 45° (Fig. 32.6).

In this case the horizontal stresses are, see Fig. 32.7

$$\sigma_{xx} = K_p \gamma z + 2c\sqrt{K_p}. \tag{32.14}$$

The total horizontal force on a wall of height h is obtained by integration of the horizontal stresses from $z = 0$ to $z = h$. This gives

$$Q = \frac{1}{2} K_p \gamma h^2 + 2ch\sqrt{K_p}. \tag{32.15}$$

In the passive case the cohesion c appears to lead to a constant factor in the expression for the horizontal stresses. There is no reason for cracks to appear, as there are no tensile stresses in this case.

The two extreme states of stress considered here are often denoted as the *Rankine states*, after the Scottish scientist Rankine (1857), who indicated that these stress states are the limiting conditions. In the case of a solid retaining wall, on a good foundation, the actual horizontal stresses will be somewhere between these two extremes. As the limits are so far apart (there may be a factor 9 between them), this leaves the horizontal stress σ_{xx} undetermined to a high degree.

32.4 Neutral Earth Pressure

It has been found that the possible states of stress in a soil may vary between fairly wide limits, especially if the friction angle is large. For a normal sand, with $\phi = 30°$, the smallest value of the horizontal stress is $\frac{1}{3}$ of the vertical stress (which usually is known from the surcharge and the weight of the overlying soil), and the largest value is 3 times the vertical stress. In case of a rigid retaining wall, the lateral stress against the wall is unknown, at least from a strictly scientific viewpoint. In reality there may be some additional information that may help to determine the probable range of the horizontal stress. If the horizontal displacements of the wall are practically zero, the ratio of the horizontal stress to the vertical stress is denoted as the *neutral earth pressure coefficient* K_0. In an elastic material this value would be $K_0 = \nu/(1-\nu)$, but this is not a very good estimate, as soil is not an elastic material, and the history of the development of stresses in the soil may be more important than the condition of zero lateral deformation. Nevertheless, in many cases it is unlikely that the coefficient K_0 would be larger than 1, as this would require some form of motion of the wall towards the soil mass. Also, the active state of stress (say $\frac{1}{3}$) may also be unlikely, if the wall is rather stiff and strong. All this suggests that the neutral stress coefficient may perhaps vary in the range from 0.5 to 1.0. For soft clay the value may be close to 1, and for sands values of about 0.6 or 0.7 have been found to give reasonable results. Lacking any better information the value may be estimated by the formula proposed in 1944 by the Hungarian engineer J. Jaky,

$$K_0 = 1 - \sin\phi, \tag{32.16}$$

but there is no real basis for this formula, except that it gives values between 0 and 1, with the value being close to 1 if the friction angle ϕ is very small (as it is for soft clays).

The best procedure is to try to measure the value of K_0, using an instrument of which the response is determined by the horizontal stress. For example, in the CAMKO-meter, developed in Cambridge (UK), a rubber membrane around a pipe is being pressurized, and the resulting deformation is measured (Fig. 32.8). The idea is that the soil response will be different for lateral pressures below and above the original neutral stress. The membrane consists of three parts, with the central part being the measuring cell. A similar instrument is Marchetti's dilatometer, which consists of a hollow circular plate that is pushed into the soil, in a vertical position. By expanding the plate the lateral response is measured, and from this response the lateral stiffness and the neutral stress coefficient may be estimated. Another method is to inject water into the soil from a tubular instrument. The idea is that a vertical crack may be produced in the soil if the water pressure exceeds the horizontal total stress,

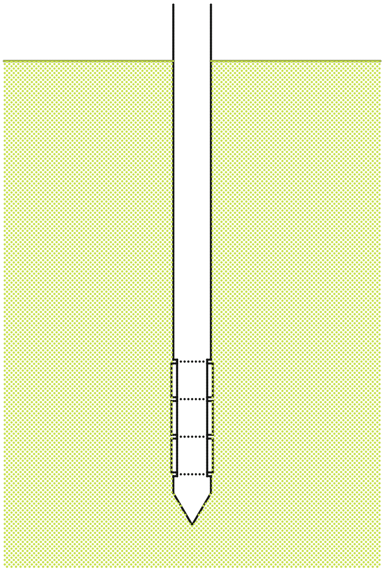

Fig. 32.8 CAMKO-meter

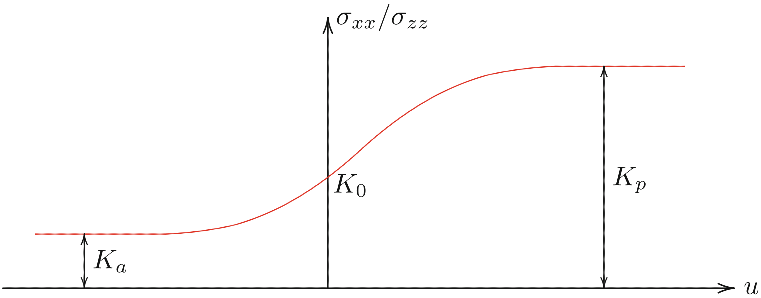

Fig. 32.9 Horizontal stress as a function of the displacement

because the soil skeleton can not transfer tensile stresses. In petroleum engineering this process is called *hydraulic fracturing*, and it is used to improve the permeability of porous rock.

A possible relation between the horizontal stress against a retaining structure and its horizontal displacement is shown in Fig. 32.9. If the displacement is zero, the lateral stress coefficient will be K_0. If the structure now is being pressed towards the soil, the lateral stress will gradually increase, until finally the passive coefficient K_p is reached. On the other hand, if the structure moves away from the soil, the lateral stress will decrease, until its lowest value is reached, as defined by the active coefficient K_a. In advanced computations a relation as shown in Fig. 32.9 may be used to determine the displacement of the structure and the stresses against it. In many cases it can be argued that one of the two limiting values can be considered as appropriate, see the next chapter.

32.5 Groundwater

In the case of a soil saturated with water it should be noted that the Mohr–Coulomb criterion describes the limiting states of *effective* stress in the soil. The correct procedure should be that the smallest and the largest horizontal stress can be deduced from the vertical effective stress, using the active or passive stress coefficient. The horizontal total stress can be obtained by adding the pore water pressure to the horizontal effective stress.

As an example a retaining wall is shown in Fig. 32.10. The wall is assumed to be 8 m high, with the groundwater level at 2 m below the soil surface. The question is to determine the horizontal total stress at a depth of 8 m, assuming that the soil is sand, with $c = 0$ and $\phi = 30°$, for the case of active earth pressure. The volumetric weight of the soil is $16\,\mathrm{kN/m^3}$ when dry, and $20\,\mathrm{kN/m^3}$ when saturated with water. It is assumed that there is no capillary rise in the sand. The vertical total stress at a depth of 8 m is the weight of 2 m dry soil and 6 m of saturated soil, which gives $\sigma_{zz} = 152$ kPa. Because the pore pressure at that depth is 60 kPa the vertical effective stress is $\sigma'_{zz} = 92$ kPa. The active stress coefficient is $K_a = \frac{1}{3}$, so that the horizontal effective stress is $\sigma'_{xx} = 31$ kPa. The total stress is found by adding the pore pressure, i.e. $\sigma_{xx} = 91$ kPa. It is interesting to note that this consists for 66% of water pressure, and for only 34% of effective stress. This illustrates that the contribution of the pore water pressure may be very large. This is especially true in the case of active earth pressure with such small effective stresses.

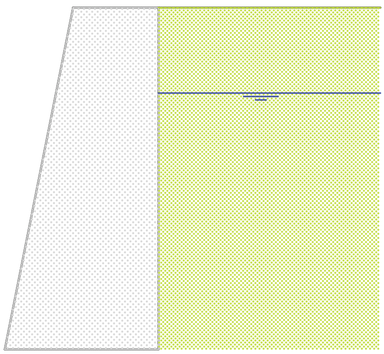

Fig. 32.10 Groundwater in the soil

Table 32.1 K_a and K_p

ϕ	$\sin\phi$	K_a	K_p
0	0.000	1.000	1.000
5	0.087	0.826	1.211
10	0.174	0.704	1.420
15	0.259	0.589	1.698
20	0.342	0.490	2.041
25	0.423	0.405	2.469
30	0.500	0.333	3.000
35	0.574	0.271	3.690
40	0.643	0.217	4.608
45	0.707	0.172	5.814

Example 32.1 A small table of the active and passive earth pressure coefficients, as a function of the angle of internal friction ϕ, is shown in Table 32.1, on the basis of Eqs. (32.4) and (32.6).

Note the large sensitivity of the values upon the value of ϕ. The conclusion must be that the friction angle ϕ should be determined with great accuracy.

Example 32.2 If the cohesion c is unequal to zero, it follows from Eq. (32.3) that the minor principal stress σ_3 can be zero, while the major principal stress σ_1 is unequal to zero. This means that in a cohesive material an excavation can be made with vertical sides. What is the maximum depth of such an excavation, on the basis of this formula?

Solution

Assuming that the major principle stress, the vertical normal stress, is $\sigma_1 = \gamma h$, where h is the depth of the excavation, it follows from Eq. (32.3) that $\sigma_3 = 0$ if $h = 2c/(\gamma K_a)$. If $\phi = 0$ this reduces to $h = 2c/\gamma$. This problem will be considered in more detail in Chap. 43.

Problem 32.1 Why is the instrument shown in Fig. 32.8 known as the CAMKO-meter?

Problem 32.2 A bulldozer, having a blade of 4 m width and 1 m height, is used to remove an amount of dry sand, of 1 m height. Estimate the total force necessary to move the sand.

Problem 32.3 A concrete wall that retained a mass of gravel, of 5 m height, has collapsed by overturning. Estimate the total horizontal force on the wall at the moment of failure, per meter width.

Reference

W.J.M. Rankine, On the stability of loose earth. Phil. Trans. Royal Soc. London **147**, 9–27 (1857)