Chapter 44
Stability of Infinite Slope

The evaluation of the stability of a slope, of an embankment or a dyke, is an important problem of applied soil mechanics. In the previous chapter this problem has been considered for a vertical slope in a purely cohesive material ($c > 0$, $\phi = 0$). As a preparation for the general case, which will be considered in the next chapter, this chapter will present some solutions for slopes of infinite extent, in a homogeneous frictional material, without cohesion ($c = 0$, $\phi > 0$).

44.1 Infinite Slope in Dry Sand

Consider an infinitely long slope, in dry sand, at inclination α, see Fig. 44.1. The equations of equilibrium can now best be expressed using coordinates parallel and perpendicular to the slope,

$$\frac{\partial \sigma_{\xi\xi}}{\partial \xi} + \frac{\partial \sigma_{\eta\xi}}{\partial \eta} + \gamma \sin \alpha = 0, \tag{44.1}$$

$$\frac{\partial \sigma_{\xi\eta}}{\partial \xi} + \frac{\partial \sigma_{\eta\eta}}{\partial \eta} - \gamma \cos \alpha = 0. \tag{44.2}$$

The stresses in these equations are total stresses, but as there are no pore pressures, they are effective stresses as well, in this case of a dry soil.

The state of stress is not uniquely determined by the equilibrium conditions. One of the possible solutions can be obtained by assuming that the state of stress is independent of ξ, the coordinate along the slope. That seems to be a reasonable assumption, because the slope extends towards infinity both in upward and in downward direction. There is nu absolute need for the independence of ξ, however, and it is not more than an assumption. Using this assumption the equilibrium equations

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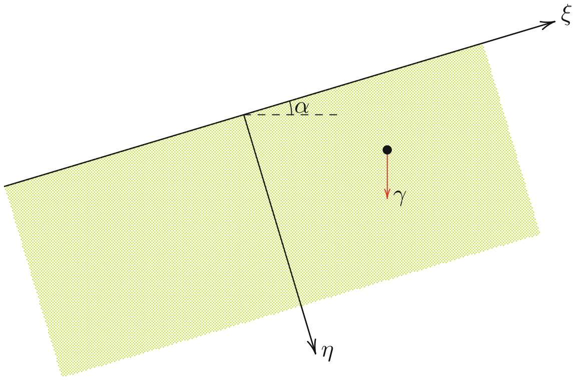

Fig. 44.1 Infinite slope in dry sand

give, when expressed in effective stresses,

$$\sigma'_{\eta\xi} = -\gamma\eta \sin\alpha, \tag{44.3}$$

$$\sigma'_{\eta\eta} = +\gamma\eta \cos\alpha. \tag{44.4}$$

The integration constants have been taken as zero, because at the surface $\eta = 0$ the stresses $\sigma'_{\eta\eta}$ and $\sigma'_{\eta\xi}$ must be zero. It follows that

$$\frac{|\ \sigma'_{\eta\xi}\ |}{|\ \sigma'_{\eta\eta}\ |} = \tan\alpha. \tag{44.5}$$

The Coulomb failure criterion states that in a cohesionless material ($c = 0$) this ratio can not be larger than $\tan\phi$. This means that α can not be larger than ϕ, $\alpha < \phi$.

A *stability factor* can be introduced as

$$F = \frac{|\ \sigma'_{\eta\xi}/\sigma'_{\eta\eta}\ |_{\max}}{|\ \sigma'_{\eta\xi}/\sigma'_{\eta\eta}\ |}. \tag{44.6}$$

The factor F may also be called the *safety factor*. In this case

$$F = \frac{\tan\phi}{\tan\alpha}. \tag{44.7}$$

If $\alpha < \phi$ this is greater than 1, the slope is stable. If $\alpha > \phi$ the value of F is smaller than one, the slope is unstable.

It should be noted that the stability factor F appears to be independent of the volumetric weight γ. That is a characteristic of frictional materials. In general the safety factor is defined as

$$F = \frac{\text{strength}}{\text{load}}. \tag{44.8}$$

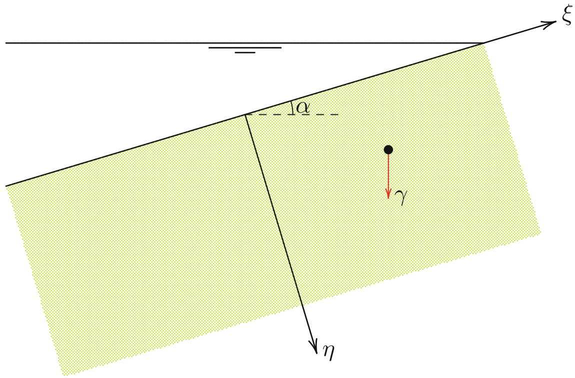

Fig. 44.2 Infinite slope under water

In case of loading by the weight of a frictional material the load is proportional to the volumetric weight, but so is the strength. The result is that the volumetric weight cancels in the ratio, so that the safety is independent of the volumetric weight.

It has been seen that the steepest possible slope in dry sand is ϕ. This property can be used as a simple method to determine the value of the friction angle ϕ of dry sand: it is the inclination of the steepest slope. It should be noted that this property holds only for a soil without cohesion, and completely dry. A small amount of water can easily disturb it.

44.2 Infinite Slope Under Water

For the case of a very long slope under water, see Fig. 44.2, the critical slope can be determined as follows.

Equilibrium is again described by the Eqs. (44.1) and (44.2), but in this case there is a certain pore pressure. Terzaghi's effective stress principle the total stresses can be expressed by $\sigma_{\xi\xi} = \sigma'_{\xi\xi} + p$ and $\sigma_{\eta\eta} = \sigma'_{\eta\eta} + p$, so that the equations of equilibrium can be expressed in terms of effective stresses as

$$\frac{\partial \sigma'_{\xi\xi}}{\partial \xi} + \frac{\partial \sigma'_{\eta\xi}}{\partial \eta} + \frac{\partial p}{\partial \xi} + \gamma \sin\alpha = 0, \tag{44.9}$$

$$\frac{\partial \sigma'_{\xi\eta}}{\partial \xi} + \frac{\partial \sigma'_{\eta\eta}}{\partial \eta} + \frac{\partial p}{\partial \eta} - \gamma \cos\alpha = 0. \tag{44.10}$$

If the groundwater is at rest, the pressure distribution is hydrostatic. If the z-axis is directed vertically upward, the pressures in the groundwater can be written as

$$p = p_0 - \gamma_w z = p_o + \gamma_w \eta \cos\alpha - \gamma_w \xi \sin\alpha. \tag{44.11}$$

The reference pressure p_0 in this expression is the pressure at the level $z = 0$. If the entire slope is located under water, the phreatic surface (the level at which $p = 0$) must be located at an infinite height. The pore pressure at the level $z = 0$ then is infinitely large, $p_0 = \infty$. The present example is not completely realistic, which is a consequence of considering an *infinite* slope. At its best the example is the limiting form of a very long slope.

Substitution of (44.11) into (44.9) and (44.10) gives

$$\frac{\partial \sigma'_{\xi\xi}}{\partial \xi} + \frac{\partial \sigma'_{\eta\xi}}{\partial \eta} + (\gamma - \gamma_w)\sin\alpha = 0, \tag{44.12}$$

$$\frac{\partial \sigma'_{\xi\eta}}{\partial \xi} + \frac{\partial \sigma'_{\eta\eta}}{\partial \eta} - (\gamma - \gamma_w)\cos\alpha = 0. \tag{44.13}$$

These are precisely the same equations as in the dry case, except that γ has been replaced by $\gamma - \gamma_w$. Because it was found earlier that the stability factor F is independent of γ, see Eq. (44.7), it follows that this is also valid in this case of a slope under water, i.e.

$$F = \frac{\tan\phi}{\tan\alpha}. \tag{44.14}$$

It appears that a slope under water can also be maintained at an inclination ϕ. This conclusion seems to be in contradiction with experimental evidence, which suggests that a slope under water usually is less steep than a slope above water, in the same material. A possible explanation is that under water other processes may disturb the stability of a slope, such as erosion by waves or by flowing groundwater. In a basin with water a slope at rest can indeed be as steep as a slope in dry sand.

44.3 Flow Parallel to the Slope

An interesting problem is the stability of an embankment or dam in which groundwater flows parallel to the slope, in downward direction, see Fig. 44.3. This may occur in a dyke that is just not high enough to retain the water in a river, so that water flows over the slope. This water penetrates into the dyke material, and after some time a flow of groundwater parallel to the slope may be created, as shown in Fig. 44.4.

If the flow is uniform the pressure distribution must be linear in ξ and η, i.e.

$$p = A\eta + B\xi + C. \tag{44.15}$$

Along the surface the pressure must be zero (this will be the case if the soil is saturated, with merely a thin film of water flowing over the slope), i.e. $p = 0$ for $\eta = 0$. It then follows that $B = C = 0$, so that the pressure distribution reduces to $p = A\eta$. This means that the groundwater head h is

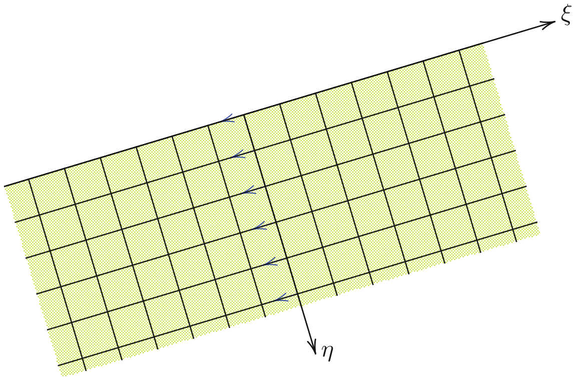

Fig. 44.3 Parallel groundwater flow

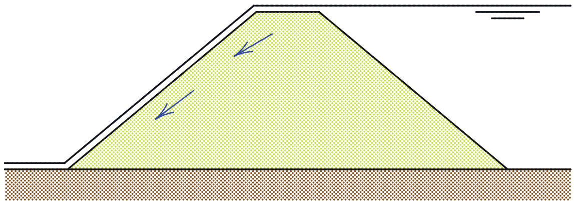

Fig. 44.4 Parallel flow

$$h = z + \frac{p}{\gamma_w} = A\frac{\eta}{\gamma_w} - \eta \cos\alpha + \xi \sin\alpha. \tag{44.16}$$

If the flow is parallel to the soil surface the component of the specific discharge vector perpendicular to the surface must be zero, $q_\eta = 0$, and therefore $\partial h/\partial\eta = 0$. It follows that $A = \gamma_w \cos\alpha$, so that the pressure p is

$$p = \gamma_w \eta \cos\alpha. \tag{44.17}$$

Substitution of this pressure distribution into the equations of equilibrium (44.9) and (44.10) gives

$$\frac{\partial\sigma'_{\xi\xi}}{\partial\xi} + \frac{\partial\sigma'_{\eta\xi}}{\partial\eta} + \gamma \sin\alpha = 0, \tag{44.18}$$

$$\frac{\partial\sigma'_{\xi\eta}}{\partial\xi} + \frac{\partial\sigma'_{\eta\eta}}{\partial\eta} - (\gamma - \gamma_w)\cos\alpha = 0. \tag{44.19}$$

A solution independent of ξ is

$$\sigma'_{\eta\xi} = -\gamma\eta \sin\alpha, \tag{44.20}$$

$$\sigma'_{\eta\eta} = (\gamma - \gamma_w)\eta \cos\alpha. \tag{44.21}$$

In this case the stability factor F is

$$F = \frac{\gamma - \gamma_w}{\gamma} \frac{\tan \phi}{\tan \alpha}. \tag{44.22}$$

Because $(\gamma - \gamma_w)/\gamma < 1$ (usually about 0.5), it follows that the steepest possible slope in this case is much smaller than ϕ. The groundwater flow appears to have a large negative influence on the stability of the slope.

It must be concluded that it is very unfavorable for the stability of the downstream slope of a dyke if groundwater flows down the slope, parallel to the slope. This may occur in the case of groundwater exiting the slope along a seepage surface, or if the level of the free water at the upstream side of the dyke is so high that it flows over the dyke, and penetrates into the downstream slope. This mechanism is considered to have been responsible for the failure of many dykes in the 1953 flood in the South–West of the Netherlands. The analysis in this section was given by Joustra and Edelman (1960).

44.4 Horizontal Outflow

Another interesting example is a dyke in which groundwater is flowing in horizontal direction, see Fig. 44.5. If groundwater flows through the dyke in horizontal direction, the groundwater head is independent of z, $\partial h/\partial z = 0$. Because $h = z + p/\gamma_w$ it then follows that $\partial p/\partial z = -\gamma_w$. Furthermore, along the surface, that is for $z = x \tan \alpha$, the pressure p must be zero. And if the flow is uniform the pressure distribution must be linear. The only pressure distribution that satisfies all these conditions is

$$p = \gamma_w x \tan \alpha - \gamma_w z. \tag{44.23}$$

This can be expressed into ξ and η using the transformation formulas for rotation of the coordinates,

$$x = \xi \cos \alpha + \eta \sin \alpha, \quad z = -\eta \cos \alpha + \xi \cos \alpha.$$

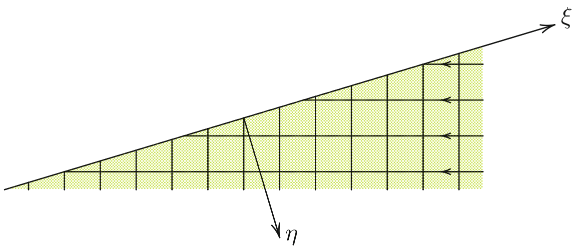

Fig. 44.5 Horizontal groundwater flow

The result is

$$p = \gamma_w \eta / \cos\alpha. \tag{44.24}$$

Substitution into the equations of equilibrium (44.9) and (44.10) in this case gives

$$\frac{\partial \sigma'_{\xi\xi}}{\partial \xi} + \frac{\partial \sigma'_{\eta\xi}}{\partial \eta} + \gamma \sin\alpha = 0, \tag{44.25}$$

$$\frac{\partial \sigma'_{\xi\eta}}{\partial \xi} + \frac{\partial \sigma'_{\eta\eta}}{\partial \eta} - \gamma \cos\alpha - \gamma_w / \cos\alpha = 0. \tag{44.26}$$

A solution independent of ξ is

$$\sigma'_{\eta\xi} = -\gamma\eta \sin\alpha, \tag{44.27}$$

$$\sigma'_{\eta\eta} = \left(\gamma - \frac{\gamma_w}{\cos^2\alpha}\right) \eta \cos\alpha. \tag{44.28}$$

The stability factor F now is

$$F = \frac{\gamma - \gamma_w / \cos^2\alpha}{\gamma} \frac{\tan\phi}{\tan\alpha}. \tag{44.29}$$

This value is even smaller than the value in the previous case, see (44.22), because the value of $\cos^2\alpha$ is always smaller than 1. It follows that a horizontal outflow of groundwater is even more dangerous than a flow parallel to the slope.

Such a horizontal flow can be considered to occur, approximately, for a permeable dyke or dam on an impermeable base. Large dams are often built on an impermeable base, to prevent leakage from the lake through the subsoil. If the dam were built from homogeneous material, see Fig. 44.6, groundwater will exit from the dam at the downstream slope, with a practically horizontal flow through the dam. This is a very unfavorable situation, and should be avoided (Fig. 44.7).

There are two good technical solutions. The first solution is to place a blanket of almost impermeable material (clay) on the upstream slope, or, even better, to construct a core of clay in the center of the dam. This is better because it can not be damaged by poor maintenance or accidental damage. The second solution is to construct a filter at the toe of the dam or dyke, consisting of very permeable material (for instance gravel). Such a filter will attract the groundwater and drain it away.

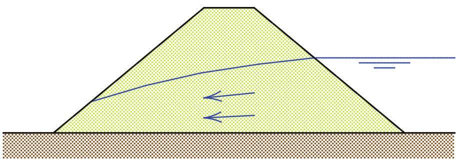

Fig. 44.6 Flow through dyke

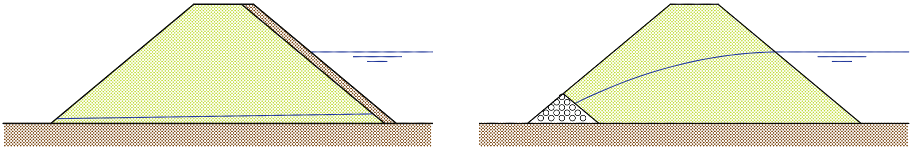

Fig. 44.7 Dam with a clay blanket, or with a drain

Great care should be taken to maintain the high permeability of the filter. Of course, the best solution is to apply both solutions: a clay core in the center, and a filter in the downstream toe. Failure of a large dam is such a catastrophe that it should be avoided at all cost.

Reference

K. Joustra, T. Edelman, The failure of dykes in 1953 (in Dutch). De Ingenieur **72**, 23–28 (1960)