Chapter 8
Dispersion and Waves on Water

Abstract This chapter asks how the time development of a wave may be described numerically. The algorithm may offer a better understanding of wave motion than the traditional treatment of the topic. We proceed by discriminating between phase and group velocities and introduce the concept of dispersion. Numerical modelling of dispersion is described in detail, computer programs are provided, and the calculations demonstrate distortion of pulses of waves when they pass through a dispersive medium. Finally, we discuss various phenomena related to gravity-driven surface waves on water, based on a formula for phase velocity of waves on water. As a curiosity, we present at the very end a fun experiment with an oscillating water drop on a hot surface.

8.1 Introduction

Waves on water and sea have fascinated people through the ages. There exists a panoply of waveforms, and the underlying physics is so complex that even today it is almost impossible to make calculations on swirling waves like those illustrated by Katsushika Hokusai almost 200 years ago; see Fig. 8.1.

The waves we treat in this chapter are extremely simple in comparison. Nevertheless, we hope that even our simple descriptions can give you a much deeper understanding of the *phenomenon of waves* than you had prior to reading this chapter, which has three main themes: numerical calculation of the time evolution of a wave, dispersion including differences between phase and group velocities, and a review of gravity-driven waves on water.

Before starting a more thorough analysis, we will undertake a brief recapitulation of oscillations and waves in general. A feature common to all such phenomena is that:

- There is an equilibrium state of the system when oscillations and waves have died out.
- There is a "restoring force" that tries to bring the system back to equilibrium when it is not at equilibrium.

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Fig. 8.1 Real waves are extremely complex, like "The Great Wave off Kanagawa". Katsushika Hokusai, Public Domain [1]

- There is an "inertial force" that causes the system to go past the equilibrium state even though the restoring force here is equal to zero.

For a swinging pendulum, the restoring force is a component of gravity; for waves on a string, the tension on the string acts as the restoring force. For sound waves in air or a liquid, pressure differences provide the restoring force through the compression of parts of the volume. The "inertial force" in all these examples is that expressed by Newton's first law. For surface waves on water, there are *two* restoring forces, namely gravity and surface tension.

8.2 Numerical Study of the Time Evolution of a Wave

It is very difficult to understand the mechanisms that lie behind the temporal development of a wave by starting from the wave equation and relying solely on analytical mathematics. If your repertoire consists of only analytical mathematics, you will find it difficult to understand why initial conditions are so crucial to how a wave develops, and how the boundary conditions affect the time development of the wave in detail. Instead, we will use numerical methods to review the mechanisms that govern the time evolution of a wave.

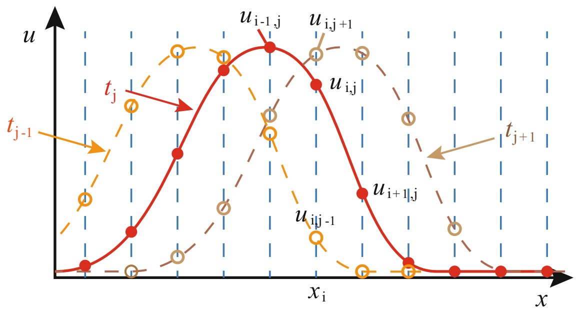

Fig. 8.2 Using numerical methods, a wave is described only at discrete positions in space and at discrete instants in time. Here one and the same wave are indicated at three different times. The first subscript specifies the position index, and the second subscript specifies the time index

There are several reasons for presenting such a review. The most important is to bring to the fore the underlying algorithm because it can provide a better understanding of wave motion in general.

The starting point is the one-dimensional wave equation for a nondispersing medium (explained later in the chapter):

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \ .$$

In a numerical calculation, the solution is stated only at discrete instants and positions:

$$u(x, t) \rightarrow u(x_i, t_j) \equiv u_{i,j}$$

where

$$x_i = x_0 + i\Delta x, \quad (i = 0, 1, 2, \ldots, N-1) \ ,$$

and

$$t_j = t_0 + j\Delta t, \quad (j = 0, 1, 2, \ldots, M-1) \ .$$

Figure 8.2 illustrates how a wave is described numerically. For each instant of time, a numerical string describes the amplitude at the selected spatial positions. In the figure, parts of the position data points are displayed for three different instants.

In the chapter on numerical methods earlier in the book, it was shown that the second derivative can be expressed in discrete form as follows:

$$\frac{\partial^2 u}{\partial x^2} \equiv u_{xx}(x_i, t_j)$$
$$= \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{\Delta x^2} \ .$$

This can be expressed more succinctly as:

$$u_{xx,i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \ . \tag{8.1}$$

In a similar way, the double derivative with respect to time can be expressed as:

$$u_{tt,i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2} \ . \tag{8.2}$$

The discretized version of the whole wave equation takes the form:

$$u_{tt,i,j} = v^2 u_{xx,i,j} \ . \tag{8.3}$$

Setting Eq. (8.2) in Eq. (8.3) and rearrangement of the terms gives:

$$u_{i,j+1} = u_{i,j} + (u_{i,j} - u_{i,j-1}) + (\Delta t \ v)^2 u_{xx,i,j} \ .$$

The expression shows that if we know the wave at an instant and at the preceding instant, we can calculate the amplitude of the wave at the next instant by using our prescription. This is an important formula that we should dwell on:

The algorithm to calculate how a wave evolves in time and space is given by the equation:

$$u_{i,j+1} = u_{i,j} + (u_{i,j} - u_{i,j-1}) + (\Delta t \ v)^2 u_{xx,i,j} \ . \tag{8.4}$$

These terms are actually quite easy to understand:

- The first term on the right-hand side states that we must begin with the current amplitude at a point in the wave when we calculate the amplitude for the next instant.
- The second term corresponds to the assumption that the time derivative of the amplitude at our given point of the wave will be about the same at the next instant as it was in the previous one. This is the "inertial term" corresponding to Newton's first law.
- The third term states that if the wave in our given point bulges (often bulging away from the equilibrium state), there is a "restoring force" that tries to pull the system back to the equilibrium state. See Fig. 8.2. This restoring force is closely related to the phase velocity of the wave. In the expression, the phase velocity appears in the second power. The phase velocity is therefore determined by how powerfully the *neighbourhood* affects the motion of any selected point in the wave. The algorithm can be visualized as shown in Fig. 8.3.

The algorithm in Eq. (8.4) shows that if we know the wave at all positions at an instant t_j and the wave as it was at the preceding t_{j-1}, then can we calculate the wave

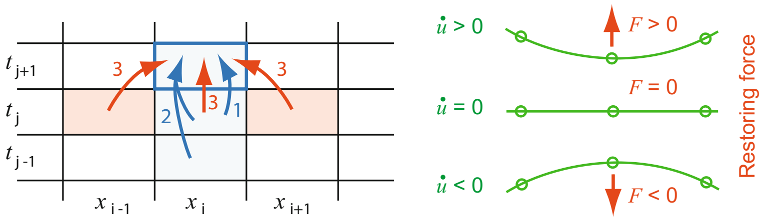

Fig. 8.3 Illustration of the cardinal algorithm that can be used for calculating the time development of a one-dimensional wave when the initial and boundary conditions are given. New amplitude at a particular point is determined by: (1) the amplitude "now" at that point; (2) the approximation that the velocity at the point will be the same at the next instant as in the previous; and (3) the restoring force from the nearest neighbours to the point, which will increase or decrease the change in position according to the sign of the curvature of the restoring force

as it will be at the next instant t_{j+1}. There are hurdles to be jumped over, presented by the initial conditions and boundary conditions, and we will get back to these shortly.

Equation (8.4) is probably the easiest expression to use, if we want to *understand* the rationale behind the algorithm developed below. The expression on the right-hand side of Eq. (8.4) is not suitable for the design of the program code itself. It is advantageous to put Eq. (8.1) into Eq. (8.4), and the result, after some rearrangement, comes out to be:

$$u_{i,j+1} = 2\left[1 - \left(\frac{v\Delta t}{\Delta x}\right)^2\right] u_{i,j} - u_{i,j-1} + \left(\frac{v\Delta t}{\Delta x}\right)^2 (u_{i+1,j} + u_{i-1,j}) \;. \tag{8.5}$$

Problem at the boundary of the region under consideration

Equation (8.5) is the central expression we use to calculate how a wave evolves in time, but the expression contains some important details that we need to look into. When we start the calculations, we assume that we know the initial conditions along the part of the wave we describe at the start of the calculations. For example, the amplitude at the instant $j = 0$ given by $\{u_{i,0}\}$ for $i = 0, 1, 2, \ldots, N$. But Eq. (8.5) also includes $x_{i+1,0}$ and $x_{i-1,0}$. The points $x_{-1,0}$ and $x_{N+1,0}$ do not exist, so our algorithm must employ some artifice for dealing with these terms. In other words, we must supply so-called boundary conditions for the particular problem at hand. These conditions apply at all instants in the calculations.

In practice, it may be almost impossible to find boundary conditions that are perfect for the calculations we want to make. The most common boundary conditions are "open/free" and "closed/fixed". In the former case, we put $x_{-1,j} = x_{0,j}$ and $x_{N+1,j} =$

$x_{N,j}$, in the latter case we set $x_{-1,j} = x_{N+1,j} = 0$. For a concrete calculation, we must choose how to set the boundary conditions, and in many cases, the response depends strongly on the physical system we try to describe.

For a wave that has zero amplitude at the boundary, we can consider, without incurring any error, the time evolution of the wave until the wave has spread to the edge of the calculation range. By making the calculation region large enough and limiting the time for which we consider the wave evolution, calculations of localized waves can be good even without worrying about boundary effects.

Problem with the starting instant

Another source of difficulty in Eq. (8.5) is the term $u_{i,j-1}$. If we start the calculations at time $t = 0$, there is no $u_{i,-1}$. Therefore, we get trouble already at the start of the calculations.

On the other hand, in all differential equations, we must use the initial conditions to arrive at the particular solution we seek. For a wave, it means that the initial conditions, for example, may be stated as the amplitude at all positions at $t = 0$, along with the time derivative of the amplitude at all positions at the same time. Based on this information, we can calculate positions at the starting instant and approximate positions one time-step earlier.

There are also other ways to specify initial conditions and procedures that can be followed for taking advantage of the initial conditions. We confine ourselves to the amplitude and its time derivative, both as a function of position.

The time derivative of the result at the point i can be specified as follows:

$$\dot{u}_{i,j} \equiv \left(\frac{\partial u}{\partial t}\right)_{i,j} \approx \frac{u_{i,j} - u_{i,j-1}}{\Delta t} .$$

Consequently,

$$u_{i,j-1} = u_{i,j} - \Delta t \; \dot{u}_{i,j} . \tag{8.6}$$

For $j = 0$ we get:

$$u_{i,-1} = u_{i,0} - \Delta t \; \dot{u}_{i,0} . \tag{8.7}$$

Threading together

Assume that the initial conditions are given by the amplitude $\{u_{i,0}\}$ at all positions along the wave and the time derivative of the amplitude $\{\dot{u}_{i,0}\}$ at all positions along the wave at the start time. Then Eq. (8.5) in combination with Eq. (8.7) can be used for the starting instant in the calculations. Equation (8.5) can be used for the remaining instants as many times as we wish. Along the way, one must take account of the boundary conditions.

8.2.1 An Example Wave

As an example, let us calculate how a Gaussian wave moves on a string. The initial conditions are a snapshot of the wave as it is at one point (both position and speed!), and we will follow its development in time.

The displacement as a function of position along the string is given analytically by:

$$u(x,t) = A \exp\left[-\frac{(x-vt)^2}{2\sigma^2}\right] = A \exp\left[f(x,t)\right] \tag{8.8}$$

where we have used the notation $\exp[f(x,t)]$ instead of the notation $e^{f(x,t)}$, since the expressions in this chapter are more complex than in previous chapters.

The time derivative of $u(x,t)$ comes out to be:

$$\begin{aligned}
\frac{\partial u}{\partial t} \equiv \dot{u} = A \exp\left[f(x,t)\right] \frac{\partial f}{\partial t} &= A \exp\left[-\frac{(x-vt)^2}{2\sigma^2}\right](-2)\left(\frac{x-vt}{\sqrt{2}\sigma}\right)\left(-\frac{v}{\sqrt{2}\sigma}\right) \\
&= \frac{(x-vt)v}{\sigma^2} A \exp\left[-\frac{(x-vt)^2}{2\sigma^2}\right], \\
&= \frac{v}{\sigma^2}(x-vt)u\ .
\end{aligned} \tag{8.9}$$

We choose to describe the wave on a string that is long in relation to the width of the Gaussian function, and we choose to follow the wave only so long that it does not come too close to a boundary. We use in the program a complete adherence to the endpoints along the way in the calculations.

We select the following parameters $A = 1$, $\sigma = 2\sqrt{2}$, $v = 0.3$ and allow x to cover the range from -20 through $+20$ in 400 equal steps. We try with $\Delta t = 0.1$ and follow the movement for 300 time increments. No units are provided, but we assume that all units are SI devices.

A computer program written in Matlab is given below. The code is also available at the "Supplementary material" web page for this book at http://www.physics.uio.no/pow. The program performs the calculations based on the expressions given above.

```
function waveAnimationX

% Generate position array
delta_x = 0.1;
x = -20:delta_x:20;
n = length(x);
nx = 1:1:n;  % Just for plotting purposes

% Generate and plot the wave at t=0
sigma = 2.0*sqrt(2.0);
u = exp(-(x/sigma).*(x/sigma)/2.0); % Gaussian shape
plot(nx,u,'-r');
```

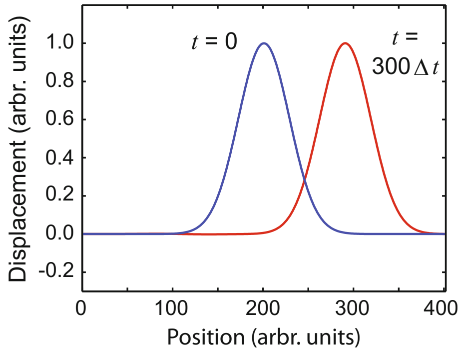

Fig. 8.4 Profiles of the wave at the start of the calculation and after a lapse of 300 time-steps for initial conditions that ensure a constant wave shape as the wave evolves in time

```
axis([1 n+1 -0.3 1.2]) % Ease comparison with animation
figure;

% Generate parameters and time derivative of the wave at t=0
v = 0.5;  delta_t = 0.1;
factor = (delta_t*v/delta_x)^2;
dudt = (v/(sigma*sigma))*x.*u;

% Calculate effective initial conditions:
u_jminus1 = u - delta_t*dudt;
u_j = u;

% The animation (one thousand time steps):
for t = 1:1000
    u_jplus1(2:n-1) = (2*(1-factor))*u_j(2:n-1) - ...
       u_jminus1(2:n-1) + factor.*(u_j(3:n)+u_j(1:n-2));

    % Handle boundary problem (fixed boundary)
    % u_j(-1) = u_j(n+1) = 0
    u_jplus1(1) = ...
      (2*(1-factor)).*u_j(1) - u_jminus1(1) + factor.*u_j(2);
    u_jplus1(n) = ...
      (2*(1-factor)).*u_j(n) - u_jminus1(n) + factor.*u_j(n-1);

    plot(u_j);
    axis([0 n+1 -0.3 1.2])
    drawnow;

    u_jminus1 = u_j;
    u_j = u_jplus1;
end;
```

Figure 8.4 shows the wave at the start and after the passage of 300 time-steps. We see that the wave moves to the right (positive v) and that the waveform remains unchanged.

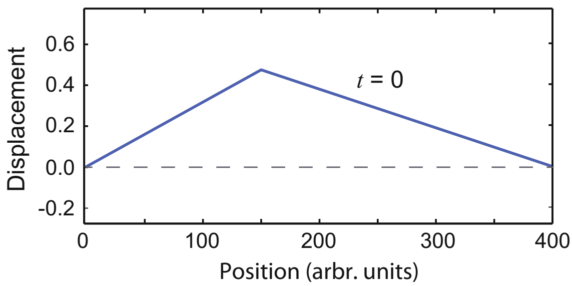

Fig. 8.5 Profiles of the initial position of the string on a guitar just before it is released and made to oscillate

In an exercise at the end of the chapter, you are asked to investigate how the wave evolves if we use a $\dot{u}$ which is either too small or too large compared to what it should have been. Which term in Eq. (8.9) is now incorrect if we want to preserve the waveform as the wave evolves? Would it be possible to explain the pattern observed in the simulations if you consider the initial condition as a sum of two different waves? It is crucial that you carry out this exercise and try to explain in your own words the mechanisms behind time evolution of a wave.

You are also urged to modify the code so that you can handle a case where the wave hits an interface between two media with different impedances (different phase velocities). It is recommended that you complete the exercise, for that would provide you with a significantly better understanding of waves.

Finally, another highly recommended exercise: When we play the guitar, we pull at the string so that the initial condition is a slanted triangle (with straight edges, see Fig. 8.5) and no motion before we release the string. Use the reasoning and algorithm that lies behind Fig. 8.3 to suggest how the string will move afterwards! It will reveal whether you have understood the algorithm or not!

Then perform a numerical calculation of the motion of the guitar string. It is easy since the string is at rest before you release it, and the endpoints are fixed (no motion is permitted). You may be surprised by the result!

It should be noted that the algorithm we use does not allow for any rigidity in the string. If the string has a certain stiffness, segments a little further than the neighbouring point will also affect the motion. A true guitar string will therefore get a little different motion than our calculations show, at least if we follow the motion over several periods. However, if we use a rubber band as a guitar string, we will observe a pretty close fit with the calculation, because the band has negligible stiffness. There are nice YouTube videos (shot with high-speed camera) showing the motion of a rubber band. Examples are "Motion of Plucked String" by Dan Russell and "Slow motion: Rubber string pulled and released" by Ravel Radzivilovsky. It is fun to compare your own calculations with these videos!

8.3 Dispersion: Phase Velocity and Group Velocity

In the previous section, we studied the mechanisms which govern the time development of a one-dimensional wave. We initially said that the calculations dealt with an idealized situation in which there was no *dispersion*. "No dispersion" means that a wave moves at the same speed no matter what its wavelength. In the calculations, the wave rate v was a constant.

For many physical wave phenomena, the restoring force will vary with the wavelength. In such situations, we say that the medium is *dispersive*. The multicoloured band we get when we send white light through a glass prism is an example of the phenomenon called *dispersion*. The spectrum is a consequence of the fact that light of different wavelengths travels with different speeds through the glass. This is the dispersion property of glass for light.

Let us take a closer look at this. We know that the refractive index of glass varies with the wavelength of light (see Fig. 8.6 for different types of glass). The refractive index increases as the frequency increases (wavelength decreases).

In Chap. 9, we will show at the phase velocity of the electromagnetic waves (light) in glass is given by the relation

$$v_\mathrm{p} = c_\mathrm{glass} = c_0/n(\lambda)$$

where c_0 is the light velocity in vacuum, $c_\mathrm{glass} = v_\mathrm{p}$ is the light velocity in glass, which by definition is the phase velocity of light in glass. $n(\lambda)$ is the refractive index which is wavelength dependent [see Eq. (9.36)].

> Phase velocity is the velocity a constant intensity laser beam (or a perfect harmonic wave) will have when it travels through a medium.

It follows from the data plotted in Fig. 8.6 that the phase velocity decreases as the wavelength decreases. Such a behaviour is called *normal dispersion*.

A slightly different graphic representation is often used to display dispersive behaviour. The alternative is to plot the angular frequency ω as a function of the wavenumber k. For a usual monochromatic wave $A\cos(kx - \omega t)$, the velocity (i.e., phase velocity) is given by:

$$v_\mathrm{p} = \frac{\omega}{k}$$

If we want to send information from one place to another, we cannot just send a constant intensity laser beam. We must make changes in the light output, and the information is conveyed through the changes.

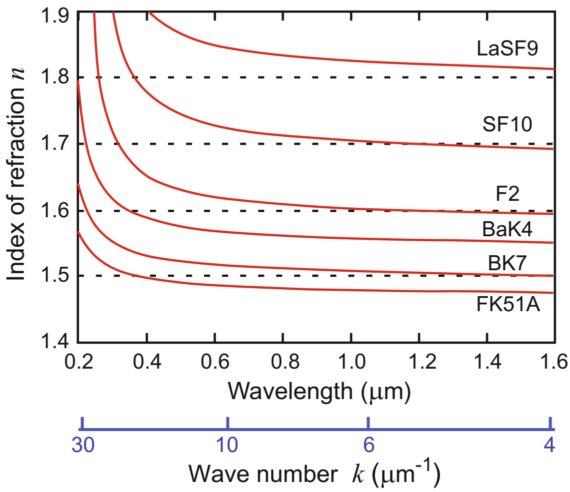

Fig. 8.6 Refractive index of light, from UV to IR, in various types of glass

In an optical fibre, we often send a number of light pulses one after the other. In the case of radio telecommunications, we also make variations in the radio wave that causes the wave to be seen as different "groups" of waves that come one after the other. It is remarkable that these pulses or wave groups propagate through the fibre or the atmosphere at slightly different velocities than a constant intensity laser beam would propagate. The pulses or wave groups propagate with what is called *group velocity*.

There is usually little difference between phase velocity and group velocity for electromagnetic waves. However, when we throw a pebble in a still body of water, the group velocity will only be half the velocity of water waves if a wave-making machine had generated continuous waves of about the same wavelength as we saw in the rings after the stone hit the water. It is therefore important to distinguish between phase velocity and group velocity!

It is dispersion that accounts for the difference between phase and group velocity. The connection between phase velocity v_p, (angular) frequency ω and wavenumber k is:

$$\omega = v_p k$$

When there is no dispersion, v_p is independent of k, and if we plot ω as a function of k, we get a straight line.

With dispersion, however, v_p will be a function of the wavelength and hence k. We can then write:

$$\omega(k) = \mathscr{F}(k) \ .$$

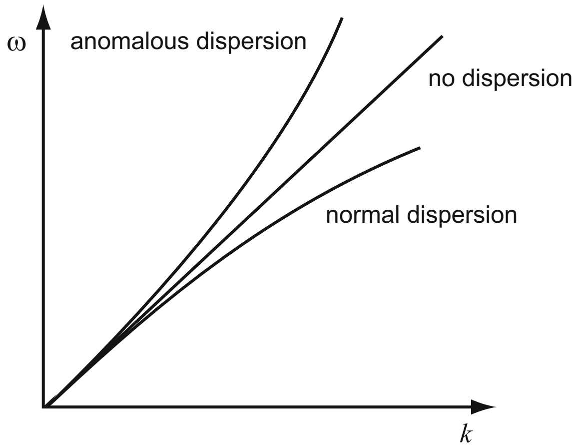

Fig. 8.7 Relation between the angular frequency ω and the wavenumber k for a given medium is called the *dispersion relation* for the medium. We distinguish between three different classes of media, as indicated in the figure. Note: The three curves represent completely different physical action mechanisms, so the three curves do not have to coincide for low k values. It is the form (curvature) that is important!

where $\mathscr{F}$ is some function. Then a plot of ω will no longer be a straight line.

We call $\mathscr{F}$, which gives us the relationship between $\omega(k)$ and k, the *dispersion relation* for the medium in question. For a dispersive medium, an ω versus k plot will be a curved line, as shown in Fig. 8.7. When the curve bends downwards, the phase velocity decreases with the wavenumber (the wavelength becoming smaller). This is called *normal dispersion*. When the curve bends upwards, the phase velocity increases with the wavenumber (the wavelength becoming smaller). This is called *anomalous dispersion*.

It can be shown that the group velocity is determined by the slope of the dispersion relation in the region under consideration:

$$v_g = \frac{\partial \omega}{\partial k} \, . \tag{8.10}$$

It can be shown that such a definition corresponds to the velocity of the "envelope" of a composite wave packet, at least where the envelope has a Gaussian shape. This corresponds to what we associate with "group velocity". For more complicated "envelopes" it is not always easy to specify a group velocity precisely, since the actual shape of the envelope will change as it moves.

The fact that the group velocity is the derivative of the dispersion relation $\omega(k)$ opens up interesting possibilities. We will use it several times in this book.

For example, let us find an expression for the variation of group velocity with the refractive index. The starting point is then the relationship:

$$\frac{\omega}{k} = \frac{c_0}{n(k)} \, .$$

Whence follows the relations:

$$v_g = \frac{\partial \omega}{\partial k} = -\frac{c_0}{n^2}\frac{\partial n}{\partial k} + \frac{c_0}{n} ,$$

$$v_g = v_p(\omega)\left(1 - \frac{k}{n}\frac{\partial n}{\partial k}\right) . \tag{8.11}$$

In normal dispersion, $dn/d\omega > 0$, which means that $v_g < v_p$, that is, the group velocity is less than the phase velocity. If we use the data from Fig. 8.6, we see that there is very little difference between the phase velocity and the group velocity when we transmit light through glass—at least as long as the wavelength is greater than 400 nm (visible light and IR). On the other hand, dispersion becomes a major problem if we try to send light of shorter wavelengths through glass.

In modern communication, we use optical fibres and wavelengths in the IR region, where the refractive index is almost completely independent of the wavelength. Then dispersion is very small, and it allows the use of short pulses, which ensures a high data transfer rate.

8.3.1 Why Is the Velocity of Light in Glass Smaller Than That in Vacuum?

This may be an appropriate moment for injecting a small aside, since in practice it has been found that relatively few know why light travels more slowly in glass than in vacuum. A clear indication is obtained by examining the expression for the velocity of light through a medium. This expression, usually given in books of general electromagnetism, is also discussed in detail in Chap. 9 in our book, and it reads:

$$c = c_0/n = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_r \mu_0 \mu_r}}$$

where c_0 is the light velocity in vacuum, n is the refractive index, ε_0 is the permittivity in vacuum, ε_r is the relative permittivity, μ_0 is the magnetic permeability of vacuum and μ_r is the relative magnetic permeability. In glass, which is diamagnetic, μ_r is approximately equal to 1, and we get:

$$c = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_r \mu_0}} = c_0 \frac{1}{\sqrt{\varepsilon_r}} .$$

When we remember that ε_r is a measure of how much polarization (shifting of positive and negative charges in each direction) we can achieve when we put a material into an electric field, we realize that polarization of glass is the reason that light goes slower through glass than in vacuum.

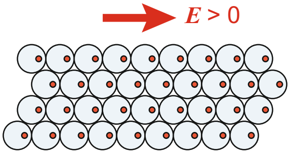

Fig. 8.8 When an electromagnetic wave passes through a slab of glass, the electromagnetic field in the wave will cause the electrons in the electron cloud around each nuclear core to shift as expected from the Coulomb force

Figure 8.8 shows what happens when an electromagnetic wave passes through glass. The electric field will alternate in a harmonic manner, with a value sometimes in one direction (across the direction of motion of the light), sometimes zero, and sometimes in the opposite direction. The electrons in the glass atoms will be affected by the electric field, and the entire "electron cloud" around each nuclear core will shift slightly relative to the core as indicated in the figure. In reality, the displacement is extremely small, since the electromagnetic field from the electromagnetic wave is usually small compared to the electrical forces between the core and electrons.[1] Yet, even in weak light, there will be a collective displacement of the electrons relative to the nuclei, and that is what really matters.

The collective displacement results in the glass being almost regarded as an antenna with oscillating currents. This oscillation in charges leads to the transmission of electromagnetic waves at the same frequency as the wave that started it all. However, we have seen in Chap. 3 (forced fluctuations) that there is generally a phase difference between the movement and the applied force. It is *the combination of the original wave and the phase shifted resonant wave from the oscillating electron clouds*, which ensures that the light velocity in glass is less than in vacuum.

It goes without saying that when the electromagnetic wave has passed the glass and gets into the air (almost like vacuum in our context), there will be no noteworthy polarization of the medium and the wave will not be delayed by the re-emitted wave. The velocity of light speed returns, of course, to (almost) the velocity of light in vacuum.

If we recall what was said in Chap. 3 about forced oscillations, we will also think of the resonance phenomenon. At certain frequencies, the amplitude became particularly large under the influence of the applied force. If you look at Fig. 8.6, you can see clear indications that something special happens to wavelengths just under 200 nm (0.2 μm). Then we are in the UV region. Several different physical processes will take place at the same time, but it may be useful to think that there will be some form of resonance in the electron oscillations around the nuclei. By thinking about the form of the resonance curve in Chap. 3, you can hopefully also imagine

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[1] In a rapid pulse laser experiment in Germany in 2013, however, the electric field was so powerful that many electrons were stripped away from the core. Then the glass is transformed from being an insulator to a good electric conductor within a few femtoseconds!.

what happens if we go through resonance and reach even shorter wavelengths. Then curves in a diagram similar to Fig. 8.6 will slope the opposite way and we get the so-called anomalous dispersion. For some materials, the supposed resonance frequency will be at much longer wavelengths, and then we can achieve anomalous dispersion even for common visible light. However, it is somewhat strained to compare dispersion unequivocally with resonance in such phenomena, because more physical interactions usually contribute.

This is one of many aspects of physics where the simple laws and patterns discussed in the early chapters of the book appear. Simple principles are often *part* of the explanation even for more complicated phenomena, but seldom the *whole* explanation!

A nice little historical episode in this context:
In Newton's corpuscular model of light, diffraction was explained by the particles being *faster* through glass than in air, but the wave description gave the opposite prediction. Measurement of the velocity of light in glass was therefore regarded, during a certain period, as an important test for seeing whether a wave model or particle model accorded better with experiments. However, we cannot measure the velocity of light velocity in a coherent monochromatic wave. We must have a "structure" in the wave that we can recognize in order to be able to measure the velocity of light. This translates into the measurement of group velocity.

However, no one was able to measure the velocity of light in this way in the eighteenth and early nineteenth centuries. Foucault was the first to carry out the experiment. That happened in 1850, and the result showed that the velocity of light in glass was smaller than that in air, which supported the wave model for light. By this time, however, most physicists had reluctantly abandoned Newton's particle model for light. Experiments of Thomas Young (1801 double-slit experiment) and a work of Fresnel around 1820 (first opposed by Poisson, but corroborated by an experiment conducted by Arago), eventually convinced physicists that the wave model of light gave a better description than particle model. Please read about "Arago spot" in Wikipedia.

8.3.2 Numerical Modelling of Dispersion

Dispersion is a phenomenon that is somewhat difficult to understand. We present here a method which can be used to model dispersion numerically. We hope that, by reading the description of the method and the results furnished by it, you will understand dispersion better. We recommend that you run the computer program and watch how the waves within a group are moving forward, backward or stand still compared to the envelope of the wave group. It is fascinating, and you can easily observe such a pattern in real life when you look, for example, at the wake behind a boat on the sea.

We start this section by pointing out that a frequency analysis of a wave may be carried out both in time domain and in space domain. It is related to Fig. 6.2 in Chap. 6.

We often use the word “wave” rather uncritically, and seldom think that a real physical wave *must* have a limited extent in time and space. This means that when we describe a wave, for example, with the following expressions:

$$y(x, t) = A\cos(k_0 x - \omega_0 t) \ ,$$

this is at best just an *approximate description* of reality within a limited range of time and space. The velocity such a wave moves with is the phase velocity $v = \omega_0/k_0$. A Fourier analysis of the time variation of the amplitude (meaning the displacement from the rest position) of such a wave at one fixed position $x = x_0$ would be

$$Y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(x_0, t)\mathrm{e}^{-\mathrm{i}\omega t}dt \tag{8.12}$$

and Y would give one sharp peak in the frequency domain for $\omega = \omega_0$. This tells us that the amplitude of the wave varies harmonically with time at the fixed position $x = x_0$ and the frequency is $f = \omega_0/2\pi$ and the time periodicity is $T = 1/f$.

We could equally well have described the wave as a snapshot at one particular time $t = t_0$. A Fourier analysis can be carried out of the amplitude *as a function of position* for this particular time. We would then have a slightly different expression:

$$Y(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(x, t_0)\mathrm{e}^{-\mathrm{i}kx}dx \ . \tag{8.13}$$

The numbers we put into the calculation would be almost identical with the numbers describing the amplitude as a function of time. So mathematically, there will be no difference (in the numbers used). As physicists, however, we need to keep track of the difference and how the analysis should be used. Even in this case Y would give one sharp peak for $k = k_0$. We call k “the wavenumber”, but that is the number of wavelengths within 2π metres and can equally well be called 2π times the “*spatial frequency*”. This tells us that the amplitude of the wave varies harmonically with *position* at the particular time $t = t_0$ and that the spatial frequency is $f_s = k/2\pi$ and the spatial periodicity is the wavelength $\lambda = 1/f_s$.

In our numerical simulations of dispersion, we will use a description based on spatial frequencies, as will be apparent in the following.

Our chosen model of a real physical wave

As mentioned above, dispersion will have no influence on the motion of a pure harmonic wave. At the same time, it is also impossible to define a “group” for a pure harmonic wave. Thus, for a simulation of dispersion, we need a different model for a physical wave.

A physical wave changes character (form) from one region in space-time to another, and it can in general be very complicated indeed.

We have chosen to base our discussion on a “wave packet” that is formed by multiplying a harmonic wave with a Gaussian envelope. We describe the wave at a

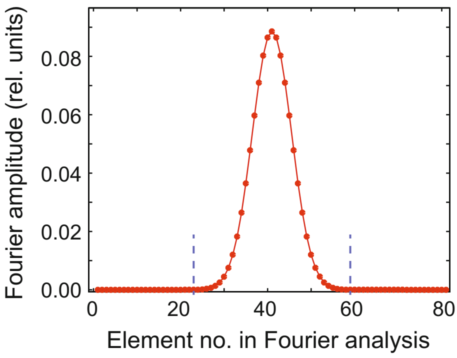

Fig. 8.9 Frequency analysis of our wave packet (absolute values). Only the small region where the coefficients are clearly different from zero is shown. Along the horizontal axis, only element number has been given, in order to pick out the indices we are interested in

particular time $t = t_0 = 0$

$$y(x, t_0) = \cos(k(x - x_r))\, e^{-\{\frac{x-x_r}{\sigma}\}^2} \,. \tag{8.14}$$

Here x_r is the position where the wave packet has its maximum and the $1/e$ width of the envelope equals $2\sigma_x$. For our choice of parameters, the wave at the starting instant ($t = 0$) is shown in the upper left of Fig. 8.10.

We may Fourier transform this description of our wave packet as described in Eq. (8.13). From Fig. 5.12 in Chap. 5 and the description in that chapter, we know that the Fourier transform of Eq. (8.14) will have contributions mainly within a band of (spatial) frequencies, which correspond to a band of wavelengths. The band for our chosen model (discrete version) is shown in Fig. 8.9.

If we also bring the inverse Fourier transform into play, we can then state that

Our model of a physical wave $y(x, t_0)$ can be described as a sum (integral) of harmonic spatial waves with different wavelengths, for wavelengths in a limited wavelength band.

The key element in our simulation of dispersion

- We now know that the wave at $t = t_0 = 0$ can be described as a sum of spatial harmonic waves with different wavelengths.
- We also know that a harmonic wave will evolve in time as if dispersion was not present.
- However, dispersion implies that the phase velocity will depend on the wavelength.

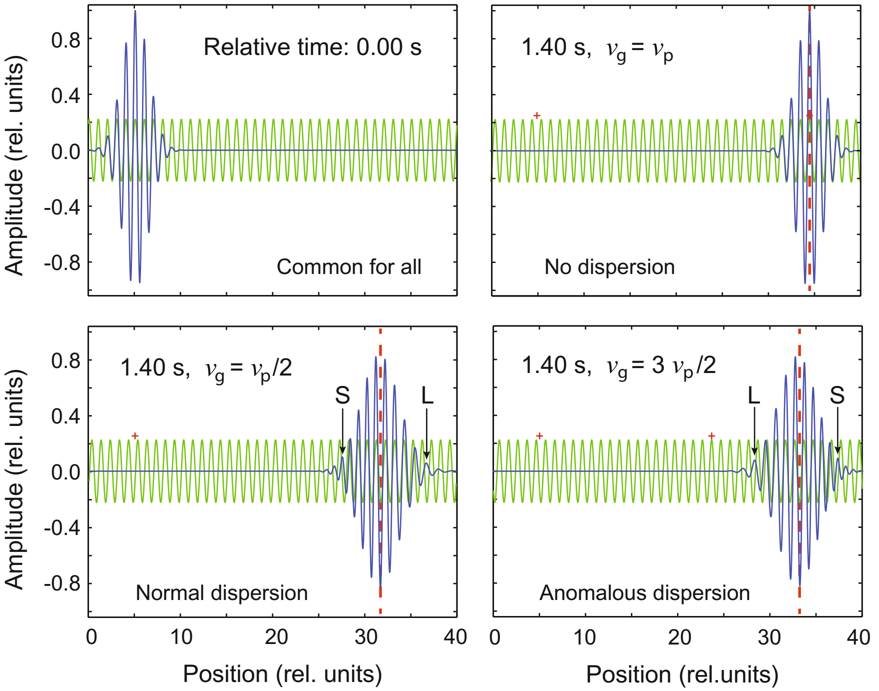

Fig. 8.10 Evolution of the wave packet with different dispersions is shown in blue. The green curve shows a wave with a wavelength corresponding to the maximum in Fig. 8.9 and is only included to show the difference between phase rate and group velocity in an animation. The wave packet at the start of time development is shown at the top left, and the next three graphs show the wave packet after it has been 1.4 s. **S** and **L** indicate shorter or larger wavelengths than the dominant one. See the text for details

- The time evolution of the total wave packet can then be calculated by adding a number of spatial harmonic waves that evolve in time with different phase velocities.

To be more specific, the last step can be implemented by replacing summation of spatial harmonic waves at one instant of time $\cos[k(i)x + \theta(i)]$ followed by a *common* time evolution—with summation of harmonic waves in space and time with *individual* time evolution and arguments $\cos[k(i)x - \omega(i)t + \theta(i)]$. However, the challenge is to determine $\omega(i)$. This is explained in the more detailed description of the actual simulation program at the end of this chapter, and it is illustrated in the Fig. 8.21 there.

Later in this chapter, we will discuss some physical wave phenomena in which the chosen behaviour is manifested.

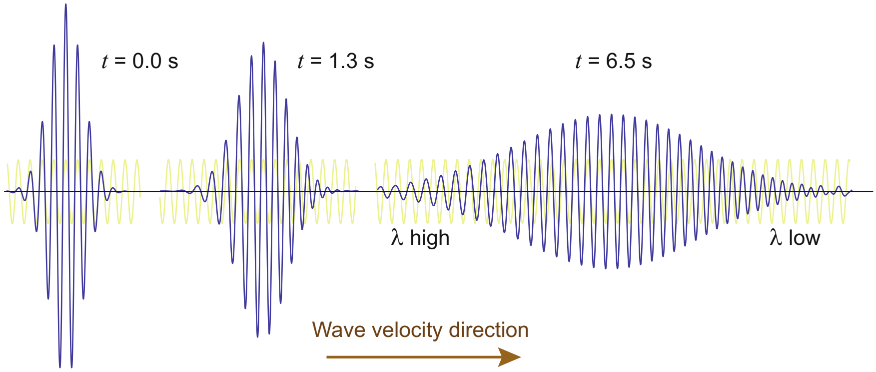

Fig. 8.11 Wave packet at the start, after it has moved (with anomalous dispersion) for a short time, and after a much longer time. Note the difference in wavelength in the front of the wave packet compared to the end. It is a fingerprint of dispersion

The results of our animation are given in three of the four plots in Fig. 8.10. A number of details emerge from the modelling/animation:

Figure 8.10 shows the important characteristics of dispersion:

- The waveform of the envelope curve does not change over time when there is no dispersion.
- When there is dispersion, the wave packet "spreads", its shape changes and the peak amplitude decreases, as shown in Fig. 8.11.
- When there is no dispersion, the group velocity equals the phase velocity, i.e. $v_g = v_p$.
- With normal dispersion, the group velocity is less than the phase velocity, more specifically $v_g = \frac{1}{2} v_p < v_p$ in our case.
- With anomalous dispersion, group velocity is greater than the phase velocity, more specifically $v_g = \frac{3}{2} v_p > v_p$ in our case.
- Group velocities are exactly as expected on the basis of Eq. (8.10).
- Although the wave packet ("the group") moves with a different velocity than the phase velocity, the *individual wave peaks* within the envelope moves approximately with the phase velocity of a wave whose wavelength corresponds to the dominant component in the frequency spectrum (shown green in Fig. 8.10).
- This means that the wave packet under the envelope curve moves forward *relative to the envelope* with normal dispersion and backward with anomalous dispersion.
- This means that at normal dispersion, wave packets will apparently disappear at the front of a wave packet as time passes and appear to grow out of nothingat the rear end of the wave packet. For anomalous dispersion, the opposite holds. See Fig. 8.12.

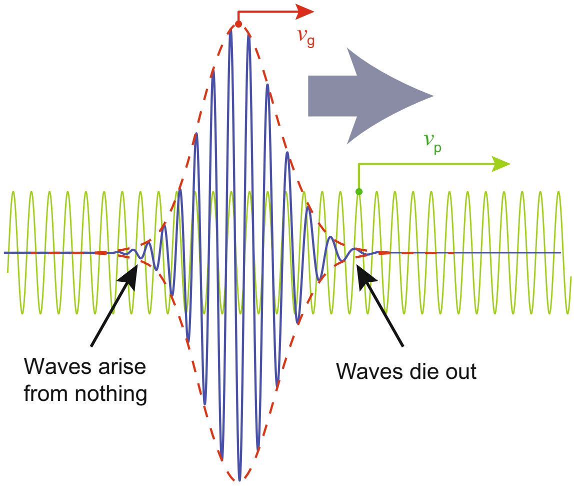

Fig. 8.12 Wave packet after it has moved for a while with normal dispersion. The phase and group velocities, v_p and v_g, respectively, are indicated by arrows

- In normal dispersion, the phase velocity for small wavenumbers, i.e. long wavelengths, is greater than the phase velocity for short wavelengths. Then the longest wavelengths will dominate the fastest part of the group, and the shortest wavelengths will dominate the last (slowest) part of the group. For anomalous dispersion, it is the opposite. In Fig. 8.10, long wavelengths are marked with L and short with S.

8.4 Waves in Water

It is time now to describe waves on the surface of water. However, we will start with qualitative descriptions before we grapple with a mathematical description where it is possible to go into more detail.

In Chap. 6, we derived the wave equation for waves on a string and waves in a medium. It would have been nice to go through a similar derivation for surface waves in water, but this will not be attempted here, since the task is rather demanding. The interested reader is referred instead to books in hydrodynamics or geophysics. We will nevertheless look at some details. In Fig. 8.13 is shown one possible model, which can be used as a basis (the model is the starting point for the derivation in, for example, the book by Persson, reference at the end of this chapter).

Here we consider a vertical volume element parallel to the wavefront has the same volume, regardless of whether it is in a trough (valley) or a crest (top). In the figure, this would mean that $V_1 = V_2$. However, since the pressure is equal to the air pressure

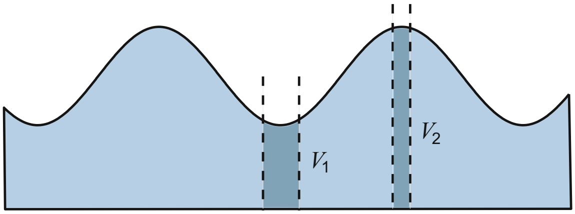

Fig. 8.13 One wave model envisages that vertical volume along the wavefront preserves its value, regardless of whether the column is in a trough (valley) or a crest (peak). The surface area of the cross sections will change, but we ignore that at first

above the water surface (approximately the same above all volume elements) and the pressure increases with the depth inside the water, the pressure at a given height above the bottom is higher in the volume element corresponding to the wave peak compared with that in the wave valley. In this way, we can regard the wave as a longitudinal pressure wave that moves with the velocity of the wave.

In Chap. 6, sound waves in air and water were described as pressure waves. The model in Fig. 8.13 looks similar to that description, but is still quite different!

For sound waves, we considered the gas or liquid as compressible fluids, that is, if we increase the pressure, the volume will decrease. The compressibility modulus was central to the derivation. Gravity, on the other hand, played no role whatsoever.

When surface waves on water are modelled, we completely ignore the compressibility. Regardless of pressure changes, a volume element retains the same volume.

In surface waves, large pressures will mean that the volume element is compressed across the wavefront, that is, in a volume element below a wave peak.

We may wonder whether it is reasonable to operate with completely different models of sound waves and surface waves, and of course there are transition zones where these descriptions will be at variance with each other. However, there are physically good reasons to operate with different models.

In the case of sound waves, we are most interested in frequencies in the audible region (and possibly ultrasound). That is, from about 20 Hz and upwards. The period is 50 ms or less (in part much less). If sound would lead to surface waves as described in this chapter, we must move significant amounts of water up to several metres in 25 ms or less! It would require enormous powers (according to Newton's second law).

On the other hand, we can transfer large amounts of water a few microns within 25 ms as required for sound waves, and still shorter times (higher audio frequencies). The powers that are needed for this task are achievable.

Surface waves on water have a much lower frequency (at least for large wave heights). Then we get time to move large amounts of water from a wave bottom to a wave peak with the available power.

One must also take into account the time scale and Newton's second law, which means that we operate with completely different models of sound waves in water and gravity-driven surface waves on water.

A better model

All in all, the model given in Fig. 8.13 does not provide a good description of surface waves. For a better description, we would like to base ourselves on one of the basic equations in fluid mechanics, namely Navier–Stokes equation:

$$\rho \left(\frac{\partial \vec{\boldsymbol{v}}}{\partial t} + \vec{\boldsymbol{v}} \cdot \nabla \vec{\boldsymbol{v}} \right) = -\nabla p + \nabla \cdot \vec{\mathscr{T}} + \vec{\mathscr{B}}$$

where ρ is mass density, $\vec{\boldsymbol{v}}$ is the flow rate, p is hydrostatic pressure, $\vec{\mathscr{T}}$ is a stress vector (may include surface tension) and $\vec{\mathscr{B}}$ stands for "body forces" that work per unit volume in the fluid. ∇ is the del operator.

It may be useful to look closely at Navier–Stokes equation and recognize that it is largely a further development of Newton's second law for a continuum fluid.

Navier–Stokes equation is nonlinear, which means that solutions of this equation do not necessarily follow the superposition principle. If two functions separately are solutions of the equation, the sum of these functions will not necessarily be a solution of the equation. Another characteristic feature of nonlinear equations is that they can have chaotic solutions, that is, solutions where we cannot predict how the solution will develop in time (in a purely deterministic way). Even the slightest change in initial conditions or boundary conditions could result in the solution after a time being having wildly different values. This has come to be called "the butterfly effect". The flap of a butterfly's wings can cause weather development after a long while to be completely different from what it would have been had the butterfly not flapped its wings.

There are some interesting mathematical challenges associated with Navier–Stokes equation today, but we will not mention it here.

My main concern is to point out that there is a wealth of different phenomena related to motion in fluids, and amazingly many physicists and mathematicians have been interested in water waves. These include Newton, Euler, Bernoulli, Laplace, Lagrange, de la Coudraye, Gerstner, Cauchy, Poisson, Fourier, Navier, Stokes, Airy, Russell, Boussinesq, Koertweg, de Vries, Zabusky, Kruskal, Beaufort, Benjamin, Feir and others. We are talking about monster waves, tsunamis, solitary waves, etc. The field has a rich tradition, also in the Norwegian research milieu, and there is still a lot to be tackled!

In our time, computers have become so powerful and so many numerical methods have been developed for use in mathematics and physics that we can now grab wave descriptions in a completely different way than could be done a few decades earlier. As an example of the development that has taken place, Professor Ron Fedkiw (born 1968), as working with Computer Sciences at Stanford University, received an Oscar award in 2008 for his efforts to animate realistic water waves for use in the film industry (including the film "Poseidon"). For those who are students today and will become familiar with numerical methods for solving mathematical and physical problems, this is extra fun. After completing your studies you will have the skills that would enable you to produce, only with modest effort, realistic animations of similar to those of Ron Fedkiw!

8.4.1 Circle Description

Let's now give a picture-and-word description of the waves themselves. Figure 8.14 shows a vertical cross-section of the wavefront. The solid curve shows the wave at a given moment, and the dotted curve shows the wave a short while later. The wave moves to the right in this case.

In the figure, arrows are drawn to show which direction the water must move in order to let the wave as it is now to become what it will be. The arrows in the upper half are quite easy to understand, while the arrows in the lower half may be harder to get hold of. However, we recall that the wave does not necessarily lead to a net transport of water in the direction of the wave, so water that moves forward in a part of the wave must move backwards into another part of the wave. And water that moves upward in part of the wave must move down in another part. If we keep these facts in mind, the directions of the arrows begin to make sense.

Note that the water must move *both* along the wave propagation direction and across it. This means that the wave is a mixture of a longitudinal and a transverse wave.

If we draw the direction of motion and relative position of the same small volume element at different times while a wave peak passes, we get a chart as in the lower part of the figure. It appears that the water in the surface moves along a vertical circle across the wavefront.

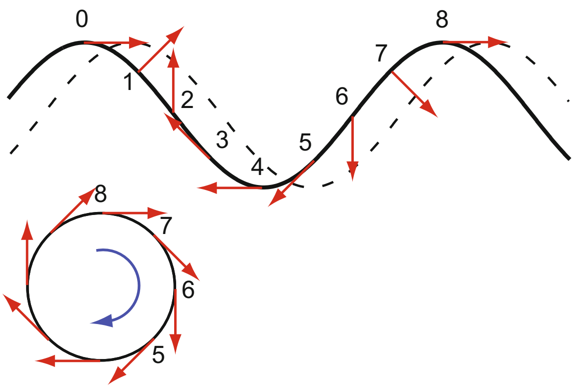

Fig. 8.14 Upper part indicates in which direction the water at the surface moves when the wave rolls to the right. In the lower part, the position and speed of one and the same volume element are drawn as a wave peak passes. The current wave element is that at position 8 at the beginning, but at the next moment it is located on the part of the wave that is in line with point 7 in the upper part. At the next moment, it has a location in the waveform that corresponds to point 6, etc. The result is that the volume element we follow appears to move in the clockwise direction as time passes

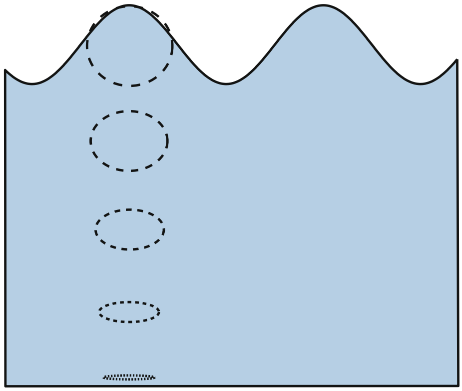

Fig. 8.15 When we want to indicate how the water moves between the surface and the bottom, simple sketches like this are used. However, sketches like this give a rather simplistic picture of what is happening

Further down into the water, the circular motion will change from being near-circular (as in the surface) to a more and more flattened ellipse, as shown in Fig. 8.15. At the bottom of the bottom, the movement is almost a pure horizontal movement back and forth. We can notice that when we snorkel at the bottom of a lake, and see aquatic plants and weeds swing slowly back and forth as waves pass on the surface.

This description, however, applies only to shallow water, that is for water that is not much deeper than the wavelength (the distance between the wave crests).

For deeper water, the waves on the surface will only propagate downwards a short distance, but near the bottom, the waves on the surface will not be noticed at all.

It is possible to see the circular motion by spraying small droplets of coloured oil into the water, provided that the mass density of these drops is about the same as that for water. We can then follow the movement of the drops in water waves, as is done in the basement of Abel's House at UiO and in Sintef's wave tanks in Norway. However, I have been told that it is harder to show these circular motions than we might infer from the textbooks.

When we portray water waves by drawing circles and ellipses at different depths, we must recognize that such a description can be easily misunderstood. How should we look at the circles and the ellipses for subsequent volumes in the wave direction? Here there must be some sort of synchronization that does not emerge from the figure and which necessarily has to give a more detailed description than can be conveyed through simple sketches.

The sinusoidal form is by no means the best model for surface waves on water. Often the wave tops are more pointed than the bottoms, as indicated in Fig. 8.16. The larger the amplitude, the steeper the top becomes. However, there is a limit to this tendency. When the wave peak becomes larger than about 1/7 of the wavelength, the wave is often unstable and can, e.g., go over to a breaking wave. At the border, the angle between the upward and downward part of the wave peak is about 120° (an angle that of course does not fully apply to the actual vertex).

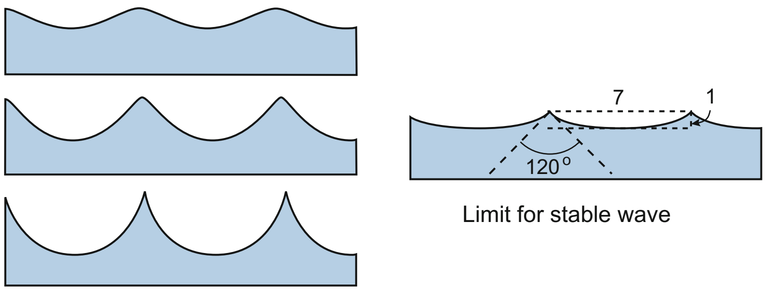

Fig. 8.16 Waveform is usually such that the top is more pointed than the bottom. The effect becomes clearer as amplitude increases. When the peak to peak amplitude is 1/7 of the wavelength, we reach a limiting value after which further increase in amplitude often gives an unstable wave

8.4.2 Phase Velocity of Water Waves

Although we have not shown how the wave equation will actually look for surface waves, we can establish an approximate expression of one characteristic of the solutions, namely the phase velocity of water waves. The expression is:

$$v_{\mathrm{p}}^2(k) = \left[\frac{g}{k} + \frac{Tk}{\rho}\right] \tanh(kh) \qquad (8.15)$$

where k is the wavenumber, g the acceleration due to gravity, T the surface tension, ρ the mass density and h the depth of water. The formula applies to a practically flat bottom (compared to the wavelength).

The first term inside the square brackets indicates the contribution of gravity to the restoring force, while the second indicates the contribution of surface tension. The first term thus corresponds to so-called gravity-driven waves, while the second term corresponds to what we call "capillary waves".

Since the wavenumber k occurs in the denominator of one term and in the numerator of the other, it follows that the gravitational term will dominate for small wavenumbers (long wavelengths), while the surface tension will dominate at large wavenumbers (small wavelengths). It may be interesting to find the wavelength where the two terms are about the same magnitude. We put then:

$$\frac{g}{k_{\mathrm{c}}} = \frac{Tk_{\mathrm{c}}}{\rho} .$$

The subscript c indicates a "critical" wavenumber where the two contributions are equal. The result is:

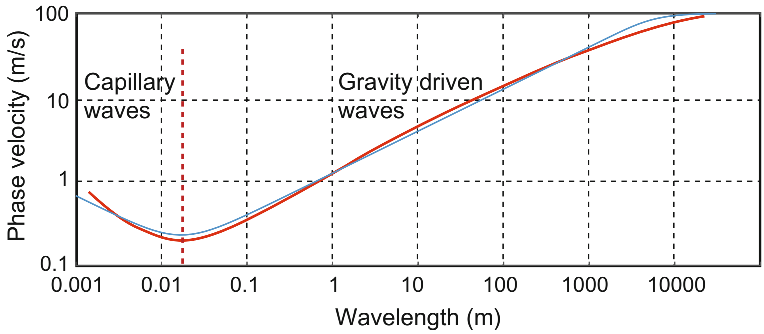

Fig. 8.17 Phase velocity of surface waves on water. The red curve in this figure is taken from R. Nave, (water depth was not specified). The blue curve is calculated using Eq. (8.15) with h = 1000 m. Inspired by [2]

$$\frac{1}{k_c^2} = \frac{T}{g\rho} \, .$$

Since $k = 2\pi/\lambda$ we find finally:

$$\lambda_c = 2\pi\sqrt{\frac{T}{g\rho}} \, .$$

For pure water at about 25 °C and 1 atmosphere, $T = 7.197 \times 10^{-2}$ N/m. Then the critical wavelength becomes:

$$\lambda_c \approx 1.7 \text{ cm}.$$

In other words, the surface tension will dominate the phase velocity of waves of wavelength appreciably smaller than 1.7 cm, while gravity will dominate for wavelengths considerably larger than 1.7 cm.

The phase velocity is actually smallest when the wavelength is about 1.7 cm, being only 0.231 m/s. Both shorter and longer wavelengths increase the phase velocity, and at very long wavelengths, the phase velocity can reach more than 100 m/s. Figure 8.17 shows the calculated phase velocity for wavelengths from 1 mm to 10 km. The calculations are essentially based on the fact that the water depth is large relative to the wavelength (something that cannot be attained here on earth for the longest wavelengths!).

We will immediately look at the expression for phase velocity, but first of all remind us of some features of the hyperbolic tangent function. The entire range of hyperbole trigonometric functions can be defined in an analogous manner to normal sine, cosine, etc. (all of which can be described by exponential functions with complex exponents). For the hyperbolic functions, the expressions look like this:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \; ,$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \; ,$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \; .$$

In what follows, we focus on how hyperbolic tangent behaves when the argument is much smaller or much larger than 1. Then it applies:

$$\tanh(x) \approx x \text{ for } |x| < 1 \; ,$$

$$\tanh(x) \approx 1 \text{ for } x > 1 \; .$$

In Eq. (8.15), the argument for tanh is equal to hk. The argument can also be written as

$$hk = \frac{2\pi h}{\lambda} \; .$$

It is then natural to distinguish between "shallow water" characterized by $h < \lambda/20$ and "deep water" characterized by $h > \lambda/2$. These limits mean that the shallow water condition corresponds to:

$$hk < \frac{2\pi\lambda}{20\lambda} = \frac{\pi}{10} < 1.0$$

and deep water condition corresponds to:

$$hk > \frac{2\pi\lambda}{2\lambda} = \pi > 1.0 \; .$$

It is time now to discuss some main features of Eq. (8.15). For shallow water first and wavelengths well above 1.7 cm (so that we may ignore the surface tension term) follow:

$$v_\mathrm{p}^2(k) = \frac{g}{k}\tanh(kh) \approx \frac{g}{k}kh = gh \; ,$$

$$v_\mathrm{p}(k) = \sqrt{gh} \; .$$

We see that the phase velocity is independent of the wavelength (wavenumber). Furthermore, we notice that phase velocity decreases as the depth decreases.

This gives a good effect. When waves come from the ocean towards a longshore beach, waves that are inclined inward will move fastest in the part where the depth is greatest. That is, the part of the longest wave will go faster than the part of the wave that is farther in. Generally, this causes the wavefront to become quite parallel to the shoreline, no matter what direction the waves had before approaching the beach.

For a deep water coast all the way down to the mountain cliffs down to the water, there is no equivalent effect and the waves can come in towards the cliffs in any direction.

For deep water waves, the phase velocity (assuming that the surface tension plays a negligible role):

$$v_f^2(k) = \frac{g}{k}\tanh(kh) \approx \frac{g}{k}1 = \frac{g\lambda}{2\pi} ,$$

$$v_f(k) = \sqrt{\frac{g}{2\pi}}\sqrt{\lambda} \approx 1.25\sqrt{\{\lambda\}}\,\text{m/s} .$$

where $\{\lambda\}$ means the value of λ (without units) measured in the number of metres.

Thus, in deep water the phase velocity will change with the wavenumber (wavelength). Such a relationship has been called *dispersion* earlier in the chapter.

Increasing the wavelength by two decades in our case, the phase velocity will increase with a decade. This is reflected approximately in Fig. 8.17.

Something to ponder over

It may be interesting to know that the ocean wave with the highest wavelength here on earth has a wavelength of 20,000 km. It rolls and goes all the time. Can you guess what sort of wave it is? Do you want to characterize it as a surface wave that is gravity driven? If so, does it fall under our description above and will it have a wavenumber given by our formulas? You can think about it for a while!

When we treated Eq. (8.15), we said that for wavelengths well over 1.7 cm, gravity dominated the wave motion. For capillary waves with wavelength significantly less than 1.7 cm, the surface tension dominated. These numbers apply at the ground surface.

A water drop will have a shape that is determined by both gravity and the surface tension. When the gravitational force disappears, such as, for example, in the weightless state of Spacelab, it is possible to make water droplets that are almost perfectly spherical, even with a diameter up to

10 cm. Waves on the surface of such water balls will in weightless state be dominated by surface tension even at wavelengths greater than 1.7 cm.

8.4.3 Group Velocity of Water Waves

We have previously given in Eq. (8.15), an expression for the phase velocity of surface waves in water, but reproduce the formula here to refresh the reader's memory.

$$v_{\mathrm{p}}^2(k) = \left[\frac{g}{k} + \frac{Tk}{\rho}\right] \tanh(kh) \ .$$

As usual, here k is the wavenumber, g the acceleration due to gravity, T the surface tension, ρ the mass density and h is the depth of the water. The expression can be derived if we start by just taking into account gravity and surface tension, and we ignore viscosity, wind and a tiny but final compressibility to the water.

We have previously found expression of the phase velocity for gravity-driven waves for shallow and deep water. Now we will also discuss group velocity and describe three of the four possible simple special cases a little more in depth:

1. Gravity-driven waves with a small depth relative to the wavelength, i.e. the product $hk << 1$:

The wavelength is assumed to be large relative to the critical (1.7 cm) and from Eq. (8.15) follows:

$$v_{\mathrm{p}}^2(k) \approx \frac{g}{k} hk \ ,$$

$$v_{\mathrm{p}} \approx \sqrt{gh} \ .$$

This has been shown earlier, but let us also look at the group velocity. We then use the relationship $v_{\mathrm{p}} = \omega/k$ and get:

$$\frac{\omega}{k} = \sqrt{gh} \ ,$$

$$\omega = \sqrt{gh}\, k \ ,$$

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \sqrt{gh} = v_{\mathrm{p}} \ .$$

Therefore,

$$v_\mathrm{g} = v_\mathrm{p}\ .$$

This means that there is simply no dispersion.

2. Gravity-driven waves in deep water

In this case, we found:

$$v_\mathrm{p}^2(k) \approx \frac{g}{k}\ .$$

We set again $v_\mathrm{p} = \omega/k$ and get:

$$\frac{\omega^2}{k^2} = \frac{g}{k}\ .$$

This leads to the following dispersion relation:

$$\omega \approx \sqrt{gk}\ .$$

The group velocity is thus seen to be:

$$v_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}\sqrt{\frac{g}{k}}\ .$$

$$v_\mathrm{g} \approx \frac{1}{2} v_\mathrm{p}\ . \tag{8.16}$$

Thus, we see that the group velocity is approximately equal to half the phase velocity.

Wake pattern from ships often fall into this category. The single waves seem to roll faster than the "plow" or "fan" that follows the boat (see Fig. 8.18). As a result, the single waves roll in a way past the "fan" and disappear soon afterwards. We will look into this in some respects.

3. Short ripples in deep water

Here the wavelength of the waves is small relative to the critical wavelength of 1.7 cm. At the same time, the wavelength is much less than the depth of the water. Then we get surface tension-driven waves and

$$v_p^2(k) \approx \frac{Tk}{\rho} \times 1 = \frac{\omega^2}{k^2} \ .$$

The dispersion relation is easily seen to be:

$$\omega \approx \left(\sqrt{\frac{T}{\rho}}\right) k^{\frac{3}{2}} \ .$$

The group velocity in this case becomes:

$$v_g = \frac{d\omega}{dk} = \left(\sqrt{\frac{T}{\rho}}\right) \frac{3}{2} k^{\frac{1}{2}} = \frac{3}{2}\sqrt{\frac{Tk}{\rho}} \ ,$$

$$v_g = \frac{3}{2} v_p \ .$$

In this case, the group velocity is actually greater than the phase velocity (corresponding to anomalous dispersion). In this case, individual waves seems to appear from nothing at the front of the group of waves, and then move "backwards" through the group and disappear. However, relative to the water, the single waves will always propagate away from the source that created the waves (as long as we do not have reflection), but the illusion of walking backwards is because the group velocity is even greater than the phase velocity.

8.4.4 Wake Pattern for Ships, an Example

Many are not used to identifying what is meant by a group of waves and what is meant by single waves. Left part of Fig. 8.19 attempts to show this. The figure refers to the photograph in Fig. 8.18. The fan with many single waves that extends slightly across the outer edge of the fan forms the group of waves. This fan is expanding at a speed that is the group velocity. However, each single wave will wander in a different direction than the fan as such and with a different wave velocity which is now the phase velocity.

We have previously concluded that for deep water waves, the group velocity is about half the phase velocity [see Eq. (8.16)]. This means that the single waves move

Fig. 8.18 Photograph of a boat with waves forming a V-shaped wake behind it. See further discussion in the next section. Arpingstone, Public Domain, [3]

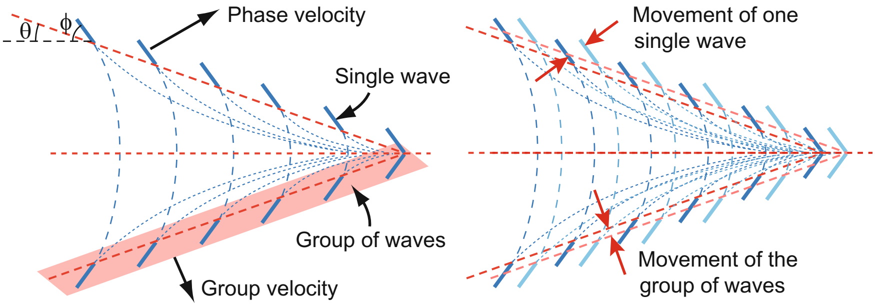

Fig. 8.19 To the left: Identification of the group moving with group velocity and single waves moving at phase velocity in waves from a boat. The figure is a continuation of Fig. 8.18. To the right: Detail showing how far the wave group and how far a single wave has moved across each of its wavefronts over a certain period of time. The figure clearly shows that the group velocity is lower than the phase velocity of these water waves

faster than the group. The single waves therefore appear to arise of almost nothing on the inside of the group, and scroll across the group, and almost disappear when they reach the outer edge of the group.

If you have ever paddled a canoe and watched a bit nervously how fast single waves approach the canoe after a boat has passed, you might have wondered that the waves which looked so scary seem to have vanished on their own before reaching the canoe. Only much later than we first became aware of them do the waves reach the canoe. The waves make it to the canoe only when the *group* reaches there, and the group moves half as fast as the single waves. This time course comes out beautifully in the animation of how a wave packet evolves in time using the computer program discussed earlier in the chapter (listed at the end, just before "Learning objectives").

In the right part of Fig. 8.19, the waves are drawn at one point and a little later. Then it becomes clear that the group has gone a much shorter distance than the single waves in the period we are studying. It always remains true that the wave pattern behind the boat is stationary relative to the boat. When the boat has moved 10 m forward, the entire wave pattern behind the boat has also moved 10 m ahead.

Lord Kelvin (W. Thomson) claimed long time ago (1887) that wakes pattern from ships are fanning out at a constant angle of 19.47°, no matter the speed of the vessel. This was regarded as an established truth. This has lately been shown not to be true. Especially, the wake pattern is quite different from what is seen in Fig. 8.18 for high-speed boats. The subject is described by Ceri Perkins in an easily readable article in Physics World May 30, 2013, and in more details, for example, in the paper "A solution to the Kelvin wake angle controversy" by A. Darmon, M. Benzaquen and E. Paphal, 2013, available at https://www.gulliver.espci.fr/sites/www.gulliver.espci.fr/IMG/pdf/darmonbenzaquen2013.pdf. It is fascinating to see the broad range of pattern a ship wake can take.

Additional comments for the most interested:
There has been a lot of research on phase and group velocities since about 1980. Much of this is related to light.

In February 2015, Giovannini and colleagues published an article in *Science* that shows taking the velocity of light in vacuum is not necessarily c as Einstein's relativity theory indicates. Light velocity equal to c applies only to light in the form of plane waves. For some other wave configurations, which the authors designate as "spatially structured photons", the velocity of light in vacuum is slightly lower than c (not a big difference, but demonstrably smaller).

We have long known that when light goes through glass (or a water drop for that matter), light of different wavelengths travels at different speeds. The refractive index is wavelength dependent, $n(\lambda)$, which is again the expression of dispersion. Light in glass shows normal dispersion.

However, in the last few decades a number of special materials have been developed, and some of these have a highly varying phase velocity for light with different wavelengths. Therefore, we can get widely varying phase and group velocities, and even materials where phase velocity has one direction and group velocity has the opposite direction.

There are also artificial materials and experimental relationships where we can slow down the light enormously, even "stop" it for shorter periods, then start it again (search for Lene Hau at Harvard University to get an insight into an exciting research field. Lene is Danish and is a favourite for the Danish press).

In some materials, a light pulse can travel—according to some people—faster than light in the vacuum, and in principle, we threaten Einstein's relativity theory in this way. When we look at what happens, we see that the claim of "faster than the light speed in vacuum" can be defeated. It all depends on how we define this or that, but Einstein's relativity theory is not exactly threatened by these experiments, on the whole. What the future will bring is something harder to contemplate!

Dispersion is also relevant to matter waves in quantum physics. Group velocity is defined through dispersion relations where $\omega(k)$ is described and we use $v_g = d\omega/dk$. For matter waves, the wavelength through the Broglie relationship is related to the momentum and the frequency of energy. For matter waves, therefore, we have dispersion if the energy does not increase with the momentum in the expected manner.

Dispersion turns up in many other contexts, among them the so-called Kramers–Kronig relation that shows that dispersion is related to the amount of absorption for different wavelengths in the medium. To a certain extent, this is linked to forced oscillations and Q-values, as we have mentioned earlier, but we do not have the time to go further in depth.

8.4.5 Capillary Waves

We all know waves at sea. Less commonly known are oscillations in small water droplets where surface tension is the dominant restoring force. When a drop falls from a tap, it will oscillate while it falls. Examples of this are found in references 1 and 2 at the end of the chapter.

Standing waves in a water drop we can observe when we place some water in the pit of an old-fashioned electric stove, provided that the plate is so hot that the drop floats atop a cushion (steam cushion) that forms. We can get beautiful quantized oscillations with an asterisk shape where an integer number of arms swings back and forth (see Fig. 8.20). Slight variation in heat or size of the drop may cause it to suddenly change the swing pattern from, for example, a five-arm to a four-arm star. The arms are shot out and pulled back in such a way that we will perceive an octagonal star (since we cannot follow the rapid movement with the unaided eye).

The purpose of this description is to recall that classical physics is full of quantified states, in an analogous manner to what is found on the atomic scale described in quantum physics. We have already seen in other chapters other examples of quantization on macroscopic scale, such as oscillations on a string and sound waves in a musical instrument.

The reason for quantization is that we are dealing with waves and the associated boundary conditions. For waves on a guitar string, the quantization is a consequence of the fact that the amplitude at the endpoints must be equal to zero. This is completely analogous to quantization of the wave function in quantum physics (e.g. for a "particle in box").

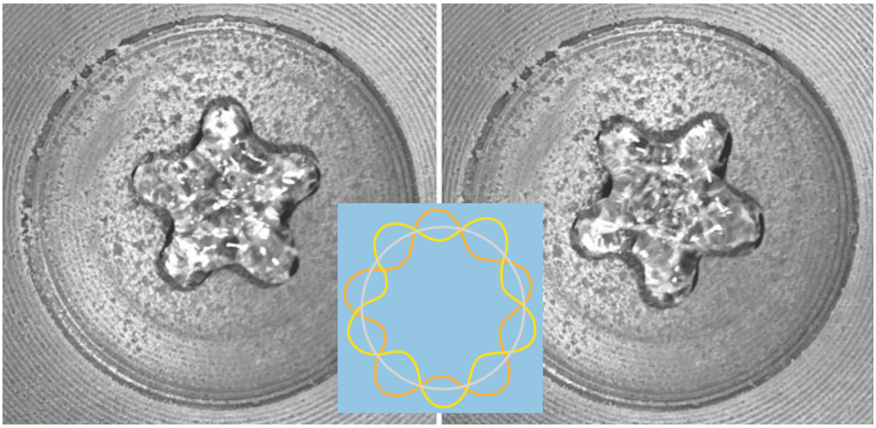

Fig. 8.20 Pictures of an oscillating water drop. The picture to the left is taken a few milliseconds before the image to the right. The two pictures show the extremes of the oscillation of this drop. The movement can be considered as a standing wave in the drop. The images are selected from a video taken by high-speed camera at the Department of Physics, University of Oslo (M. Baziewich and A. I. Vistnes). The video is available at the "Supplementary material" web page for this book at http://www.physics.uio.no/pow

8.5 Program Details and Listing

Given below is a Matlab program that can be used to explore how dispersion affects the time development of a wave packet (a wave group). The program consists of a main program that calls on four functions. One must find out oneself which parameters are to be changed when one wants to switch between no dispersion, normal dispersion and anomalous dispersion. It is natural to change several of the functions if you want to enter the appropriate parameters to model completely specific physical wave phenomena. However, it is imperative to understand the parts of the program that you want to change; otherwise, the result may turn out to be meaningless.

A description of the different parts of the actual program

The function/script *pg3* is the main program that activates the different modules in the complete program. We have to choose in the program code whether we want normal, anomal or no dispersion.

The wave packet is calculated in function *pg_wpack*.

Since the wave is limited in extent, a Fourier analysis of $y(x)$ in the function *pg_fft* will yield a range of spatial frequencies. For our selection of parameters, the result is illustrated in Fig. 8.9. The components with indices 23–59 (marked with blue vertical lines) contain all the components that are notably different from zero (compared to the value of the most powerful component). The different indices correspond to each (spatial) frequency (as explained in detail in the chapter on Fourier analysis). This range of components has to be stated in the code of *fg3*.

Fourier analysis gives us (spatial) frequency, amplitude and phase of each components of interest.

$$k(i) = (i-1)\frac{2\pi x_{\max}}{N} \quad A(i) = 2\,\text{abs}(Y(i)) \quad \theta(i) = \text{atan2}(\text{imag}(Y(i)), \text{real}(Y(i)))$$

where $Y(i)$ is the ith element in the Fourier transform of $y(x)$, and $x_{\max}$ and N are, respectively, the greatest value of the position and number of points in our description. The factor of 2 in A is because we use only the lower half of the Fourier spectrum (not the folded part). The expressions “abs”, “atan2”, “imag” and “real” are all Matlab functions.

We know from Chap. 5 on Fourier transformation that we can describe the same function as in Eq. (8.14) by a “reverse Fourier transform”:

$$z(x) = \sum_{i=23}^{59} A(i)\cos[k(i)x + \theta(i)] \tag{8.17}$$

where we have included only the components that are worth mentioning for the result. Plotting $z(x)$ calculated with Eq. (8.17) and comparing it to a plot based on Eq. (8.14) will not reveal any difference visually.

Note that in Eq. (8.17), we add cosine functions, each contribution having the same amplitude over the *entire region* for which we calculate. There is no specific information about where the peak of the wave packet should be or how wide it is. All of this is information hidden in amplitudes and phases of the various frequency components that are included.

Both Eqs. (8.14) and (8.17) describe the wave at time $t = 0$. Equation (8.17) is most useful in our context because it is well suited to describe how the wave will *evolve* over time when we have dispersion. Then the algorithm we used first in the chapter will not suffice because there is no simple wave equation when the phase velocity is not constant.

If we use Eq. (8.17), we can get the time evolution simply by replacing $\cos[k(i)x + \theta(i)]$ with $\cos[k(i)x - \omega(i)t + \theta(i)]$. Then each spatial frequency component will evolve with its individual phase velocity. This demonstrates why Fourier analysis sometimes is very useful! However, the challenge is to determine $\omega(i)$. The function *pg_omega* takes care of that challenge.

We have chosen the following three variants:

$$\omega(k) = v_\mathrm{p}\frac{k}{k_\mathrm{dom}} \quad \text{No dispersion}$$

$$\omega(k) = \kappa_1 v_\mathrm{p}\sqrt{\frac{k}{k_\mathrm{dom}}} \quad \text{Normal dispersion}$$

$$\omega(k) = \kappa_2 v_\mathrm{p}\left(\frac{k}{k_\mathrm{dom}}\right)^{3/2} \quad \text{Anomalous dispersion}$$

where v_p is the phase velocity of a contemplated harmonic wave with wavenumber k_dom (dominant wavenumber in our Fig. 8.21). Note: We have chosen the parameters so that the group velocities are roughly the same for all three cases. The phase velocities are quite different. κ_1 and κ_2 are small correction factors (1.04 and 1.10)

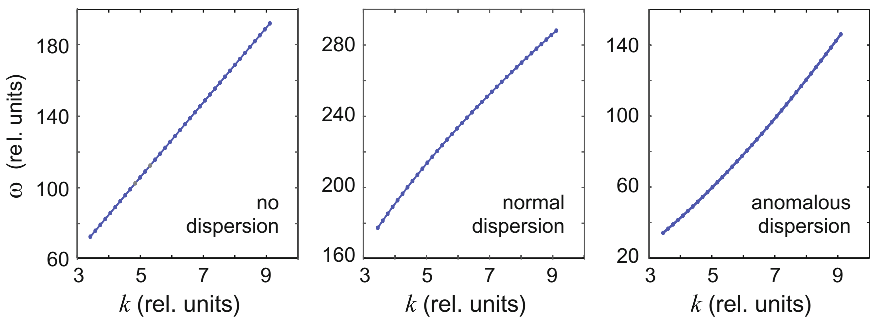

Fig. 8.21 Relation between frequency and wavenumber used in our calculations for the next figure

to optimize Fig. 8.10 and do not matter for the argument that follows. Figure 8.21) shows how ω varies with the wavenumber k.

In the function *fg_omega*, we also defines a parameter "deltat" that is used in the animation of the time evolution of the wave packet. The numbers in the computer code at the end of this chapter are chosen just so that the final position of the wave packet is convenient for plotting. Can be adjusted as wanted to have a different time at the end of the animation.

The animation including plotting is carried out by the function *pg_animer*. This function uses the function *pg_wave* to generate a complete spatial wave for each wavelength component and time in the animation.

Main Program

The code is available at the "Supplementary material" web page for this book at http://www.physics.uio.no/pow.

```
function pg3

% Program to illustrate the difference between phase and group
% velocity. Movement of a wave package is animated (blue). To
% ease the understanding of the difference between phase and
% group velocity, a pure monochromatic wave with the central
% wavelength is animated along with the wave package (in green).
% This will move with the phase velocity.
% Version: 5. October 2017, AIV

% NOTE: Due to the periodicity buried in a FFT, the wave pattern
% will only be valid if the animation stops before the wave
% pattern reaches the right end of the animation plot. If it
% turns up again at the left, the result is not valid.

% NOTE 2: You have to choose several values for the parameters
% in the program listing below!

% Choose first type of dispersion:
disp = -1.0;   % -1,0,+1: normal, no, anomal dispersion

% Create a wave package IN SPACE (!)
N = 4000;
xmax = 40.0;
xlambda = 1.0;      % Spatial wavelength
xsigma = 2.0;       % Width of the package
[x,z] = pg_wpack(N,xmax,xlambda,xsigma);
plot(x,z,'-b');

% Spatial frequency analysis, find amplitude and phase as a
% function of the wavenumber k
 [A,phase,k] = pg_fft(z,N,xmax);

% Pick manually those points in the frequency plot with
% considerable amplitude (use pg_fft, last part).
```

```
imin = 23;
imax = 59;

% Determines omega(k) using the dispersion relation
[omega,deltat] = pg_omega(imin,imax,k,disp);

% Now the movement can be animated
[xavt] = ...
   pg_animer(x,deltat,N,A,phase,k,omega,imin,imax,xmax,disp);
xavt;  % Position to a peak in a monochromatic wave (moving at
       % the phase velocity) after finishing the animation.
       % Start value is xmax/8.Remove the semicolon to have
       % this value written to the screen.
```

Create Wave Package in Space (at t = 0)

```
function [x,z] = pg_wpack(N,xmax,xlambda,xsigma)

% Create a wave package in space (!). Version Oct 5 2017 AIV
% Input parameters: N: Points in the description,
% xmax: Defines the interval x is defined ( |0,xmax>),
% xlambda: spatial wavelength for the central wavelengh,
% xsigma: the width in the gaussian shaped wave package.
% Returns: x array as well as the wave package array.

x = linspace(0,xmax*(N-1)/N,N);
xr = xmax/8.0; % Startpoint for the centre of the wave package
xfreq = 1/xlambda;  % Spatial frequency
y = cos((x-xr)*2*pi*xfreq);
convol = exp(-((x-xr)/xsigma).*((x-xr)/xsigma));
z = y.*convol;
return;
```

Frequency Analysis of Wave Package in Space

```
function [A,theta,k] = pg_fft(z,N,xmax)

% Frequency analysis of a wave package in space.
% Version Oct 5 2017 AIV
% Input parameters: z: the array describing the wave package,
% N: number point in this description, xmax: describe the x
% interval |0, xmax>. Returns: Amplitude A and phase (theta)
% in the frequency analysis as a function of the wavenumber k.

Zf = fft(z)/N;
A = 2.0*abs(Zf);  % Ignore the error in Zf(1), don't use it
theta = atan2(imag(Zf),real(Zf));
xsamplf = N/xmax;   % Spatial sampling frequency
xfreq = linspace(0,xsamplf*(N-1)/N,N); % Spatial frequency
k = zeros(1,N);
k = 2.0*pi*xfreq;
```

```
% NOTE: Use the reminder of this function when you need to
% pick frequency components for your wave package. You need
% this in order to choose imin and imax in the program pg3.m
%figure;
%plot(A,'.-r'); % Plot to be able to choose points to be used
%plot(xfreq,A,'.-r');  % Alternative plot
%plot(xfreq,fase,'.-k');
return;
```

Generate the Dispersion Relation omega(k)

```
function [omega,deltat] = pg_omega(imin,imax,k,disp)

% Generate the dispersion relation omega(k).
% Version Oct 5 2017, AIV
% Input parameters: imin, imax: first and last index that
% will be used in the function that creates the animation,
% k: the wavenumber array created by the function pg_fft,
% disp: -1, 0, or +1 represent normal, no and anomalous
% dispersion.
% Returns: omega: the dispersion relation omega(k),
% deltat: a suitable delta_t for the animation in order to
% get useful animation/plots.

if (disp==-1)   % Normal dispersion (here vg = vp/2)
    deltat = 0.015;
    omegafactor = 44.0;
    for i = imin:imax
        omega(i) = omegafactor*sqrt(k(i));
    end;
end;

if (disp==0)   % No dispersion (here vf = const)
    deltat = 0.015;
    omegafactor = 9.5;
    for i = imin:imax
        omega(i) = omegafactor*k(i);
    end;
end;

if (disp==1)   % Anomal dispersion (here vg = 3vp/2)
    deltat = 0.0065;
    omegafactor = 5.5;
    for i = imin:imax
        omega(i) = omegafactor*(k(i)^1.5);
    end;
end;

figure;
plot(k(imin:imax),omega(imin:imax),'.-b');
xlabel('k (rel. units)');
ylabel('omega (rel. units)');
return;
```

Create a Sum of Harmonic Spatial Waves at a Given Time t

```
function [zrecon] = pg_wave(x,t,N,A,phase,k,omega,imin,imax)

% Generate the complete spatial wave using the Fourier
% coefficients. Version Oct 5 2017 AIV
% Input parameters: x: position array, t: current time,
% N: number of points, [A, phase, k]: amplitude, phase and
% wavenumber arrays, respectively, omega: the dispersion
% relation omega(k), [imin, imax]: minimum andmaximum index
% that will be used in the arrays A, phase and k.
% Returns: zrecon: the position of the marker which gives the
% position to where a peak with the central wavelength would
% have ended up (for verification of proper functioning).

zrecon = zeros(1,N);
for i = imin:imax  % Sum over Fourier elements
    arg = k(i)*x - omega(i)*t + phase(i);
    zrecon = zrecon + A(i)*cos(arg);
end;
return;
```

Make an Animation of All Spatial waves

```
function [xavt] = ...
   pg_animer(x,deltat,N,A,phase,k,omega,imin,imax,xmax,disp)

% Animation of a wave package during some time. To ease the
% understanding of the difference between phase and group
% velocity, a pure monochromatic wave with the central
% wavelength is animated along with the wave package.
% Returns how far the monochromatic wave has moved during
% the animation (indicates the phase velocity).
% Input parameters: See the explanations given in the
% functions pg3.m, pd:wpack.m, pg_fft.m, pg_omega.m and
% pg:wave.m  Version: Oct 5 2017 AIV

figure;
count=1;
% The animation loop
for n = 1:200
    % Calculate the wave at time t (manual IFFT)
    t = deltat*n;
    [zrecon] = pg_wave(x,t,N,A,phase,k,omega,imin,imax);
    % Calculate also the wave with central spatial frequency
    % in the distribution
    imean = round((imin+imax)/2.0);
    [zrecon0] = pg_wave(x,t,N,A,phase,k,omega,imean,imean);
    % Calculate marking positions, start and end of movement
    % at phase velocity
    x00 = xmax/8.0;
    xavt = x00 + t*omega(imean)/k(imean);
```

```
    % Plots everything
    plot(x,2.5*zrecon0,'-g', x,zrecon,'-b', x00,0.25,'+r', ...
       xavt,0.25,'+r');
          xlabel('Position (rel)');
          ylabel('Amplitude (rel)');
          axis([0,xmax,-1.04,1.04])
          title('Movement to a blue wave package');
          S = sprintf('Time: %.2f s',t);
          text(3.0, 0.8,S);
          S = sprintf('Xref: %.2f',xavt);
          text(3.0, 0.65,S);
          S = sprintf('Dispersion code: %.1f',disp);
          text(3.0, -0.8,S);
          M(count)=getframe;
          count=count+1;
          M(count)=getframe;
          count=count+1;
end;
% Animation is played with (1 x 20 frames per sec)
movie(M,1,20);
return;
```

8.6 References

1. R.E. Apfel, Y. Tian et al. : *Free Oscillations and Surfactant Studies of Superdeformed Drops in Microgravity. Phys. Rev. Lett.* 78 (1997) 1912–1915 (Large water drop analysed in the spaceship Columbia.).
2. H. Azuma and S. Yoshihara: *Three-dimensional large-amplitude drop oscillations: Experiments and theoretical analysis. J. Fluid Mech.* 393 (1999) 309–332.
3. For oscillating drops, see an elegant piece of work from: C-T. Chang, S. Daniel, P.H. Steen: *Excited sessile drops dance harmonically*, described in *Phys. Rev. E* 88 (2013) 023015.
   For a system that resembles ours, see for example:
   http://www.youtube.com/watch?v=YcF009w4HEE (Leidenfrost-effect: The dancing druppel (version 2)) or the last half of the video
   http://www.youtube.com/watch?v=b7KpHGgfHkc (JuliusGyula_HotPot 1.3). Both were accessible on 4 October 2017.
   We have also made our own high-speed films of oscillating water droplets, and a few of these will be available at the "Supplementary material" web pages for this book. The films were taken by Michael Baziljevich and Arnt Inge Vistnes 2014.
4. J. Persson: Vågrörelselära, akustik och optik. Studentlitteratur 2007.

8.7 Learning Objectives

After going through this chapter, you should be able to:

- Perform numerical calculations of the time course of a one-dimensional wave (with arbitrary shape) when there is no dispersion, based directly on the wave equation.
- Explain the contents of the algorithm for such calculations.
- Explain the difference between phase and group velocity in general and know how each is calculated.
- Explain how we can animate the time development of a wave packet.
- Know typical characteristics of how a wave packet develops over time when there is no dispersion, normal dispersion and anomalous dispersion.
- Provide examples of physical systems with dispersive behaviour, both normal and anomalous.
- Perform numerical calculations of the time course for a one-dimensional wave in dispersive media.
- Explain the contents of the algorithm for such calculations.
- Explain differences in gravity-driven waves in water and sound waves through water.
- Explain the two different "restoring forces" of surface waves on water.
- Enter an approximate criterion for whether it is the surface tension or gravity that dominates in a given case.
- Give examples of surface tension-driven waves and gravity-driven waves.
- Explain a model where we explain/describe waves by (small volumes of) water following a circular motion.
- Find approximate expression of phase velocity and group velocity of waves both in shallow and deep water, starting from the formula

$$v_{\rm p}^2(k) = \left[\frac{g}{k} + \frac{Tk}{\rho}\right] \tanh(kh) \ .$$

- Recapitulate the main features in Fig. 8.17.

8.8 Exercises

Suggested concepts for student active learning activities: Dispersion, group velocity, phase velocity, anomal/normal/no dispersion, mechanism for wavelength dependence of the speed of light in glass, wave packet, wave envelope, gravity-driven waves, capillary waves, high-speed camera, V-shaped wake.

Comprehension/discussion questions

1. What do we mean by a dispersive medium? How will dispersion affect the wave motion of *a*) a harmonic wave and *b*) a nonharmonic wave?
2. What is the difference between normal and anomalous dispersion?
3. What characterizes dispersion? What is a dispersion relation? Is the dispersion responsible for the phenomenon that waves often come in almost parallel to a sandy beach?
4. Indicate how a guitar string looks (amplitude vs position) just before we release the string. Use the algorithm given in Fig. 8.3 to tell which parts of the string will move in the first time step, second time step and third time step. Perhaps you can guess how the string will actually vibrate?
5. The oscillatory pattern we encounter in the previous task (and in our calculations based on the program *bolgeanimationX*) corresponds to the real motion of a guitar string during the first few oscillations. Eventually, sharp transitions disappear. Can you imagine what physical characteristics of the string, not taken into account here, would affect the fluctuations quite quickly? (Hint: Videos on YouTube, which show a motion entirely in accord with our calculations, use a rubber band rather than a proper guitar string to get the particular vibrating behaviour found in our calculations.)
6. A common misconception about why light goes slower through glass than in vacuum is that the photons are impeded by the glass. Such a view is confronted with a problem when we come to explain that the velocity returns to its value in air as soon as the light has passed the glass. Why is this hard to explain with the aforementioned explanation/model?
7. Why do we use a wave packet in the calculations that give us animation of dispersion?
8. Could you give some kind of explanation that the wavelength is different at the beginning of a wave packet compared to the wavelength at the end of the pack if we have normal or anomalous dispersion?
9. Are surface waves in water transverse or longitudinal waves? Explain.
10. Try to explain why we do not notice any effect of surface waves on water at a depth that is large relative to the wavelength.
11. Explain why waves roll in with the wave peaks parallel to the water's edge on a longshore beach.
12. See the video of Chang and coworkers in Reference 3 above. Find a diagram from the web that shows electron orbitals for the hydrogen atom. Compare the diagrams of Chang et al. with quantum descriptions of atomic orbitals. Comment on similarities and dissimilarities (Hint 1: 2D vs 3D; Hint 2: Quantum Physics operates with *wave* functions; Hint 3: Quantization).

Problems

13. Set up a mathematical expression (based on wavenumber and angular frequency) for a plane, monochromatic harmonic wave. Comment on the phase velocity and group velocity to the extent they are defined.

14. Create your own program to calculate numerical solutions of the wave equation. Feel free to take a look at the program shown under point Sect. 8.2.1 above. Test that a wave described by Eqs.(8.8) and (8.9) appears as shown in Fig. 8.4. Then make the following changes:
(a) Change the time derivative of the amplitude at the starting instant to the negative of what it should have been. Complete the calculations and describe what you observe.
(b) Reduce the time derivative of the amplitude at the starting instant to a half of what it should have been. Complete the calculations and describe what you observe.
(c) Use instead twice the time derivative of the amplitude instead of the correct one at the starting instant. Complete the calculations and see what you observe this time, paying attention to both amplitudes and phases.
(d) How do you want to create the initial conditions to simulate standing waves? [Optional]
(e) What conclusion can you deduce from all the calculations in this task? With a pendulum motion, we can choose position and velocity independently and always get an oscillatory motion that is easy to understand. Does the same apply to waves?
15. Modify the program you used in the previous task so that it can handle the case of a one-dimensional wave along a string meets a material with a different phase velocity. The wave should be able to continue into the new material and may also be reflected at the point where the string changes property (may correspond to the string changing mass per unit length). Attempt both with a 30% increase in phase velocity and a 30% reduction in phase velocity. Describe the results and comment on whether the results are consistent with what is described in Chap. 7 or not.
16. Make some simple sketches that show how you, before you do the calculations (or get to know the results found by fellow students), envisage a guitar string to vibrate. Write *afterwards* a computer program that calculates the motion of a guitar string for at least a couple of vibration periods after the string has been pulled, by means of a plectrum or fingernail, at a point that is at a distance of about 1/3 of the string length from one end, and released from there (after being at rest). Feel free to look at the program shown under point Sect. 8.2.1 above. Describe the motion.
[Check after you have done the calculations, whether there is a match between your calculations and YouTube movies mentioned in the text.]
17. Try to modify the computer program *waveAnimationX* based on the algorithm in Eq. (8.5) early in this chapter so that it can be used to describe the movement of a triangular wave as depicted in Fig. 8.22 for the case where the waveform is conserved during the movement. The wave will eventually hit a fixed/closed boundary and is reflected. Compare the result with the left side of Fig. 7.2.
18. Try to describe how the triangular wave in the previous problem will develop if it runs through a dispersive medium. In this case, a procedure based on Fourier

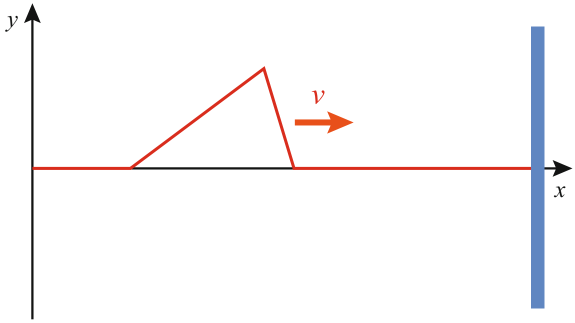

Fig. 8.22 A triangular wave that moves towards higher x-values. Suppose that the shape is unchanged at first until it hits a wall. See the problem text

decomposition in spatial components described in the end of this chapter should be used. Do not include the reflecting wall in this case.

19. Check through your own calculation that the wavelength is about 1.7 cm when surface waves on water are controlled as much by surface tension as by gravity. Surface tension for clean water at 25 °C is 7.197×10^{-2} N/m.
20. Determine the phase velocity of surface waves on "deep" water at a wavelength of 1.7 cm (Tip: Use the information from the previous task.).

References

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