

Chapter 39

Strip Footing

One of the simplest problems for which lower limits and upper limits can be determined is the case of an infinitely long strip load on a layer of homogeneous cohesive material.

39.1 Strip Load

The problem considered in this chapter concerns a half space of homogeneous cohesive material ($\phi = 0$), see Fig. 39.1, loaded by a strip load. The weight of the soil will be disregarded, at least in this chapter. That means that it is assumed that $\gamma = 0$. The problem is a first schematization of the foundation of a structure, using a long strip foundation, made of concrete, for instance.

It will first be attempted to obtain a lower bound for the failure load, using an equilibrium system. Such a system should consist of a field of stresses that satisfies the conditions of equilibrium in all points of the field, that agrees with the given stress distribution on the soil surface, and that does not violate the yield condition in any point.

39.2 Lower Bound

An elementary solution of the conditions of equilibrium in a certain region is that the stresses in that region are constant, because then all conditions are indeed satisfied. In a two-dimensional field these equilibrium conditions are, in the absence of gravity,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} = 0, \quad (39.1)$$

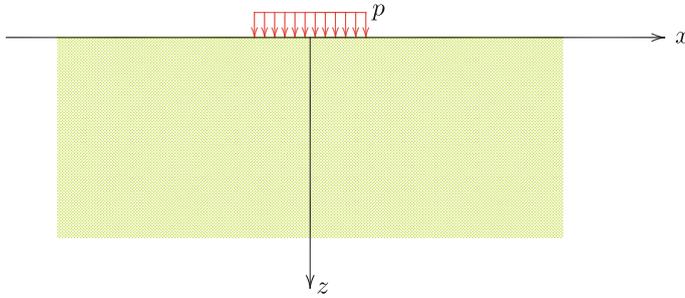


Fig. 39.1 Strip load on half space

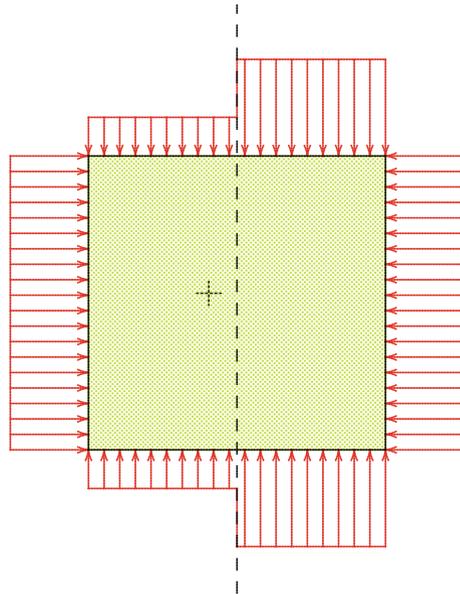
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \tag{39.2}$$

$$\sigma_{xz} = \sigma_{zx}. \tag{39.3}$$

The main difficulty is to satisfy the boundary condition, because the normal stress σ_{zz} is discontinuous along the surface, see Fig. 39.1. This difficulty can be surmounted by noting that in a statically admissible field of stresses (an equilibrium system), not all stresses need be continuous. Formally this can be recognized by inspection of the equations of equilibrium, Eqs. (39.1)–(39.3). All partial derivatives in these equations must exist, which means that the stresses must at least be continuous in the directions in which they have to be differentiated. It follows that the shear stress σ_{xz} must be continuous in both directions, that the normal stress σ_{xx} must be continuous in x -direction, and the normal stress σ_{zz} must be continuous in z -direction. However, two of the partial derivatives, $\partial \sigma_{xx} / \partial z$ and $\partial \sigma_{zz} / \partial x$, do not appear in the equations of equilibrium, and therefore no conditions have to be imposed on the continuity of these two normal stresses in these directions. This means that σ_{xx} may be discontinuous in z -direction, and that σ_{zz} may be discontinuous in x -direction. Such a discontinuity is shown, for the vertical direction, in Fig. 39.2. This figure shows a small element, with all the stresses acting upon its boundaries. The normal stress σ_{xx} must be continuous in x -direction, because of equilibrium, as can most easily be seen by letting the width of the element approach zero. Then the continuity of the stress σ_{xx} can be seen as a consequence of Newton’s principle of equality of action and reaction. The normal stress σ_{zz} , however, may jump across the vertical line, without disturbing equilibrium. In Fig. 39.2 the stress σ_{zz} is discontinuous in x -direction. The partial derivative $\partial \sigma_{zz} / \partial x$ is infinitely large at the location of the vertical axis, but the element, and all of its parts, are perfectly well in equilibrium.

This property of equilibrium systems has been applied by Drucker, one of the originators of the theory of plasticity, to construct equilibrium fields for practical problems. In this method the field is subdivided into regions of simple form, in each of which the stress is constant, so that the equations of equilibrium are automatically satisfied. The various subregions then are connected by requiring that all the stresses

Fig. 39.2 Stress discontinuity



transferred on the boundary surfaces are continuous, allowing the normal stresses in the direction of these boundaries to be discontinuous. An example is shown in Fig. 39.3, for the case of a strip footing. In a vertical strip below the load the stresses are supposed to be $\sigma_{xx} = 2c$, $\sigma_{zz} = 4c$, and $\sigma_{xz} = 0$. In the two regions to the left and right of this strip the stresses are $\sigma_{xx} = 2c$, $\sigma_{zz} = 0$, and $\sigma_{xz} = 0$. On the two vertical discontinuity lines only the vertical normal stress σ_{zz} is discontinuous. The other stresses are continuous, as required by equilibrium. This field of stresses satisfies all the conditions of equilibrium, and satisfies the boundary conditions on the upper surface. The shear stress $\sigma_{zx} = 0$, and the normal stress $\sigma_{zz} = 0$ if $|x| > a$, and $\sigma_{zz} = p = 4c$ if $|x| < a$, where $2a$ is the width of the loaded strip. The stress distribution should also satisfy the condition that the yield condition is never violated. This can be checked most conveniently by considering the Mohr circles for this case, as shown in the right half of Fig. 39.3. In order that all circles remain within the yield envelope the value of the load p should be such that $p < 4c$. The stress distribution satisfies all the conditions for a statically admissible stress field, and it can be concluded that $p = 4c$ is a lower bound for the failure load. If the true failure load is denoted by p_c , it now has been shown that

$$p_c \geq 4c. \tag{39.4}$$

It is possible that by considering more than two discontinuity lines slightly higher lower bounds can be found. This will not be investigated here, however.

Another method to obtain a statically admissible stress field is to use an elastic solution, when available. Such a solution satisfies the equilibrium equations and the

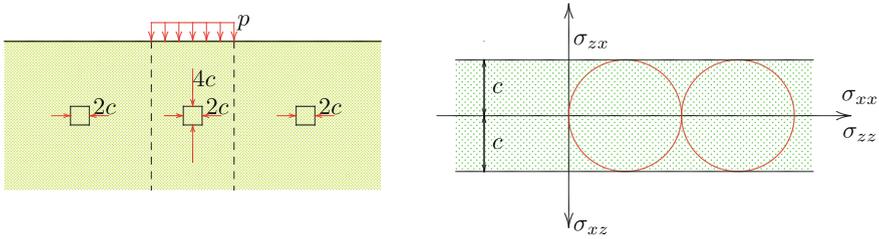


Fig. 39.3 Equilibrium system

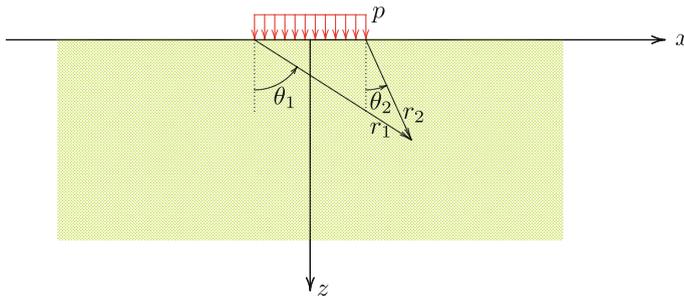


Fig. 39.4 Elastic solution

boundary conditions. It also satisfies Hooke’s law and the compatibility equations, which is not required for a statically admissible stress field, but not forbidden either. If the stress field is such that the maximum shear stress is not larger than the strength c , a lower bound of the failure load has been obtained. For the case of a strip load, see Fig. 39.4, the elastic solution has been given in Chap. 29. It can be shown that the maximum shear stress is

$$\tau = \frac{p}{\pi} | \sin(\theta_1 - \theta_2) | . \tag{39.5}$$

This equation can be derived from the formulas (29.4)–(29.6) by noting that

$$\tau^2 = \left(\frac{\sigma_{xx} - \sigma_{zz}}{2} \right)^2 + \sigma_{xz}^2 . \tag{39.6}$$

The maximum value of $| \sin(\theta_1 - \theta_2) |$ is 1, so that the maximum elastic shear stress is p/π . If this is taken equal to c , the load is $p = \pi c$. For this value of the load the elastic solution is a statically admissible stress field, and the corresponding load is a lower bound for the failure load, i.e.

$$p_c \geq 3.14c . \tag{39.7}$$

Unfortunately, this is a lower value than the value found before (4c), so that this elastic lower bound does not contribute to a better approximation of the failure load.

39.3 Upper Bound

An upper bound for the failure load can be obtained by considering the mechanism shown in Fig. 39.5. This mechanism consists of a displacement field in which half a circle, of radius a , rotates over a small angle, without internal deformations. This half circle slides along the remaining part of the body. The displacement field is compatible, and satisfies the boundary conditions on the displacements (that is very simple: there are none). The load corresponding to this deformation can be determined using the virtual work principle. If the circle rotates over a small angle θ , the displacement along the circle is θa . The work done by the internal stresses on the virtual deformations (which are concentrated at the circle's circumference) is, assuming that the shear stresses along the circle attain their maximum value c ,

$$\pi c a^2 \theta,$$

because the length of the circular arc is πa . The average displacement of the external load is $\frac{1}{2} a \theta$, so that the work done by the load is

$$\frac{1}{2} p a^2 \theta.$$

Equating these two forms of work gives

$$p = 2\pi c.$$

This is an upper bound for the failure load p_c ,

$$p_c \leq 6.28c. \tag{39.8}$$

A somewhat lower upper bound can be found by choosing the center of the circle somewhat higher, see Fig. 39.6. If the angle at the top is 2α and the rotation again is θ , the virtual work equation gives

Fig. 39.5 Mechanism 1

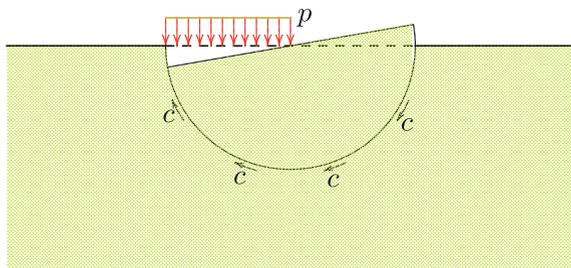
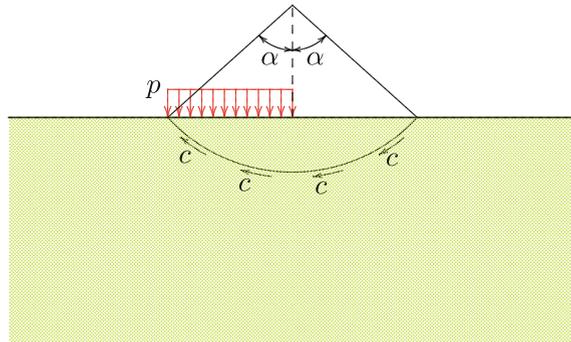


Fig. 39.6 Mechanism 2



$$2cR^2\alpha\theta = \frac{1}{2}pa^2\theta,$$

and because $a = R \sin \alpha$, in which R is the radius of the circle and a the width of the load,

$$p = \frac{4c\alpha}{\sin^2 \alpha}.$$

For $\alpha = \frac{1}{2}\pi$ the previous upper bound is recovered, but the smallest value is obtained for $\alpha = 1.165562$, or $\alpha = 66.78^\circ$. The center of the circle then is located at a height $0.429a$. The corresponding value of p is $5.52c$. This is an upper bound, hence

$$p_c \leq 5.52c. \tag{39.9}$$

It can be concluded at this stage that it has been shown that

$$4c \leq p_c \leq 5.52c. \tag{39.10}$$

In the next chapter the failure load will be limited within even closer bounds.

It should be emphasized that for the determination of an equilibrium system the deformations are not relevant. And in a mechanism internal equilibrium is irrelevant, except that the virtual work equation can be considered as the equilibrium condition corresponding to the assumed failure mode.

In the two examples considered here, of a rotation along a circular slip surface, that equilibrium condition is the equilibrium of moments with respect to the center of the circle. This is a general result: in an analysis on the basis of a circular slip surface, the failure load can be calculated by considering equilibrium of moments with respect to the center of the circle. This equation is equivalent to the virtual work equation. Because in a mechanism other equilibrium conditions are irrelevant, and need not be satisfied, it is not allowed to determine the failure load from any other type of equilibrium condition, not even moment equilibrium with respect to some other point than the circle's center.