Chapter 20
Shear Strength

One of the main characteristics of soils is that the shear deformations increase progressively when the shear stresses increase, and that for sufficiently large shear stresses the soil may eventually fail. In nature, or in engineering practice, dams, dikes, or embankments for railroads or highways may fail by part of the soil mass sliding over the soil below it. In this chapter the states of stresses causing such failures of the soil are described. In later chapters the laboratory tests to determine the shear strength of soils will be presented.

20.1 Coulomb

A slope in a soft soil may fail if the slope is too steep or the soil has insufficient strength. A very small cause, such as a small load, or a small local disturbance, may result in a large landslide. Other causes for such a landslide may be water waves against the slope, or a rising groundwater table in the interior of a dam. A spectacular case, in Norway, is shown in the file "Rissa landslide" on the internet site YouTube. A dramatic failure occurred in Aberfan, Wales, in 1966, when a coal tip failed due to large rain fall destroying a young childrens school, see "Aberfan disaster" on YouTube.

It seems reasonable to assume that a sliding failure of a soil will occur if on a certain plane the shear stress is too large, compared to the normal stress. On other planes the shear stress is sufficiently small compared to the normal stress to prevent sliding failure. It may be illustrative to compare the analogous situation of a rigid block on a slope, see Fig. 20.1. Equilibrium of forces shows that the shear force in the plane of the slope is $T = W \sin \alpha$ and that the normal force acting on the slope is $N = W \cos \alpha$, where W is the weight of the block. The ratio of shear force to normal force is $T/N = \tan \alpha$. As long as this is smaller than a certain critical value, the friction coefficient f, the block will remain in equilibrium. However, if the slope

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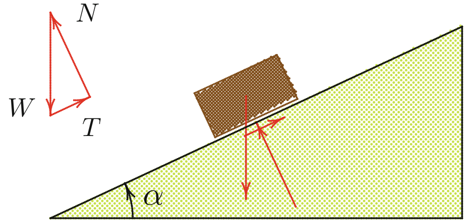
Fig. 20.1 Block on slope



angle α becomes so large that $\tan\alpha = f$, the block will slide down the slope. On steeper slopes the block can never be in equilibrium.

In 1776 Charles-Augustin de Coulomb, a French scientist who also made important contributions to the theory of electricity, used the analogy with a sliding block load to propose that the maximum possible shear stress τ_f in a soil body is

$$\tau_f = c + \sigma' \tan\phi. \tag{20.1}$$

Here σ' is the normal (effective) stress on the plane considered. The quantity c is the *cohesion*, and ϕ is the *angle of internal friction* or the *friction angle*. An elementary interpretation is that if the shear stress on a certain plane is smaller than the critical value τ_f, then the deformations will be limited, but if the shear stresses on any single plane reaches the critical value, then the shear deformations are unlimited, indicating shear failure. The cohesion c indicates that even when the normal stress is zero, a certain shear stress is necessary to produce shear failure. In the case of two rough surfaces sliding over each other (e.g. two blocks of wood), this may be due to small irregularities in the surface. In the case of two very smooth surfaces molecular attractions may play a role.

For soils the formula (20.1) should be expressed in terms of effective stresses, as the stresses acting from one soil particle on another determine the eventual sliding. For this reason the soil properties are often denoted as c' and ϕ', in order to stress that these quantities refer to effective stresses.

20.2 Mohr's Circle

From the theory of stresses (see Appendix A) it is known that the stresses acting in a certain point on different planes can be related by analytical formulas, based upon the equilibrium equations. In these formulas the basic variable is the angle of rotation of the plane with respect to the principal directions. These principal directions are the directions in which the shear stress is zero, and in which the normal stresses are maximal or minimal. The stresses on a sample of soil are shown in Fig. 20.2. It is assumed here that the maximum principal stress, σ_1, is acting in vertical direction, and hence that the smallest principal stress, σ_3 acts in horizontal direction. The

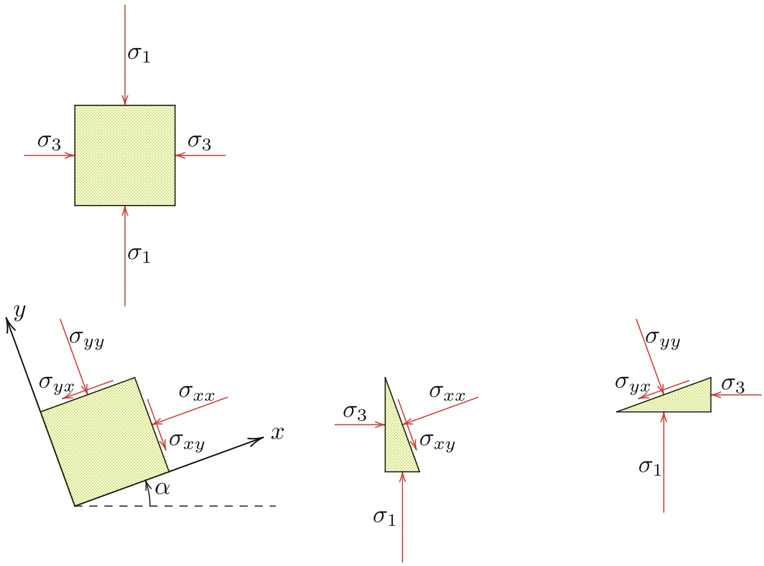


Fig. 20.2 Stresses on a rotated plane

intermediate principal stress (acting in a direction normal to the plane of the figure) is denoted by σ_2. It is possible that $\sigma_2 = \sigma_1$ or $\sigma_2 = \sigma_3$, otherwise $\sigma_3 < \sigma_2 < \sigma_1$. The stresses on two planes having their normal vectors in the x-direction and the y-direction, which make an angle α with the directions of the major and the minor principal stresses, can be expressed into the major and the minor principal stresses by means of the equations of equilibrium, see Fig. 20.2.

The stress components σ_{xx} and σ_{xy}, acting on a plane with its normal in the x-direction, can be found from the equations of equilibrium of a small elementary triangle, formed by a plane perpendicular to the x-direction and a vertical and a horizontal plane, see the small triangle in the center of Fig. 20.2. The small wedge drawn is a part of the rotated element shown in the lower left part of the figure. If the area of the oblique surface is A, the area of the vertical surface is $A \cos\alpha$, and the area of the horizontal plane is $A \sin\alpha$. Equilibrium of forces in the x-direction now gives

$$\sigma_{xx} = \sigma_1 \sin^2\alpha + \sigma_3 \cos^2\alpha. \tag{20.2}$$

Equilibrium of the forces acting upon the small wedge in the y-direction gives

$$\sigma_{xy} = \sigma_1 \sin\alpha\, \cos\alpha - \sigma_3 \sin\alpha\, \cos\alpha. \tag{20.3}$$

The stress components σ_{yy} and σ_{yx}, acting upon a plane having its normal in the y-direction, can be found by considering equilibrium of a small triangular wedge, formed by a plane perpendicular to the y-direction and a vertical and a horizontal plane, see the small triangle in the lower right part of Fig. 20.2. Equilibrium in y-direction gives

$$\sigma_{yy} = \sigma_1 \cos^2\alpha + \sigma_3 \sin^2\alpha. \tag{20.4}$$

And Equilibrium in x-direction gives

$$\sigma_{yx} = \sigma_1 \sin\alpha\, \cos\alpha - \sigma_3 \sin\alpha\, \cos\alpha. \tag{20.5}$$

Comparison of (20.5) and (20.3) shows that $\sigma_{xy} = \sigma_{yx}$, which is in agreement with equilibrium of moments of the element in the lower left part of Fig. 20.2.

It should be noted that the transformation formulas for rotation of a plane all contain two factors $\sin\alpha$ or $\cos\alpha$. This is a characteristic property of quantities such as stresses and strains, which are second order *tensors*. Unlike a vector (sometimes denoted as a first order tensor), which can be described by a magnitude and a single direction, a (second order) tensor refers to two directions: in this case the direction of the plane on which the stresses are acting, and the direction of the stress vector on that plane. In the equations of equilibrium this is seen in the appearance of a factor $\cos\alpha$ or $\sin\alpha$ because of taking the component of a force in x- or y-direction, but another such factor appears because of the size of the area on which the stress component is acting.

Using the trigonometric formulas

$$\sin 2\alpha = 2\sin\alpha\, \cos\alpha, \tag{20.6}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha, \tag{20.7}$$

the transformation formulas can be expressed in 2α,

$$\sigma_{xx} = \tfrac{1}{2}(\sigma_1 + \sigma_3) - \tfrac{1}{2}(\sigma_1 - \sigma_3)\cos 2\alpha, \tag{20.8}$$

$$\sigma_{yy} = \tfrac{1}{2}(\sigma_1 + \sigma_3) + \tfrac{1}{2}(\sigma_1 - \sigma_3)\cos 2\alpha, \tag{20.9}$$

$$\sigma_{xy} = \sigma_{yx} = \tfrac{1}{2}(\sigma_1 - \sigma_3)\sin 2\alpha. \tag{20.10}$$

The stress components on planes with different orientations can be represented graphically using Mohr's circle, see Fig. 20.3. This useful graphical representation was first presented by the German scientist Otto Mohr in 1914. A simple form of Mohr's diagram occurs if it the positive normal stresses σ_{xx} and σ_{yy} are plotted towards the right on the horizontal axis, that a positive shear stress σ_{xy} is plotted vertically downward, and that a positive shear stress σ_{yx} is plotted vertically upward.

The circle is constructed by first indicating distances corresponding to σ_1 and σ_3 on the horizontal axis. These two points define a circle, with its center on the horizontal axis, at a distance $\frac{1}{2}(\sigma_1 + \sigma_3)$ from the origin. The radius of the circle is $\frac{1}{2}(\sigma_1 - \sigma_3)$. These happen to be the two values appearing in the formulas (20.8),

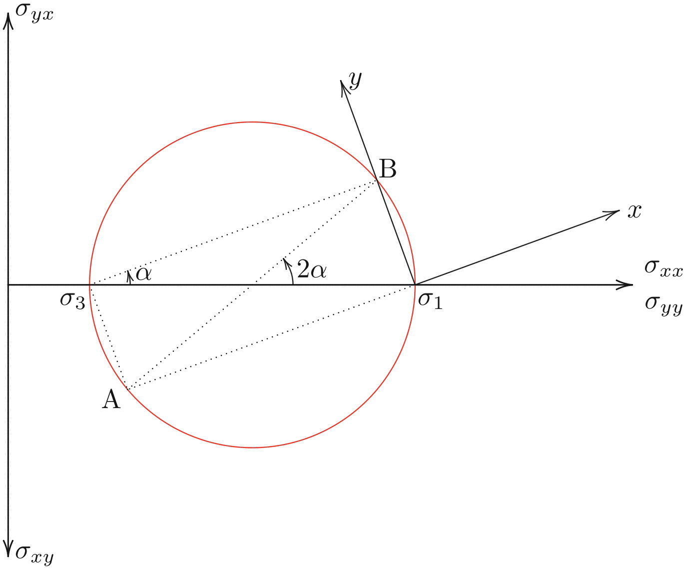


Fig. 20.3 Mohr's circle

(20.9) and (20.10). If in the center of the circle an angle of magnitude 2α is measured, it follows that the point A on the circle has the coordinates σ_{xx} and σ_{xy}. The point B, on the opposite side on the circle, has coordinates σ_{yy} and σ_{yx}. It should be noted that this is true only if on the vertical axis σ_{xy} is considered positive in downward direction, and σ_{yx} is considered positive in upward direction. The formulas (20.8), (20.9) and (20.10) now all are represented by the graphical construction.

Because an inscribed angle on a certain arc is just one half of the central angle, it follows that point B can also be found by drawing a line at an angle α from the leftmost point of the circle, and intersecting that line with the circle. In the same way the point A can be found by drawing a line from the same point perpendicular to the previous line.

The point A, which defines the stress components on a plane with its normal in the x-direction, can also be found by drawing a line from the rightmost point of the circle in the direction of the x-axis. Similarly, the point B, which defines the stress components on a plane with its normal in the y-direction, can be found by drawing a line from that point in the direction of the y-axis, see Fig. 20.2. The rightmost point of the circle is therefore sometimes denoted as the *pole* of the circle. Drawing lines in the directions of two perpendicular axes x and y will lead to two opposite intersection points on the circle, which define the values of the stress components in these two directions. If the axes rotate, i.e. when α increases, these intersection points travel along the circle.

For $\alpha = 0$ the x-axis coincides with the direction of σ_3, and the y-axis coincides with the direction of σ_1. The point A then is located in the leftmost point of the circle, and the point B in the rightmost point. If the angle α now increases from 0 to $\pi/2$ the two stress points A and B travel along the circle, in a half circle. When $\alpha = \pi/2$ point A arrives in the rightmost point and point B arrives in the leftmost point. Then the x-axis points vertically upward, and the y-axis points horizontally towards the left. If α varies from 0 to π the stress points travel along the entire circle.

20.3 Mohr–Coulomb

A point of Mohr's circle defines the normal stress and the shear stress on a certain plane. The stresses on all planes together form the circle, because when the plane rotates the stress points traverse the circle. It appears that the ratio of shear stress to normal stress varies along the circle, i.e. this ratio is different for different planes. It is possible that for certain planes the failure criterion (20.1) is satisfied. In Fig. 20.4 this failure criterion has also been indicated, in the form of two straight lines, making an angle ϕ with the horizontal axis. Their intersections with the vertical axis is at distances c. In order to underline that failure of a soil is determined by the effective stresses, the stresses in this figure have been indicated as σ'. There are two planes, defined by the points C and D in Fig. 20.4, in which the stress state is critical. On all other planes the shear stress remains below the critical value. Thus it can be conjectured that failure will start to occur whenever Mohr's circle just touches the Coulomb envelope. This is called the *Mohr–Coulomb failure criterion*. If the stress circle is completely within the envelope no failure will occur, because on all planes

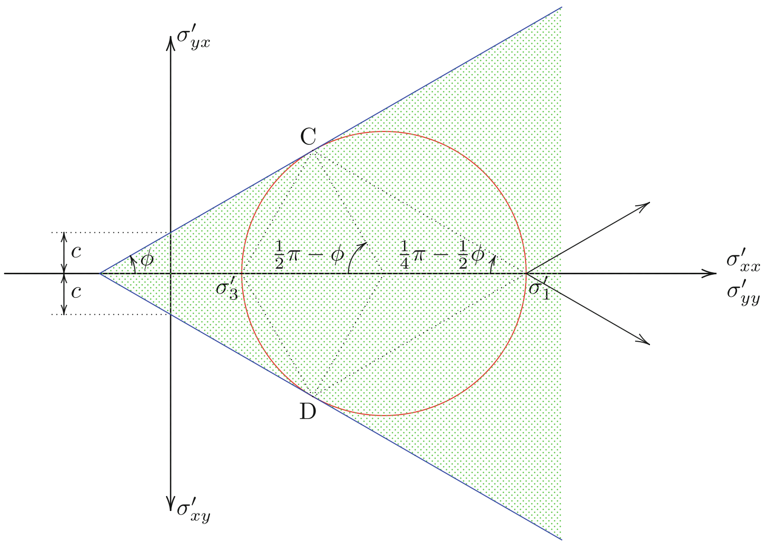


Fig. 20.4 Mohr–Coulomb failure criterion

the shear stress remains well below the critical value, as given by Eq. (20.1). Circles partly outside the envelope are impossible, as the shear stress on some planes would be larger than the critical value.

When the circle just touches the envelope there are two planes making angles $\pi/4 - \phi/2$ with the direction of the major principal stress, on which the stresses are critical. Sliding failure may occur on these planes. It can be expected that the soil may slide in the directions of these two critical planes. In the case represented by the figures in this chapter, in which it is assumed that the vertical direction is the direction of the major principal stress, see Fig. 20.2, the planes on which the stresses are most critical make an angle $\pi/4 - \phi/2$ with the vertical direction. Thus it can be expected that sliding failure will occur in planes that are somewhat steeper than 45°. If for instance $\phi = 30°$, which is a normal value for sands, failure will occur by sliding along planes that make an angle of 30° with the vertical direction.

20.4 The Mohr–Coulomb Criterion

The mathematical formulation of the Mohr–Coulomb failure criterion can be found by noting that the radius of Mohr's circle is $\frac{1}{2}(\sigma'_1 - \sigma'_3)$, and that the distance from the origin to the center is $\frac{1}{2}(\sigma'_1 + \sigma'_3)$. Failure will occur if

$$\sin\phi = \frac{\frac{1}{2}(\sigma'_1 - \sigma'_3)}{c\cot\phi + \frac{1}{2}(\sigma'_1 + \sigma'_3)}. \qquad (20.11)$$

This can also be written as

$$\left(\frac{\sigma'_1 - \sigma'_3}{2}\right) - \left(\frac{\sigma'_1 + \sigma'_3}{2}\right)\sin\phi - c\cos\phi = 0. \qquad (20.12)$$

Using this equation the value of σ'_3 in the failure state can be expressed into σ'_1,

$$\sigma'_3 = \sigma'_1\,\frac{1-\sin\phi}{1+\sin\phi} - 2c\,\frac{\cos\phi}{1+\sin\phi}. \qquad (20.13)$$

On the other hand, the value of σ'_1 in the failure state can also be expressed into σ'_3, of course,

$$\sigma'_1 = \sigma'_3\,\frac{1+\sin\phi}{1-\sin\phi} + 2c\,\frac{\cos\phi}{1-\sin\phi}. \qquad (20.14)$$

These formulas will be used very often in later chapters.

20.5 Remarks

The Mohr–Coulomb criterion is a rather good criterion for the failure state of sands. For such soils the cohesion usually is practically zero, $c = 0$, and the friction angle usually varies from $\phi = 30$ to 45°, depending upon the angularity and the roundness of the particles. Clay soils usually have some cohesion, and a certain friction angle, but usually somewhat smaller than sands.

Great care is needed in the application of the Mohr–Coulomb criterion for very small stresses. For clay one might find that a Mohr's circle would be possible in the extreme left corner of the diagram, with tensile normal stresses. It is usually assumed that this is not possible, and therefore the criterion should be extended by a vertical cut-off at the vertical axis. To express that the cohesion of soils does not necessarily mean that the soil can withstand tensile stresses, the property is sometimes denoted as *apparent cohesion*, indicating that it is merely a first order schematization.

In metallurgy it is usually found that the shear strength of metals is independent of the normal stress. The failure criterion then is that there is a given maximum shear stress, $\tau_f = c$. The Mohr–Coulomb criterion reduces to the criterion for metals by taking $\phi = 0$.

The Mohr–Coulomb criterion can also be used, at least in a first approximation, for materials such as rock and concrete. In such materials a tension cut-off is not necessary, as they can indeed withstand considerable tensile stresses. In such materials the cohesion may be quite large, at least compared to soils. The contribution of friction is not so dominant as it is in soils. Also it often appears that the friction angle is not constant, but decreases at increasing stress levels.

In some locations, for instance in offshore coastal areas near Brazil and Australia, calcareous soils are found. These are mostly sands, but the particles have been glued together, by the presence of the calcium. Such materials have very high values of the cohesion c, which may easily be destroyed, however, by a certain deformation. This deformation may occur during the construction of a structure, for instance the piles of an offshore platform. During the exploration of the soil this may have been found to be very strong, but after installation much of the strength has been destroyed. An advantage of true frictional materials is that the friction usually is maintained, also after very large deformations. Soils such as sands may not be very strong, but at least they maintain their strength.

For clays the Mohr–Coulomb criterion is reasonably well applicable, provided that proper care is taken of the influence of the pore pressures, which may be a function of time, so that the soil strength is also a function of time. Many clays also have the property that the cohesion increases with time during consolidation. This leads to a higher strength because of overconsolidation. For very soft clays the Mohr–Coulomb criterion may not be applicable, as the soil behaves more like a viscous liquid.

Example 20.1 In a sample of sand (c = 0) a stress state appears to be possible with $\sigma_{xx} = 10$ kPa, $\sigma_{yy} = 20$ kPa and $\sigma_{xy} = 5$ kPa, without any sign of failure. What can you say of the friction angle ϕ?

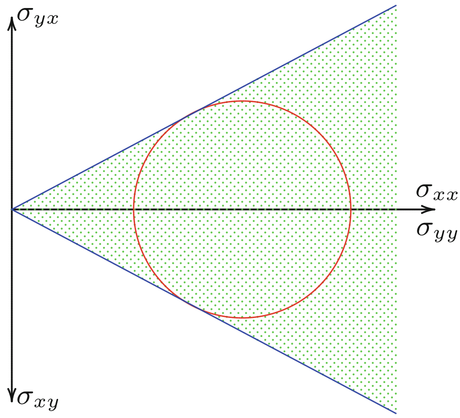


Fig. 20.5 Example 20.1

Solution

In this case $\frac{1}{2}(\sigma_{xx} + \sigma_{yy}) = 15$, $\frac{1}{2}(\sigma_{yy} - \sigma_{xx}) = 5$, $\sigma_{xy} = 5$. With Eq. (20.8), (20.9) and (20.10) it follows that $\frac{1}{2}(\sigma_1 + \sigma_3) = 15$, $\frac{1}{2}(\sigma_1 - \sigma_3)\cos 2\alpha = 5$, $\frac{1}{2}(\sigma_1 - \sigma_3)\sin 2\alpha = 5$. The (unknown) value of α can be eliminated from the last two equations using the relation $\sin^2 2\alpha + \cos^2 2\alpha = 1$. This gives $\frac{1}{2}(\sigma_1 - \sigma_3) = 5\sqrt{2}$. With Eq. (20.11) it now follows that failure would occur if $\sin\phi = \sqrt{2}/3 = 0.4714$, or $\phi = 28°$. Because the stress state is possible without any sign of failure it follows that $\phi > 28°$.

The solution can be verified by constructing Mohr's circle for this state of stress, see Fig. 20.5.

Example 20.2 A sand, with $c = 0$ and $\phi = 30°$ is on the limit of failure. The minor principal stress is 10 kPa. What is the major principal stress?

Solution

The answer can be obtained from Eq. (20.14). This gives, because $\sin\phi = 0.5$: $\sigma_1 = 30$ kPa.

Problem 20.1 In a soil sample the state of stress is such that the major principal stress is the vertical normal stress, at a value $3p$. The horizontal normal stress is p. Determine the normal stress and the shear stress on a plane making an angle of 45° with the horizontal direction, analytically or graphically using Mohr's circle.

Problem 20.2 Also determine the normal stress and the shear stress on a plane making an angle of 30° with the vertical direction, and determine the angle of the resulting force with the normal vector to that plane.