Chapter 13
Tangent Modulus

The difference in soil behavior in compression and in shear suggests to separate the stresses and deformations into two parts, one describing compression, and another describing shear. This will be presented in this chapter. Dilatancy will be disregarded, at least initially.

13.1 Deformations

The components of the *displacement vector* will be denoted by u_x, u_y and u_z. If these displacements are not constant throughout the field there will be *deformations*, or *strains*. In Fig. 13.1 the strains in the x, y-plane are shown.

The change of length of an element of original length Δx, divided by that original length, is the horizontal strain ε_{xx}. This strain can be expressed into the displacement difference, see Fig. 13.1, by

$$\varepsilon_{xx} = \partial u_x / \partial x.$$

The change of length of an element of original length Δy, divided by that original length, is the vertical strain ε_{yy}. Its definition in terms of the displacement is, see Fig. 13.1,

$$\varepsilon_{yy} = \partial u_y / \partial y.$$

Because u_x can increase in y-direction, and u_y in x-direction, the right angle in the lower left corner of the element may become somewhat smaller. One half of this decrease is denoted as the *shear strain* ε_{xy},

$$\varepsilon_{xy} = \frac{1}{2}(\partial u_x / \partial y + \partial u_y / \partial x).$$

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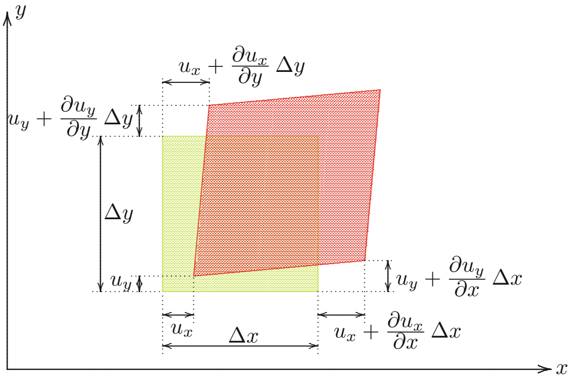


Fig. 13.1 Strains

Similar strains may occur in the other planes, of course, with similar definitions. In the general three dimensional case the definitions of the strain components are

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_x}{\partial x}, & \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right),\\
\varepsilon_{yy} &= \frac{\partial u_y}{\partial y}, & \varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right),\\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z}, & \varepsilon_{zx} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right).
\end{aligned}
\tag{13.1}
$$

All derivatives, $\partial u_x/\partial x$, $\partial u_x/\partial y$, etc., are assumed to be small compared to 1. Then the strains are also small compared to 1. Even in soils, in which considerable deformations may occur, this is usually valid, at least as a first approximation.

The volume of an elementary small block may increase if its length increases, or it width increases, or its height increases. The total volume strain is the sum of the strains in the three coordinate directions,

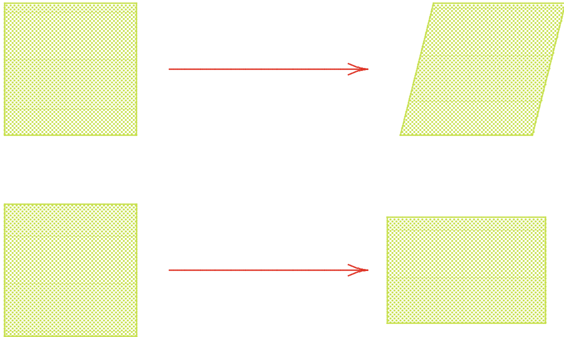
$$
\varepsilon_{\text{vol}} = \frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}. \tag{13.2}
$$

This volume strain describes the *compression* of the material, if it is negative.

The remaining part of the strain tensor describes the *distortion*. For this purpose the *deviator strains* are defined as

$$
\begin{aligned}
e_{xx} &= \varepsilon_{xx} - \frac{1}{3}\varepsilon_{\text{vol}}, & e_{xy} &= \varepsilon_{xy},\\
e_{yy} &= \varepsilon_{yy} - \frac{1}{3}\varepsilon_{\text{vol}}, & e_{yz} &= \varepsilon_{yz},\\
e_{zz} &= \varepsilon_{zz} - \frac{1}{3}\varepsilon_{\text{vol}}, & e_{zx} &= \varepsilon_{zx}.
\end{aligned}
\tag{13.3}
$$

Fig. 13.2 Distortion



These deviator strains do not contain any volume change, because $e_{xx} + e_{yy} + e_{zz} = 0$.

In a similar way *deviator stresses* can be defined,

$$\begin{aligned} \tau_{xx} &= \sigma_{xx} - \overline{\sigma}, & \tau_{xy} &= \sigma_{xy}, \\ \tau_{yy} &= \sigma_{yy} - \overline{\sigma}, & \tau_{yz} &= \sigma_{yz}, \\ \tau_{zz} &= \sigma_{zz} - \overline{\sigma}, & \tau_{zx} &= \sigma_{zx}. \end{aligned} \tag{13.4}$$

Here $\overline{\sigma}$ is the *isotropic stress*,

$$\overline{\sigma} = \tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}). \tag{13.5}$$

The isotropic stress $\overline{\sigma}$ is the average normal stress. In an isotropic material volume changes are determined primarily by changes of the isotropic stress. This means that the volume strain ε_{vol} is a function of the isotropic stress $\overline{\sigma}$ only.

Even though this may seem almost trivial, for soils it is in general not true, as it excludes dilatancy and contractancy. It is nevertheless assumed here, as a first approximation.

The remaining part of the stress tensor, after subtraction of the isotropic stress, see (13.4), consists of the deviator stresses. These are responsible for the distortion, i.e. changes in shape, at constant volume.

There are many forms of distortion: shear strains in the three directions, but also a positive normal strain in one direction and a negative normal strain in a second direction, such that the volume remains constant. Some of these possibilities are shown in Fig. 13.2. In the other three planes similar forms of distortion may occur.

13.2 Linear Elastic Material

The simplest possible relation between stresses and strains in a deformable continuum is the linear elastic relation for an isotropic material. This can be described by two positive constants, the *compression modulus* K and the *shear modulus* G. The compression modulus K gives the relation between the volume strain and the isotropic stress,

$$\overline{\sigma} = -K\,\varepsilon_{vol}. \tag{13.6}$$

The minus sign has been introduced because stresses are considered positive for compression, whereas strains are considered positive for extension. This is the sign convention that is often used in soil mechanics, in contrast with the theoretically more balanced sign conventions of continuum mechanics, in which stresses are considered positive for tension.

The shear modulus G (perhaps distortion modulus would be a better word) gives the relation between the deviator strains and the deviator stresses,

$$\tau_{ij} = -2\,G\,e_{ij}. \tag{13.7}$$

Here i and j can be all combinations of x, y or z, so that, for instance, $\tau_{xx} = -2\,G\,e_{xx}$ and $\tau_{xy} = -2\,G\,e_{xy}$. The factor 2 appears in the equations for historical reasons.

In applied mechanics the relation between stresses and strains of an isotropic linear elastic material is usually described by *Young's modulus* E, and *Poisson's ratio* ν. The usual form of the equations for the normal strains then is

$$\begin{aligned}
\varepsilon_{xx} &= -\frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})],\\
\varepsilon_{yy} &= -\frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})],\\
\varepsilon_{zz} &= -\frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})].
\end{aligned} \tag{13.8}$$

The minus sign has again been introduced to account for the sign convention for the stresses of soil mechanics.

It can easily be verified that the Eq. (13.8) are equivalent to (13.6) and (13.7) if

$$K = \frac{E}{3(1-2\nu)}, \tag{13.9}$$

$$G = \frac{E}{2(1+\nu)}. \tag{13.10}$$

For the description of compression and distortion, which are so basically different in soil mechanics, the parameters K and G are more suitable than E and ν. In continuum mechanics they are sometimes preferred as well, for instance because it can be argued, on thermodynamical grounds, that they both must be positive, $K > 0$ and $G > 0$.

13.3 A Non-linear Material

In the previous chapter it has been argued that soils are non-linear and non-elastic. Furthermore, soils are often not isotropic, because during the formation of soil deposits it may be expected that there will be a difference between the direction of deposition (the vertical direction) and the horizontal directions. As a simplification this anisotropy will be disregarded here, and the irreversible deformations due to a difference in loading and unloading are also disregarded. The behavior in compression and distortion will be considered separately, but they will no longer be described by constant parameters. As a first improvement on the linear elastic model the modulus will be assumed to be dependent upon the stresses. A non-linear relation between stresses and strains is shown schematically in Fig. 13.3. For a small change in stress the tangent to the curve might be used. This means that one could write, for the incremental volume change,

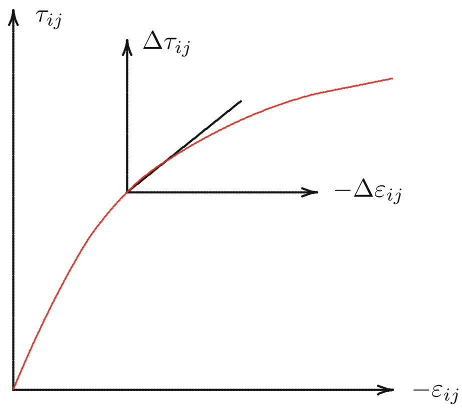
$$\Delta\overline{\sigma} = -K\,\Delta\varepsilon_{vol}, \tag{13.11}$$

Similarly, for the incremental shear strain one could write

$$\Delta\tau_{ij} = -2\,G\,\Delta e_{ij}. \tag{13.12}$$

The parameters K and G in these equations are not constants, but they depend upon the initial stress, as expressed by the location on the curve in Fig. 13.3. These type of constants are denoted as *tangent moduli*, to indicate that they actually represent the tangent to a non-linear curve. They depend upon the initial stress, and perhaps also on some other physical quantities, such as time, or temperature. As mentioned in the previous chapter, it can be expected that the value of K increases with an increasing value of the isotropic stress, see Fig. 12.3. Many researchers have found,

Fig. 13.3 Tangent modulus



from laboratory tests, that the stiffness of soils increases approximately linear with the initial stress, although others seem to have found that the increase is not so strong, approximately proportional to the square root of the initial stress. If it is assumed that the stiffness in compression indeed increases linearly with the initial stress, it follows that the stiffness in a homogeneous soil deposit will increase about linearly with depth. This has also been confirmed by tests in the field, at least approximately.

For distortion it can be expected that the shear modulus G will decrease if the shear stress increases. It may even tend towards zero when the shear stress reaches its maximum possible value, see Fig. 12.5.

It should be emphasized that a linearization with two tangent moduli K and G, dependent upon the initial stresses, can only be valid in case of small stress increments. That is not an impractical restriction, as in many cases the initial stresses in a soil are already relatively large, because of the weight of the material. It should also be mentioned, however, that many effects have been disregarded, such as anisotropy, irreversible (plastic) deformations, creep and dilatancy. An elastic analysis using K and G, or E and ν, at its best is merely a first approximate approach. It may be quite valuable, however, as it may indicate the trend of the development of stresses. In the last decades of the 20th century more advanced non-linear methods of analysis have been developed, for instance using finite element modelling, that offer more realistic computations.

Example 13.1 Consult a Handbook of Physics, a Handbook of Engineering, or an Encyclopedia, and search for a chapter on Young's modulus. Note that such a chapter usually presents useful definitions of the quantities involved, and often also contains a table of values for a large number of materials, including construction materials such as steel, concrete and wood. Some engineers from disciplines other than geotechnical engineering may be surprised that these tables do not give values for soils such as sand or clay.

The reason for this is that the stiffness of soils depends upon the initial stress, as presented in this chapter and the previous one. Or, in other words, Hooke's law does not apply to soils. Only for small stress increments Hooke's law and an appropriate value of Young's modulus E may be used, with the modulus practically proportional to the initial stress level.

Problem 13.1 A colleague in a foreign country reports that the Young's modulus of a certain layer has been back-calculated from the deformations of a stress increase due to a surcharge, from 20 to 40 kPa. This modulus is given as $E = 2000\,\text{kPa}$. A new surcharge is being planned, from 40 to 60 kPa, and your colleague (who is not a geotechnical engineer) asks your advice on the value of E to be used then. What is your suggestion?

Problem 13.2 A soil sample is being tested in the laboratory by cyclic shear stresses, of constant amplitude. In each cycle there are relatively large shear strains. What do you expect for the volume change in the 100th cycle? And what would that mean for the value of Poisson's ratio ν?