Chapter 9
Electromagnetic Waves

Abstract This chapter starts with the integral form of Maxwell's equations—one of the greatest achievements in physics. The equations are transformed to differential form and used, in concert with important assumptions, to derive the wave equation for a plane, linearly polarized electromagnetic wave. The wave velocity is determined by electric and magnetic constants valid also at static/stationary conditions. The electromagnetic spectrum is presented as well as expressions for energy transport and radiation pressure. It is emphasized that the simple plane-wave solution is often invalid due to the effect of boundary conditions; we need to discriminate between near- and far-field conditions. At the end, a brief comment on the photon picture of light is given.

9.1 Introduction

Of all the wave phenomena that are consequential to us humans, sound waves and electromagnetic waves occupy a prominent position. Technologically speaking, the electromagnetic waves rank the highest.

We are going to meet, in many of the remaining chapters, electromagnetic waves in various contexts. It is therefore natural that we go into some depth for the sake of describing these waves; it is not true that all electromagnetism can be reduced to electromagnetic waves. That means, we must be careful to avoid mistakes when we treat this material.

Of all the chapters in the book, this is the most mathematical. We start with Maxwell's equations in integral form and show how their differential versions may be deduced. It will then be shown that Maxwell's equations can lead, under certain conditions, to a simple wave equation. Electromagnetic waves are transverse, which means that the complexity of the treatment is somewhat larger than for longitudinal sound waves.

The chapter takes it for granted that the reader has previously taken a course in electromagnetism, and is familiar with such relevant mathematical concepts as line integrals and surface integrals. It is also a great advantage to know Stokes's theorem, the divergence theorem and those parts of vector calculus which relate to divergence,

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gradient and curl, and it is vital that the reader knows the difference between scalar and vector fields before embarking on the chapter.

As already mentioned, mathematics pervades this chapter. Nonetheless, we have tried to point to the physics behind mathematics, and we recommend that you too devote much time for grappling with this part. It is a challenge to grasp the orderliness of Maxwell's equations in its entirety.

Experience has shown that misunderstandings related to electromagnetism arise frequently. A common misconception, incredibly enough, is that an electromagnetic wave is an electron that oscillates up and down in a direction perpendicular to the direction of propagation of the wave. Other misapprehensions are harder to dispel. For example, many believe that the solution of the wave equation is "plane waves" and that the Poynting vector always describes energy transport in the wave. We spend some time discussing such misunderstandings and hope that some readers will find this material useful.

At the end of the chapter is a list of useful mathematical relations and a memorandum of how electrical and magnetic fields and flux densities relate to each other. It may be useful for a quick reference and for refreshing material previously learned.

Let us kick off with Maxwell's stupendous systematization (and extension) of all that was known about electrical and magnetic laws in 1864.

9.2 Maxwell's Equations in Integral Form

Four equations connect electric and magnetic fields:

1. Gauss's law for electric field:

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = \frac{Q_{\text{inside}}}{\varepsilon_r \varepsilon_0} \tag{9.1}$$

2. Gauss's law for magnetic field:

$$\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{A}} = 0 \tag{9.2}$$

3. The Faraday–Henry law:

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = -\left(\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}\right)_{\text{inside}} \tag{9.3}$$

3. The Ampère–Maxwell law:

$$\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_r \mu_0 \left[i_f + \varepsilon_r \varepsilon_0 \left(\frac{\mathrm{d}\Phi_E}{\mathrm{d}t}\right)_{\text{inside}} \right] \tag{9.4}$$

We expect that you are familiar with these laws, and therefore do not go into great detail about how to interpret them or what the symbols mean. In the first two equations, the flux is integrated over a closed surface and compared with the source within the enclosed volume (electrical monopole, i.e. charge, and magnetic monopole, which are non-existent). The vector $d\vec{A}$ is positive if it points outward of the enclosed volume.

In the last two equations, the line integral is calculated for electrical or magnetic fields along a curve that limits an open surface. The line integral is compared with the flux of magnetic flux density or electrical flux density as well as flux of electrical currents due to free charges through the open surface. The signs are then determined by the right-hand rule (when the four fingers on the right hand point in the direction of integration along the curve, the thumb points in the direction corresponding to the positive flux).

Prior knowledge of these details is taken for granted.

The symmetry is best revealed if the last equation is written in the following form:

$$\oint \vec{H} \cdot d\vec{l} = \left[i_f + \left(\frac{d\Phi_D}{dt} \right)_{\text{inside}} \right] . \tag{9.5}$$

Here, use has been made of the following relationship between the magnetic field strength $\vec{H}$ and the magnetic flux density $\vec{B}$:

$$\vec{H} = \vec{B}/(\mu_r \mu_0)$$

where μ_0 is (magnetic) permeability in vacuum and μ_r is relative permeability.

Use has also been made of the following relationship between the electric field strength $\vec{E}$ and the electric flux density $\vec{D}$ (also called "displacement vector"):

$$\vec{E} = \vec{D}/(\varepsilon_r \varepsilon_0)$$

where ε_0 is the (electrical) permittivity in vacuum and ε_r the relative permittivity.

The left-hand sides of Eqs. (9.3) and (9.5) are line integrals of field strengths ($\vec{E}$ and $\vec{H}$), whereas the right-hand sides are the time derivative of the flux through the enclosed surface plus, for the latter equation, the current density due to free charges. The flux is obtained by integrating the pertinent flux density ($\vec{B}$ or $\vec{D}$) over the surface.

The contents of Maxwell's equations are given an approximate verbal rendering below:

- There are two sources of electric field. One source is the existence of electrical charges (which may be regarded as monopoles). Electric fields due to charges are radially directed away from or towards the charge, depending on the signs of the charges. (This is the content of Gauss's law for electric field.)
- The second source of electric field is a time-varying magnetic field. Electrical fields that arise in this way have a curl (or circulation); that is, the field lines tend

Fig. 9.1 James Clerk Maxwell (1831–1879). Public domain [1]

to form circles across the direction along which the magnetic field changes in time. Whether there are circles or some other shape in space depends on the boundary conditions. (This is the content of Faraday's law.)

- There are two contributions to magnetic fields as well, but there are no magnetic monopoles. Therefore, magnetic fields will never flow out radially from a source point similar to electric field lines near an electrical charge. (This is the content of Gauss's law for magnetic fields.)
- On the other hand, magnetic fields can arise, as in the case of electric fields, because an electric field varies over time. An alternative way of generating a magnetic field is to have free charges in motion that form a net electric current. Both these sources provide magnetic fields that tend to form closed curves across the direction of the time variation of the electric field or the direction of the net electrical current. However, the shape of these closed curves in practice is entirely dependent on the boundary conditions. (This is the content of the Ampère–Maxwell law.)

It was the physicist and mathematician James Clerk Maxwell (1831–1879, Fig. 9.1)) who distilled all knowledge of electrical and magnetic laws then available in one comprehensive formalism. His publication "A Dynamical Theory of Electromagnetic Field", published in 1865, shows that it is possible to generate electromagnetic waves and that they travel with the speed of light. His electromagnetic theory is considered to be on a par with Newton's laws and Einstein's relativity theory. The original 54-page long article (https://doi.org/10.1098/rstl.1865.0008 Phil. Trans. R. Soc. Lond. 1865 vol. 155, pp. 459–512) can be downloaded for free from: The Royal Society.

Maxwell–Heaviside–Hertz: However, the four Maxwell's equations, as we know them today, are far from the equations Maxwell himself presented in "A Treatise on Electricity and Magnetism"

Fig. 9.2 Michael Faraday (1791–1897). Parts of an old 10-pound banknote from Great Britain

in 1865. His original article consisted of 20 equations with 20 unknowns. Maxwell did not use the vector field formalism familiar to us.

Oliver Heaviside (1850–1925) gave in 1884 the equations in about the form we use today. Heaviside, who was from a poor home, left school when he was 16 and receive no formal education subsequently. Nevertheless, he made a number of important contributions to physics. It is fascinating to read about him, for example, in Wikipedia. (There are certain similarities between Heaviside and Faraday. Faraday's story is also fascinating, and highly recommended to read, and is even honoured on a British banknote; see Fig. 9.2. Heaviside did not receive similar recognition.)

The German physicist Heinrich Hertz (1857–1894) was the first to demonstrate how we can send and receive electromagnetic waves. It happened in 1887 when Hertz was 30 years old.

It is interesting that Hertz is honoured by a number of stamps from many different countries, while Maxwell is far from getting the same honour.

Recapitulation from electromagnetism: It might be appropriate to begin with a little repetition of some details here. We will later see that magnetic permeability and, in particular, electrical permittivity play an important role in electromagnetic waves. The values in vacuum μ_0 and ε_0 are rather uninteresting. They are primarily related to the choice of units for electrical and magnetic fields.

The relative values, however, are of far more interest. The relative (magnetic) permeability is related to how much magnetic field is generated in a material when it is exposed to an external magnetic field. In a diamagnetic material, a tiny magnetic field is generated in the material, and the field is directed opposite the external magnetic field. Even in a paramagnetic material, only a tiny magnetic field is generated, but it is in the same direction as the extraneous field. The magnetic field generated in the material is only of the order of 10^{-5} times the external magnetic field in each of these cases. In a ferromagnetic material, a significant magnetic field is generated inside the material, and it is in the same direction as the applied field. There are many details related to these processes, and we do not deal with these here.

Since most of the substances we come in contact with are either diamagnetic or paramagnetic, we can simply set the relative permeability equal to 1.0 and ignore the interaction of the magnetic field with materials in the processes we are going to discuss.

For the electrical field, it is different. The relative (electrical) permittivity tells us something about the amount of electrical field that occurs inside a material when subjected to an external electric field. In Fig. 9.3, a schematic representation of what is happening is given. An external electric field will cause the electron cloud around an atomic core to shift slightly. However, since there are so

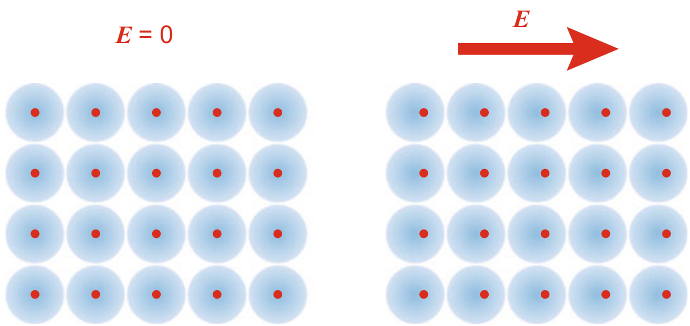

Fig. 9.3 In a material such as glass, an external electric field can easily cause a polarization of the charge distribution in each individual atom of the material. This polarization leads to an electric field inside the material directed opposite to the applied electric field

many atoms, even an almost negligible shift in position of the negatively charged electron clouds relative to the positively charged nuclei, the generated electric field inside the material can easily reach the same magnitude as the outer electric field (e.g. half the size).

There is no talk of moving free charges here, only of a local distortion of the charge distribution of each individual atom, which imparts, nonetheless, a polarization to the entire material. Note that we are talking about "polarization" in a certain sense. We will soon talk about polarization in a completely different context, so it is imperative that you do not mix different terms with the same name!.

9.3 Differential Form

We will now show how we can go from the integral form of Maxwell's equations to the differential form. The integral form can be applied to macroscopic geometries, for example to find the magnetic field at a distance of 5 m from a straight conductor where there is a net electrical current. The differential form applies to a small region of space. How "small" this might be is a matter for discussion. Maxwell's equations were developed before we had a proper knowledge of the structure of atoms and substances on the microscopic level. Maxwell's equations in differential form are often used in practice *on an intermediate length scale that is small in relation to the macroscopic world and yet large compared to atomic dimensions*.

In going over from the integral to differential form, two mathematical relationships are invoked that apply to an arbitrary vector field $\vec{G}$ in general:

Stokes's theorem (more correctly the Kelvin–Stokes theorem, since the theorem first became known through a letter from Lord Kelvin. George Stokes (1819–1903) was a British mathematician/physicist. Lord Kelvin (1824–1907), whose real name was William Thomson, was a physicist/mathematician contemporary of Stokes.)

Stokes's theorem:

$$\oint \vec{G} \cdot \mathrm{d}\vec{l} = \int_A (\nabla \times \vec{G}) \cdot \mathrm{d}\vec{A} \,. \tag{9.6}$$

The theorem provides a relation between the line integral of a vector field and the flux of the curl of the vector field through the plane limited by the line.

The second relationship we use is the *divergence theorem* (discovered by Lagrange and rediscovered by several others). Joseph Louis Lagrange (1736–1813) was an Italian/French mathematician and astronomer:

Divergence theorem:

$$\int \nabla \cdot \vec{G} \mathrm{d}v = \oint_A \vec{G} \cdot \mathrm{d}\vec{A} \,. \tag{9.7}$$

The divergence theorem provides the connection between divergence to a vector field in a volume and the flux of the vector field through the surface which bounds the volume.

Gauss's law for electric field:
We start with Gauss's law for electric field.

$$\varepsilon_r \varepsilon_0 \oint \vec{E} \cdot \mathrm{d}\vec{A} = Q_{\text{inside}} \,.$$

By using the divergence theorem, we get:

$$\oint \varepsilon_r \varepsilon_0 \vec{E} \cdot \mathrm{d}\vec{A} = \int \nabla \cdot (\varepsilon_r \varepsilon_0 \vec{E}) \,\mathrm{d}v = Q_{\text{inside}} \,.$$

We now choose such a small volume that $\nabla \cdot (\varepsilon_r \varepsilon_0 \vec{E})$ is approximately constant over the entire volume. This constant can then be taken outside the integral sign, and integration over the volume element simply gives the small volume Δv under consideration. Accordingly:

$$\int \nabla \cdot (\varepsilon_r \varepsilon_0 \,\vec{E}) \,\mathrm{d}v \approx (\nabla \cdot \vec{D}) \Delta v = Q_{\text{inside}}$$

$$\nabla \cdot \vec{D} = \frac{Q_{\text{inside}}}{\Delta v} = \rho$$

where ρ is the local charge density. We are led thereby to Gauss's law for electric fields in differential form:

$$\nabla \cdot \vec{\boldsymbol{D}} = \rho \; . \tag{9.8}$$

Gauss's law for magnetic field:
The same approach leads us to the differential form of Gauss's law for magnetic field:

$$\nabla \cdot \vec{\boldsymbol{B}} = 0 \; . \tag{9.9}$$

The Faraday–Henry law:
We will now rephrase Faraday's law. The starting point is thus:

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = -\left(\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}\right)_{\text{inside}} \; .$$

The application of Stokes's theorem now gives:

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \int_A (\nabla \times \vec{\boldsymbol{E}}) \cdot \mathrm{d}\vec{\boldsymbol{A}} = -\left(\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}\right)_{\text{inside}} \; .$$

The magnetic flux through the surface can be expressed as:

$$\Phi_B = \int_A \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{A}} \; .$$

Hence,

$$\begin{aligned} \int_A (\nabla \times \vec{\boldsymbol{E}}) \cdot \mathrm{d}\vec{\boldsymbol{A}} &= -\frac{\mathrm{d}}{\mathrm{d}t} \int_A \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{A}} \\ &= -\int_A \frac{\partial \vec{\boldsymbol{B}}}{\partial t} \cdot \mathrm{d}\vec{\boldsymbol{A}} \; . \end{aligned}$$

In taking the last step, we have assumed that the area element $\mathrm{d}A$ does not change with time. In addition, we have changed the ordinary derivative to partial derivative since the magnetic flux density $\vec{\boldsymbol{B}}$ depends on both time and spatial relationships, but we assume that spatial conditions do not change in time. Again, for a small enough area A, the functions to be integrated can be regarded as constants and placed outside the integral signs, which leads to the result:

$$\nabla \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \; . \tag{9.10}$$

This is Faraday's law in differential form.

The Ampère–Maxwell law:
The same procedure can be used to show the last of Maxwell's equations in a differential form, namely the Ampère–Maxwell law. The result is:

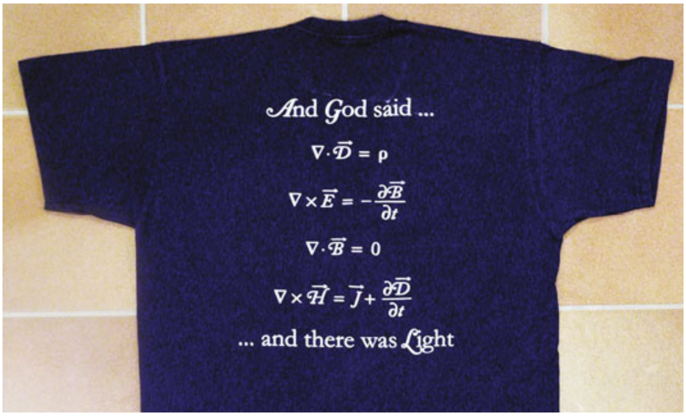

Fig. 9.4 Maxwell's equations on a T-shirt

$$\nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{j}}_f + \frac{\partial \vec{\boldsymbol{D}}}{\partial t} \tag{9.11}$$

where $\vec{\boldsymbol{j}}_f$ is the electric current density of the free charges.

Einstein had pictures of Newton, Maxwell and Faraday in his office, indicating how important he thought their works to be. It is therefore not surprising that the Physics Association at UofO has chosen Maxwell's equations on their T-shirts (see picture 9.4) as a symbol of a high point in physics, a high point as regards both how powerful equations are and how mathematically elegant they are! (It should be noted, however, that mathematical elegance did not seem to be as polished at Maxwell's time as it is today.)

Collected:

Let us assemble all of Maxwell's equations in differential form:

$$\nabla \cdot \vec{\boldsymbol{D}} = \rho \tag{9.12}$$

$$\nabla \cdot \vec{\boldsymbol{B}} = 0 \tag{9.13}$$

$$\nabla \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \tag{9.14}$$

$$\nabla \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{j}}_f + \frac{\partial \vec{\boldsymbol{D}}}{\partial t} \tag{9.15}$$

Maxwell's equations in the presence of the Lorentz force

$$\mathbf{F} = q(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}})$$

form the full basis for classical electrodynamic theory.

9.4 Derivation of the Wave Equation

The wave equation can be derived from Maxwell's equations using primarily the last two equations along with a general relation that applies to an arbitrary vector field **G**:

$$\nabla \times (\nabla \times \vec{\boldsymbol{G}}) = -\nabla^2 \vec{\boldsymbol{G}} + \nabla(\nabla \cdot \vec{\boldsymbol{G}}) \ . \qquad (9.16)$$

In words, the relation states that "the curl of the curl of a vector field is equal to the negative of the Laplacian applied to the vector field plus the gradient of the divergence of the vector field" (pause for breath).

Application of this relation to the electric field yields:

$$\nabla \times (\nabla \times \vec{\boldsymbol{E}}) = -\nabla^2 \vec{\boldsymbol{E}} + \nabla(\nabla \cdot \vec{\boldsymbol{E}}) \ .$$

We recognize the curl of the electric field in the expression on the left-hand side. Replacing it by Faraday's law, interchanging the right and left side of the equation, and changing the sign, we get:

$$\nabla^2 \vec{\boldsymbol{E}} - \nabla(\nabla \cdot \vec{\boldsymbol{E}}) = -\nabla \times \left(-\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \right) \ . \qquad (9.17)$$

On the right-hand side, we change the order of differentiation to find:

$$= \frac{\partial}{\partial t} (\nabla \times \vec{\boldsymbol{B}}) \ .$$

Applying now the Ampère–Maxwell law, and using the relation

$$\vec{\boldsymbol{B}} = \mu_r \mu_0 \vec{\boldsymbol{H}}$$

we get:

$$= \frac{\partial}{\partial t} \left[\mu_r \mu_0 \left(\frac{\partial \vec{\boldsymbol{D}}}{\partial t} + \vec{\boldsymbol{j}}_f \right) \right] \ . \qquad (9.18)$$

For the left-hand side of Eq. (9.17), Gauss's law is used for electric field to replace the divergence of electric field in the second link on the left side with charge density ρ divided by total permittivity.

$$\nabla^2 \vec{\boldsymbol{E}} - \frac{\nabla \rho}{\varepsilon_r \varepsilon_0} \; . \tag{9.19}$$

By equating the right-hand side of (9.19) to the left-hand side of (9.18), and transposing some terms, we end up with:

$$\nabla^2 \vec{\boldsymbol{E}} - \varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2} = \frac{\nabla \rho}{\varepsilon_r \varepsilon_0} + \mu_r \mu_0 \frac{\partial \vec{\boldsymbol{j}}_f}{\partial t} \; . \tag{9.20}$$

This is a nonhomogeneous wave equation for electric fields. The source terms are on the right side of the equality sign.

In areas where the gradient of charge density ρ is equal to zero (i.e. no change in electrical charge density), while there is no time variation in electrical current density $\vec{\boldsymbol{j}}_f$ of free charges, the inhomogeneous equation is reduced to one simple wave equation:

$$\nabla^2 \vec{\boldsymbol{E}} - \varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2} = 0$$

or in the more familiar form:

$$\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2} = \frac{1}{\varepsilon_r \varepsilon_0 \mu_r \mu_0} \nabla^2 \vec{\boldsymbol{E}} \; . \tag{9.21}$$

Well, to be honest, this is not an ordinary wave equation, as we have seen before, since we have used the Laplacian on the right-hand field. In detail, we have:

$$\begin{aligned} \nabla^2 \vec{\boldsymbol{E}} = & \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right) \vec{\boldsymbol{i}} \\ & + \left(\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \right) \vec{\boldsymbol{j}} \\ & + \left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) \vec{\boldsymbol{k}} \; . \end{aligned} \tag{9.22}$$

We search now for the simplest possible solution and investigate if there is a solution where $\vec{\boldsymbol{E}}$ is independent of both x and y. In that case, all terms of the type $\partial^2 E_u / \partial v^2$ will vanish, where $u = x, y, z$ and $v = x, y$. If such a solution is possible, it will involve a planar wave that moves in the z-direction, since a plane wave is just unchanged in an entire plane perpendicular to the wave direction of motion.

Equation (9.22) then reduces to the following simplified form:

$$\nabla^2 \vec{\boldsymbol{E}} = \left(\frac{\partial^2 E_x}{\partial z^2}\right) \vec{\boldsymbol{i}} + \left(\frac{\partial^2 E_y}{\partial z^2}\right) \vec{\boldsymbol{j}} + \left(\frac{\partial^2 E_z}{\partial z^2}\right) \vec{\boldsymbol{k}} = \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2} \tag{9.23}$$

and Eq. (9.21) along with Eq. (9.23) finally gives us a common wave equation:

$$\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2} = c^2 \frac{\partial^2 \vec{\boldsymbol{E}}}{\partial z^2} \tag{9.24}$$

where

$$c = \frac{1}{\sqrt{\varepsilon_r \varepsilon_0 \mu_r \mu_0}} . \tag{9.25}$$

is the phase velocity of the wave.

We already know that one solution of the wave equation, Eq. (9.24), is

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_0 \cos(kz - \omega t) \tag{9.26}$$

where $\vec{\boldsymbol{E}}_0$ is a constant vector whose direction can be chosen arbitrarily within the $x - y$-plane.

Let us now see whether we are able to derive a wave equation for the magnetic field. To this end, we start with Eq. (9.16), but apply it to the magnetic flux density and write

$$-\nabla^2 \vec{\boldsymbol{B}} + \nabla(\nabla \cdot \vec{\boldsymbol{B}}) = \nabla \times (\nabla \times \vec{\boldsymbol{B}}) .$$

We use next the Ampère–Maxwell law in order to replace the curl of $\vec{\boldsymbol{B}}$ with the time derivative of the electric flux density $\vec{\boldsymbol{D}}$ plus the current density of free charges. As in the corresponding derivation for the electric field, we interchange the order of time and space derivatives, obtaining thereby a term containing the curl of $\vec{\boldsymbol{E}}$. We invoke Faraday's law and the vanishing of the divergence of $\vec{\boldsymbol{B}}$ (Gauss's law for magnetic fields), to arrive finally at the following equation for $\vec{\boldsymbol{B}}$:

$$\nabla^2 \vec{\boldsymbol{B}} - \varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \vec{\boldsymbol{B}}}{\partial t^2} = -\mu_r \mu_0 \nabla \times \vec{\boldsymbol{j}}_f . \tag{9.27}$$

We observe that the magnetic flux density also satisfies an inhomogeneous wave equation, in which the source term is the curl of the current density of free charges. For regions of space which are source-free, we obtain a homogeneous wave equation. We seek the simplest solution to the equation, and ask, as we did in the case of the electric field, whether a plane-wave solution exists for a wave propagating in the z-direction. With the same simplifications as those used earlier, we obtain

$$\frac{\partial^2 \vec{B}}{\partial t^2} = c^2 \frac{\partial^2 \vec{B}}{\partial z^2} \tag{9.28}$$

where the wave velocity c is precisely that given in Eq. (9.25), applicable to the electric field.

We already know that, in this case as well, the equation does have a solution, which can be written in the form:

$$\vec{B} = \vec{B}_0 \cos(kz - \omega t) \tag{9.29}$$

where $\vec{B}_0$ is a constant vector whose direction is essentially arbitrary.

9.5 A Solution of the Wave Equation

Equations (9.26) and (9.29) are valid solutions of the two wave Eqs. (9.24) and (9.28), respectively. But the solutions must also satisfy Maxwell's equations individually, in practice the Ampère–Maxwell law and Faraday's law.

We start with Faraday's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

and substitute the solution for the electric field (9.26) (in determinant form):

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial \vec{B}}{\partial t}$$

$$\left\{ \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \vec{i} - \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \vec{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \vec{k} \right\} = -\frac{\partial \vec{B}}{\partial t} .$$

For the plane-wave solution sought by us, partial derivatives with respect to x or y will vanish, and the expression takes the simpler form shown below:

$$-\frac{\partial E_y}{\partial z} \vec{i} + \frac{\partial E_x}{\partial z} \vec{j} = -\frac{\partial \vec{B}}{\partial t} .$$

We already notice that $\vec{B}$ cannot have any component in the z-direction, that is, the direction of propagation of the wave. A similar analysis using Ampère–Maxwell's law shows that nor can $\vec{E}$ have a z-component (except for a static homogeneous field, which is of no interest in the present context).

We choose now the following solution for $\vec{E}$:

$$\vec{E} = E_0 \cos(kz - \omega t)\,\vec{i} \tag{9.30}$$

which means that $E_y = 0$, hence also $E_y = 0$ and also $\partial E_y/\partial z = 0$

Consequently,

$$\frac{\partial E_x}{\partial z}\vec{j} = kE_0 \sin(kz - \omega t)\,\vec{j} = -\frac{\partial \vec{B}}{\partial t} .$$

This equation will be satisfied if

$$\vec{B} = B_0 \cos(kz - \omega t)\,\vec{j} . \tag{9.31}$$

Furthermore, since

$$-\frac{\partial \vec{B}}{\partial t} = \omega B_0 \sin(kz - \omega t)\,\vec{j}$$

and the (phase) velocity of this plane wave is ω/k which must be equal to c from Eq. (9.25), we also get an important connection between electric and magnetic field in an electromagnetic wave:

$$E_0 = cB_0 . \tag{9.32}$$

We have then shown that *one* possible solution to Maxwell's equations for a space where no charges are present is a planar electromagnetic wave

$$\vec{E} = E_0 \cos(kz - \omega t)\,\vec{i} \tag{9.33a}$$

$$\vec{B} = B_0 \cos(kz - \omega t)\,\vec{j} \tag{9.33b}$$

where

$$E_0 = cB_0$$

You are reminded that solutions of wave equations generally depend on a great extent on boundary conditions. In our derivation, we have searched for a solution that gives a planar wave. In practice, this amounts to assuming that the area under consideration is located *far* from the source of the wave, as well as free charges and currents generated by such charges. The plane-wave solution is therefore, in principle, *never a perfect solution* of Maxwell's equations, but an exact solution may in some cases be quite close to a planar wave solution. It is our task as physicists to decide whether or not we can model real field distribution with a plane-wave description in each case. See the description of the near field and far field given below.

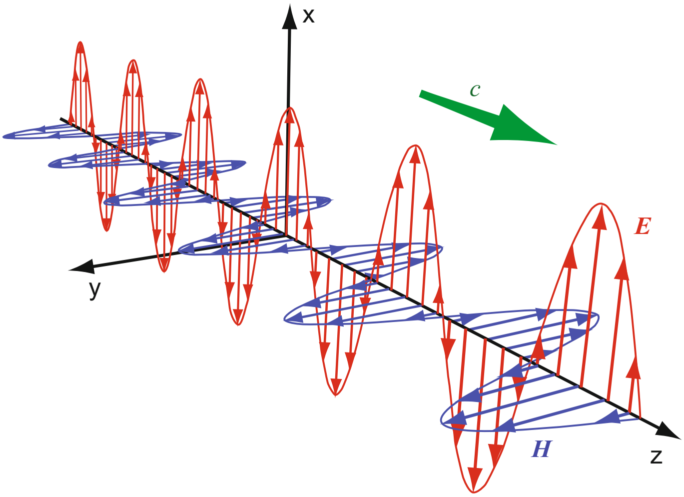

Fig. 9.5 A snapshot of the simplest form of electromagnetic wave, namely a plane wave. Such a wave can be realized sufficiently far from the source of the wave and from materials that can perturb the wave. Experience has shown that figures of this type cause many misunderstandings, which are discussed in the last part of this chapter

Since Maxwell's equations are linear, we can have plane electromagnetic waves *in addition* to other electrical or magnetic fields with completely different characteristics. *The sum* of all contributions will then not follow the relationships given in the blue box above!

Figure 9.5 shows a snapshot of an electromagnetic wave with properties as given in Eq. (9.33). Such a static figure does not give a good picture of the wave. It is therefore advisable to consider an animation to get an understanding of time development. There are several animations of a simple electromagnetic wave on the Web (but some of them have wrong directions of the vectors!).

9.6 Interesting Details

What determines the speed of light?

We saw in the derivation of the wave equation that electromagnetic waves have a phase velocity

$$c = \frac{1}{\sqrt{\varepsilon_r \varepsilon_0 \mu_r \mu_0}} \,. \tag{9.34}$$

In vacuum, $\varepsilon_r = 1$ and $\mu_r = 1$, and the velocity of the wave becomes

$$c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \,. \tag{9.35}$$

This is simply an expression for the velocity of light in vacuum.

It must have been a wonderful experience for Maxwell when he first understood this. The speed of light was known, but not associated with any other physical parameters. Then, Maxwell derives the wave equation and finds that the equations allow for the existence of electromagnetic waves, and—as it happens—these waves will propagate with the familiar speed of light! The surprise must have been particularly high because the speed of light closely follows the electrical and magnetic constants ε_0 and μ_0, which were determined by *static* electrical and static magnetic phenomena.

In glass, the velocity of light is given by Eq. (9.34), but for glass μ_r is practically equal to 1. That means that it is simply the relative electrical permittivity of the glass, which causes light to be slower in glass than in vacuum. This too is surprising since the relative permittivity can be determined by putting a glass plate between two metal plates and measuring change in capacitance between the metal plates. Even this measurement can be made by using static fields, and equally this quantity plays an important role for light oscillating with a frequency of the order of 10^{15} times per second!

There is no dispersion in vacuum, but in a dielectric material dispersion may occur because ε_r (and/or μ_r) is wavelength dependent. We discussed this in Chap. 8 when treated dispersion and the difference between phase velocity and group velocity.

It will be noted that when we discuss the passage of light through glass, we are dealing with a material constant called *refractive index n* which varies slightly from one glass to another. The phase velocity of light is lower in glass than in vacuum. The refractive index n can be defined as the ratio of the light velocity in vacuum to the light velocity in glass: $n = c_0/c$. The word *refractive* index will be explained in detail when we in Chap. 10 describe how light rays change direction when the beam is inclined towards an interface between air and glass or the other way round (Snel's law).

Glass is diamagnetic and $\mu_r \approx 1.0$. From the above expressions, then the refractive index is approximately equal to the square root of the relative permittivity:

$$n \approx \sqrt{\varepsilon_r} \,. \tag{9.36}$$

Relative permittivity is also called dielectric constant.

Plane wave

The wave we have described is plane because the electric field at a given instant is identical everywhere in an infinite plane normal to the wave propagation direction z. Another way of expressing this is to say that the "wavefront" is plane. The wavefront of a plane wave is a surface of constant phase (i.e. the argument of the sine or cosine function is identical at all points of the surface at a given time).

The fact that the electric field everywhere is directed in the $\pm x$-direction is a characteristic feature of the solution we have found. We say that the wave is *linearly polarized* in the x-direction. We return to polarization in Chap. 10, but already mention here that another solution to Maxwell's equations is a so-called circularly polarized wave. For such a solution, the electric field vectors in a snapshot corresponding to Fig. 9.5 will look like the steps in a spiral staircase, and the arrows themselves will form a "spiral" whose axis coincides with the z-axis. The magnetic field will also form a spiral, and in this case too the electric and magnetic fields will be perpendicular to each other and perpendicular to the direction of propagation. You can find nice animations of circularly polarized electromagnetic waves on the Web.

In addition, we will return later to an important discussion of the validity of the simple electromagnetic waves we have described so far.

9.7 The Electromagnetic Spectrum

In deriving the wave equation for electromagnetic waves, we placed (initially) no restrictions on the frequencies and wavelengths. In principle, more or less "all" frequencies (with the corresponding wavelengths) were eligible for consideration.

It turns out also in practice that we can generate electromagnetic waves for a wide range of frequencies (and wavelengths). Figure 9.6 shows an approximate overview of the frequency ranges/wavelength ranges we operate in, what we call the waves at different frequencies and what such waves are used for. We say that figures like 9.6 present "the electromagnetic spectrum".

Figures of this type must be taken with a large pinch of salt. They seduce many people into thinking that there exist tidy plane waves at each of the specified frequencies, but that is not the case. The spreading of waves in time and space, energy transport (or its absence) and several other factors vary widely from one frequency to another. We will come back to this a little later in this chapter.

9.8 Energy Transport

When we discussed sound, we saw that a sound wave carry energy many metres away from the source, although the molecules that contributed to the transmission through oscillatory motion only moved back and forth over distances of the order of 1 μm or less (when we ignore the diffusive motion of the molecules).

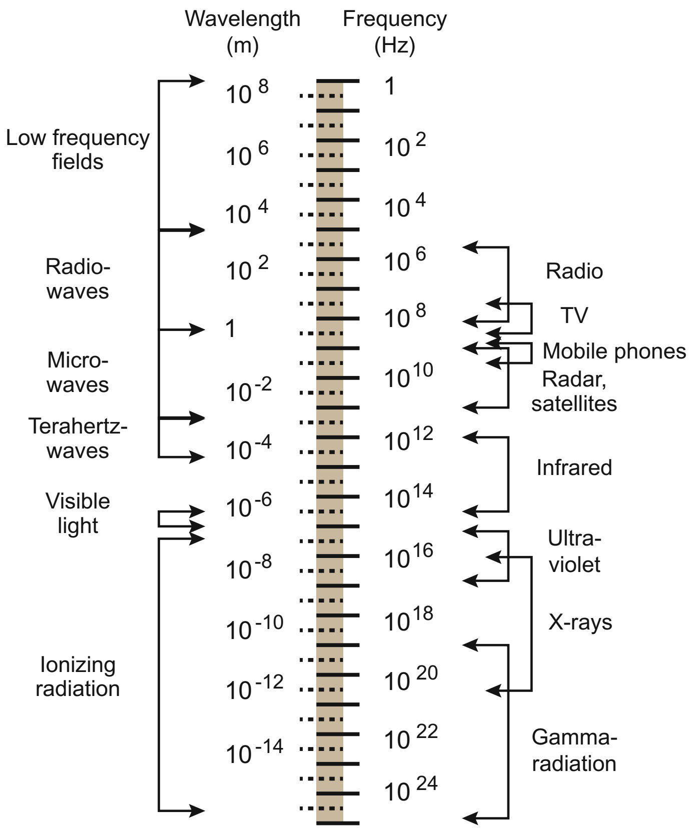

Fig. 9.6 Electromagnetic waves can exist in an impressive range of frequencies (and corresponding wavelengths). Surveys such as this may, however, give an impression of a greater degree of similarity between different phenomena than it is in practice. We will come back to this, for example, when we discuss the difference between near field and far field later in the chapter

In a similar manner, an electromagnetic wave can carry energy, something we all experience when we bask in the Easter sun on a mountain or when we lie on a sunny beach in summer.

An electric field has an energy density given by:

$$u_E(z,t) = \tfrac{1}{2}E(z,t)D(z,t) \ .$$

Similarly, the energy density of a magnetic field is given by:

$$u_H(z,t) = \tfrac{1}{2}H(z,t)B(z,t) \ .$$

When a plane electromagnetic wave (as described above) passes, the instantaneous energy density will be:

$$u_{\text{tot}}(z,t) = \tfrac{1}{2}E(z,t)D(z,t) + \tfrac{1}{2}H(z,t)B(z,t)$$
$$= \frac{1}{2}E_0\cos()\ \varepsilon E_0\cos() + \frac{1}{2}B_0\cos()\ \frac{B_0}{\mu}\cos()\ .$$

The arguments of the cosine function have been omitted in order to avoid clutter.

But we know that $E_0 = cB_0$. In addition, we want to look at the *time-averaged* energy density, and we know that the mean value of $\cos^2()$ is equal to half. Consequently, we find for the time-averaged energy density:

$$\bar{u}_{tot} = \frac{1}{4}\varepsilon E_0^2 + \frac{1}{4\mu}\left(\frac{E_0}{c}\right)^2\ .$$

Now, energy density is energy per unit volume. How much energy will cross a hypothetical surface A perpendicular to the direction of wave motion over a time Δt? Such a quantity defines the (time-averaged) wave *intensity*:

$$I = \text{intensity} = \frac{\text{Energy passed by}}{\text{Area} \times \text{Time}} = u_{\text{tot}} \times c\ .$$

The expression is relevant only when we consider a long time compared to the time a wavelength needs to pass our surface. Intended for the energy density we found in town, we get:

$$I = \frac{1}{4}\left(c\varepsilon E_0^2 + c\frac{1}{c^2\mu}E_0^2\right)\ .$$

But we know that

$$c = \frac{1}{\sqrt{\varepsilon\mu}}$$

from which follows the relation

$$\frac{1}{c^2\mu} = \varepsilon$$

and we see that the energy contributions from the electric field and the from the magnetic field are precisely equal!

Consequently, the intensity of an electromagnetic wave is given by the expression:

$$I = \tfrac{1}{2}c\varepsilon E_0^2 = \tfrac{1}{2}cE_0D_0\ . \tag{9.37}$$

By using the familiar ratio between electric and magnetic fields, the result can also be written as follows:

$$I = \frac{1}{2} c \frac{1}{\mu} B_0^2 = \frac{1}{2} c H_0 B_0 \ . \tag{9.38}$$

If we choose to specify the strength of the electric and magnetic fields in terms of the effective values instead of the amplitudes, Eqs. (9.37) and (9.38) can be recast as:

$$I = c\varepsilon E_{\text{eff}}^2 = c E_{\text{eff}} D_{\text{eff}} \tag{9.39}$$

and

$$I = \frac{c}{\mu} B_{\text{eff}}^2 = c H_{\text{eff}} B_{\text{eff}} \ . \tag{9.40}$$

A small digression: The term "effective value" can be traced to alternating current terminology. We can then state the amplitudes of harmonically varying current and voltage, but we can also specify the equivalent value of direct current and direct voltage that supply the same power to a given load; these direct current/voltage values are called effective values. In our case of electromagnetic waves, it is rather artificial to speak of direct currents and suchlike, yet we speak of effective values in the same way as for alternating currents and voltages in a wire.

We can also derive another expression that connects electrical and magnetic fields to an electromagnetic wave in the remote field. Going back to Eqs. (9.39) and (9.40), and using the relationship $B = \mu H$, we get:

$$c\varepsilon E_{\text{eff}}^2 = \frac{c}{\mu} B_{\text{eff}}^2 = c\mu H_{\text{eff}}^2$$

and are led thereby to the relation:

$$\frac{E_{\text{eff}}}{H_{\text{eff}}} = \sqrt{\mu/\varepsilon} \ .$$

For vacuum, we obtain:

$$\frac{E_{\text{eff}}}{H_{\text{eff}}} = \sqrt{\mu_0/\varepsilon_0} \equiv Z_0 = 376.7\,\Omega \tag{9.41}$$

where Z_0 is called *the (intrinsic) impedance of free space*.

The expressions have a greater scope than that warranted by our derivation. However, we must be careful about using the terms of electromagnetic waves in regions near sources and near materials that can interfere with the waves. We refer to the so-called near field and far field a little later in this chapter.

9.8.1 Poynting Vector

There is a more elegant way to specify energy density (equivalent to intensity) than the expressions presented in the previous section. The elegance is a consequence of the fact that plane electromagnetic waves are transverse, with the electrical and magnetic vectors perpendicular to each other and to the direction of propagation of the wave.

We saw that if the electric field was directed in x-direction and magnetic field in y-direction, the wave moved in z-direction. We know that for the cross-product, the relation $\vec{i} \times \vec{j} = \vec{k}$ holds, which suggests that we may be able to utilize this relationship in a smart way.

We try to calculate:

$$\begin{aligned}\vec{E} \times \vec{B} &= E_0 \cos() \vec{i} \times \frac{E_0}{c} \cos() \vec{j} \\ &= \frac{c E_0^2}{c^2} \cos^2() \vec{k} \\ &= \mu (c \varepsilon E_0^2) \cos^2() \vec{k} \ .\end{aligned}$$

The time-averaged values are (using Eq. (9.37) in the last part):

$$\overline{\vec{E} \times \vec{B}} = \mu (\frac{1}{2} c \varepsilon E_0^2) \vec{k} = \mu I \vec{k} \ .$$

Since $B = \mu H$, it follows that:

$$\vec{i} = \overline{\vec{E} \times \vec{H}} \ . \tag{9.42}$$

Here, we have introduced an intensity vector that points in the same direction as the energy flow.

More often, we operate with the *instantaneous intensity* in the form of a "Poynting vector". This is usually designated by the symbol S or P. We choose the first variant and write:

$$\vec{S} = \vec{E} \times \vec{H} \ . \tag{9.43}$$

Poynting vector provides us with a nice expression of energy flow in an electromagnetic wave.

However, the Poynting vector can be used only in the trouble-free cases where we have a simple plane electromagnetic wave far from the source and far away from disturbing elements. Put in another way: it can only be used in the far-field region (see below) where the electromagnetic fields are totally dominated by pure electrodynamics.

The English physicist John Henry Poynting (1852–1914) deduced this expression in 1884, 20 years after Maxwell wrote his most famous work.

9.9 Radiation Pressure

The electric and magnetic fields will exert a force on particles/objects struck by an electromagnetic wave. It is possible to argue that the electric field in the wave causes "forced oscillations" of charges, and that moving charge, in turn, experiences a force $\vec{F} = q\vec{v} \times \vec{B}$. This force works in the same direction as that in which the electromagnetic wave moves.

It can be shown that an electromagnetic wave causes a radiation pressure given by:

$$p_{\text{radiation}} = S_{\text{time-avg}}/c = I/c$$

if the wave is completely absorbed by the body being taken. If the body reflects the waves completely, the radiation pressure becomes twice as large, i.e.

$$p_{\text{radiation}} = 2S_{\text{time-avg}}/c = 2I/c \ .$$

In both of these terms, $S_{\text{time-avg}}$ is the absolute value of the time-averaged Poynting vector. The direction of the radiation pressure is usually identical to the direction of the Poynting vector.

It is the radiation pressure that causes the dust in a comet to always turn away from the sun. The gravitational pull exerted by sun on the dust is proportional to the mass, which in turn is proportional to the cube of the radius. The force due to the radiation pressure is proportional to the *surface* (cross section) that can absorb or reflect the wave, and the cross section goes as the square of the radius. This results in gravity dominating over radiation pressure for large particles, while the converse happens for small particles.

It is possible to regard radiation pressure as a flow rate of electromagnetic momentum. In such a picture, it can be said that the momentum per time and per unit surface moving with the wave is equal

$$S_{\text{time-avg}}/c$$

which is the same expression as for radiation pressure when the body absorbs the wave completely.

The description above applies in the event that light is either absorbed or totally reflected on the surface of a material. The situation is different for light passing through a transparent medium.

There are two different descriptions of how the momentum of light changes when light enters a transparent medium. In one description, it is claimed that the momentum increases, and in another description, the opposite is claimed. This is an optical dilemma that partly depends on whether light is regarded as waves or as particles. In this way, there is a clear parallel between the dilemma we have today and the dilemma that existed from the seventeenth century to about 1850 mentioned in the previous chapter, where we wondered whether the group velocity of light in glass was larger or smaller than the phase velocity.

If you want to learn a little more about today's dilemma, start by reading a popular scientific article by Edwin Cartlidge in Physics World.

9.10 Misconceptions

First: A small reminder …

Note that nothing actually protrudes from an electromagnetic wave. For any arbitrary point in space, the field itself changes the value. The field has a direction in space, but no arrows shoot out to the side and no sinusoidal curves are found along the wave. It is therefore a totally different situation than when, for example, we pluck a guitar string where the string actually moves across the longitudinal direction.

9.10.1 Near Field and Far Field

We have repeatedly reminded the reader of this chapter that the electromagnetic waves we have derived in Eq. (9.33) and illustrated in Fig. 9.5 are the *simplest* wave solutions of Maxwell's equations. Usually, these relationships *do not* apply to time-dependent electromagnetic phenomena in general! To understand this, we need to look more closely at the details in our derivation.

First, we ended up with inhomogeneous differential equations in Eqs. (9.20) and (9.27) as a result of combining Maxwell's equations. Only by ignoring the source terms did we arrive at the simple homogeneous wave equations that became the starting point for the plane-wave solution.

Even if there are no charges and currents in the region of our interest, fields from nearby regions can have a big influence. For example, will electric fields from charge distributions in an antenna and magnetic fields from electric currents in an antenna dominate the electromagnetic fields pattern nearby the antenna, even if it is placed in vacuum. This pattern is *not* what we find in an electromagnetic wave.

A rule of thumb in this context is that we use the word "nearby" for distances up to a few times the calculated wavelength ($\lambda = c/f$), *and/or* up to several times

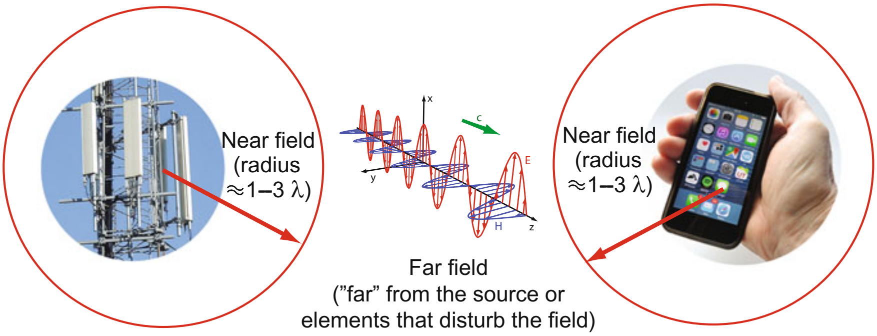

Fig. 9.7 "Near fields" dominate the electromagnetic field pattern at a distance up to the order of a calculated wavelength $\lambda = c/f$ away from charges/electrical currents. In the near-field zone, the solution of Maxwell's equations is often very different from the solution in the far-field zone (far from the source of the fields and far from disturbing elements)

the extent of the object in space away from a region where there are free charges or currents. In regions that are influenced by boundary conditions in the broadest sense, we find "*near fields*", as opposed to "*far fields*", which we find in areas where boundary conditions have almost no influence (Fig. 9.7).

It may be useful to think about how far the near-field region extends from different sources. For a light source, the wavelength is about 500 nm. The near-field range extends a few times this distance away from the source, i.e. of the order of a few microns (thousands of millimetres) away from the source.

For a mobile phone that operates at 1800 MHz, the calculated wavelength is about 16 cm. A few times this distance takes one over to the far-field zone.

To sum up:

For the *far-field region*, the following relationships we have established for simple plane electromagnetic waves are:

1. The electric and magnetic fields are perpendicular to each other.
2. There is a fixed ratio between electric and magnetic fields.
3. The Poynting vector provides a measure of transport of electromagnetic energy.
4. The energy that passes a cross section has left the source once and for all and does not (normally) return.
5. It may therefore be natural to use the word "radiation" for energy transport.

For the *near-field zone*, however, the following applies:

1. The electric and magnetic fields are normally *not* perpendicular to each other.
2. There is *no* a fixed ratio between electrical and magnetic fields.

3. The Poynting vector does *not* provide a measure of transport of electromagnetic energy.

4. Energy can build up in the vicinity of the source for some periods of time, but retracts again in other periods of time. Only, a tiny part of the energy that goes back and forth to the vicinity will leave the source like waves (and this energy transport is generally not apparent before we get into the far-field zone).
5. It is therefore not natural to use the word "radiation". We describe the situation more like "fields".

9.10.2 The Concept of the Photon

I would like to append a few comments concerning the term "photon".

The majority of today's physicists believe that light is best described as elementary particles, called photons.

A photon is perceived as an "indivisible wave packet or energy packet" with a limited extension in time and space. The word photon was originally used for visible light where the wavelength is of the order of 500 nm (the Greek word "phos" means "light"). This means that even a wave packet containing quite a few wavelengths will be tiny compared to macroscopic dimensions. In this case, then, it is not particularly odd that we perceive this as a "particle". The notion of the indivisible energy packet is assigned the energy $E = h\nu$ where h denotes Planck's constant and ν is the frequency.

Problems soon arise with "photons" in the realm of mobile telephony (and radio waves). In that case, a wave packet consisting of a few wavelengths will inevitably occupy a spatial extent of several metres (up to kilometres). Does it make sense to regard such a packet as "indivisible" and to think that energy is exchanged instantaneously from the antenna to the packet and from the latter to surrounding space?

For power lines and 50 Hz fields, the problem is even worse. For 50 Hz, a wave packet of several times the wavelength would soon extend to dimensions comparable to the perimeter of the earth! We then get serious problems imagining a photon that extends several times the wavelength. And if we consider the photon as small particles instead of an extended wave packet, it will be problematic to explain wavelengths and a variety of other properties. Furthermore, the distribution of electrical and magnetic fields near power lines is significantly different from that of light. This can be grafted into a quantum mechanical description, but then one has to resort to strange special variants where the quantum mechanical description really only mimics classic electromagnetism.

A description based on Maxwell's equations gives us a formalism that scales smoothly from STATIC electric and magnetic fields to electromagnetic waves with frequencies even larger than the frequency of visible light. Electromagnetism also provides a powerful explanation of the speed of light and what affects it, and the transformations of the special theory of relativity also come naturally out of electromagnetism.

Nevertheless, problems arise with the description of the interaction between a classical electromagnetic wave (or an electromagnetic field) and atomic processes. This is because classical electromagnetism cannot be used to describe atomic transitions.

In spite of this, I am among the physicists who believe that Maxwell's equations and electromagnetism are by far preferable to the photon concept for describing the vast majority of currently known phenomena, but not those involving atomic transitions. In my opinion, we have so far not reviewed the interaction between electromagnetic waves and atomic transitions with sufficient thoroughness. I represent a minority, but this minority is not getting smaller—quite the contrary, in fact. I mean that the last word has not been written about how physicists will think in the coming 50 years.

In a separate tract, I will explore this knotty issue and will not delve into it here.

9.10.3 A Challenge

Hitherto, we have seen that for both oscillations and mechanical waves there is an alternation between two energy forms as the oscillation/wave evolves. For example, for the mass–spring oscillator the energy changed in time between potential and kinetic energy, and the sum was always constant. In a travelling sound wave, at every point in space where the wave is passing, the energy density changes between potential energy (pressure) and kinetic energy, and the sum is always constant. For a travelling wave along a string, it is likewise.

For a travelling electromagnetic wave, it is not easy to see the same pattern. The electric field has the maximum at the same time and place as the magnetic field, at least for a plane. Have we overlooked something?

I suspect something is missing in our standard descriptions of electromagnetic waves. I have an idea I will follow up in the coming years. Perhaps this is a gauntlet you too want to take up?

9.11 Helpful Material

9.11.1 Useful Mathematical Relations

Here, we list some useful relationships from the mathematics you have hopefully met earlier:

Common to all expressions is that we operate with a scale field:

$$\phi = \phi(x, y, z)$$

and a vector field

$$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} \ .$$

A gradient is defined as:

$$\text{grad}\phi \equiv \nabla\phi \equiv \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k} \ .$$

The divergence is defined as:

$$\text{div } \vec{a} \equiv \nabla \cdot \vec{a} \equiv \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \ .$$

The divergence of a gradient is:

$$\text{div grad } \phi \equiv \nabla \cdot (\nabla\phi) \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \equiv \triangle\phi \ .$$

The curl is defined as:

$$\text{curl } \vec{a} \equiv \nabla \times \vec{a} \equiv$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} =$$

$$\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{k} \ .$$

Notice what are vector fields and what are scalar fields. In general:

- A gradient converts a scalar field into a vector field.
- A divergence works the other way.
- Div-grad starts with a scalar field, passes through a vector field and ends with a scalar field again.
- A curl, in contrast, starts with a vector field and ends with a vector field.

The symbol ∇ is involved in different operations depending on whether it works on a scalar field or a vector field, and it is especially challenging to use ∇^2 on a vector since we must then use the Laplacian on each of the components in the vector separately:

$$\nabla^2 \mathbf{a} = \left(\frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a_x}{\partial y^2} + \frac{\partial^2 a_x}{\partial z^2}\right)\vec{i}$$

$$+ \left(\frac{\partial^2 a_y}{\partial x^2} + \frac{\partial^2 a_y}{\partial y^2} + \frac{\partial^2 a_y}{\partial z^2}\right)\vec{j}$$

$$+\left(\frac{\partial^2 a_z}{\partial x^2} + \frac{\partial^2 a_z}{\partial y^2} + \frac{\partial^2 a_z}{\partial z^2}\right)\vec{\boldsymbol{k}} \ .$$

Some other useful relations appear below:

$$\text{curl grad}\phi = \nabla \times (\nabla\phi) = 0 \ ,$$

$$\text{div curl } \mathbf{a} = \nabla(\nabla \times \mathbf{a}) = 0 \ ,$$

$$\text{curl}(\text{curl}\mathbf{a}) = \text{grad}(\text{div}\mathbf{a}) - \triangle\mathbf{a} = \nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a} \ .$$

9.11.2 Useful Relations and Quantities in Electromagnetism

Here are some relationships from electromagnetism as a refresher of prior knowledge:

- Electric field strength $\vec{\boldsymbol{E}}$ is measured in V/m.
- Electric flux density $\vec{\boldsymbol{D}}$ is measured in C/m^2.
- Magnetic field strength $\vec{\boldsymbol{H}}$ is measured in A/m.
- Magnetic flux density $\vec{\boldsymbol{B}}$ is measured in T.
- E-flux density is also often referred to as electric displacement.
- Free space electrical permittivity ε_0 is measured in F/m = (As)/(Vm) and defined as

$$\varepsilon_0 \equiv \frac{1}{\mu_0 c_0^2} \approx 8.854188 \times 10^{-12}\,\text{F/m}$$

- The relative permittivity ε_r is usually a number larger than 1.0.
- Free space magnetic permeability μ_0 is measured in H/m and defined as:

$$\mu_0 \equiv 4\pi \times 10^{-7}\,\text{H/m} \approx 1.256637 \times 10^{-6}\,\text{H/m}$$

- The relative permeability μ_r is close to 1.0 for most materials. Ferromagnetic materials are an exception.
- The speed of light in vacuum is given exactly as:

$$c_0 \equiv 299{,}792{,}458\,\text{m/s}$$

  The SI basic units are now the speed of light in vacuum and the second. The length 1 metre is no longer one of the basic units!
- The relation between field strengths and flux densities is as follows:

$$\vec{\boldsymbol{D}} = \varepsilon_r \varepsilon_0 \vec{\boldsymbol{E}}$$

$$\vec{\boldsymbol{B}} = \mu_r \mu_0 \vec{\boldsymbol{H}}$$

9.12 Learning Objectives

After working through this chapter, you should be able to:

- Convert Maxwell's equations from integral to differential form (assuming Stokes's theorem and divergence theorem are given).
- Derive the wave equation for electromagnetic field in vacuum provided that Eq. (9.16) is given.
- Explain what simplifications are introduced in the derivation of the wave equation for electromagnetic fields in vacuum.
- Explain which part of Maxwell's equations is responsible for an electromagnetic wave to travel through free space.
- Explain carefully the difference between "plane wave" and polarization.
- Specify the amount of energy transport in a plane electromagnetic wave.
- Apply the Poynting vector and know the limitations of this concept.
- State and apply expression of radiation pressure in an electromagnetic field in a plane wave.
- Explain what we mean by near field and far field and why these sometimes are very different.
- Explain the characteristics of electromagnetic fields that differ in the two zones.
- Explain some problems using the photon term for all electromagnetic fields/waves.

9.13 Exercises

Suggested concepts for student active learning activities: Electromagnetic wave, line integral, surface integral, vector field, near field, far field, pure electrodynamics, polarization, dielectric, index of refraction, relative permittivity, electromagnetic spectrum, energy density, energy transport, radiation pressure.

Comprehension/discussion questions

1. It is not easy to comprehend Fig. 9.5 correctly. It is so easy to think of waves in a material way, similar to surface waves on water. However, an electromagnetic wave is much more abstract, since it is just the abstract quantities of electric and magnetic fields that changes with position and time. Discuss if it becomes easier to comprehend Fig. 9.5 if we state that an electric and magnetic field actually *change the property of the space* locally (even in vacuum) and that it is this *changed property of space* that moves as the electromagnetic wave passes by.
2. Explain briefly how to characterize a region in space where the divergence of the electric field is different from zero. Similarly, explain briefly how to characterize a region in space where the curl of the electric field is different from zero.

3. In going from the integral form of Maxwell's equations to the differential form, we use an argument based on the "intermediate" scale of length/volume. What do we mean by this?
4. Suppose we measure the electric and magnetic fields in an electromagnetic wave in the far-field zone. Can we determine the direction of the waves from these measurements?
5. We apply an alternating voltage across a capacitor, or we send an alternating current through a solenoid. Attempt to find the direction of the electric and magnetic fields and relative magnitudes. Will these fields follow the well-known laws that apply to the electric and magnetic fields for plane electromagnetic waves?
6. It is sometimes said that for an electromagnetic wave in vacuum, the electric and magnetic fields are perpendicular to each other. Magnetic fields and electric fields do not have this relationship to one another a short distance from a solenoid ("coil"), even if it is in vacuum and high-frequency electric and magnetic fields are present. What causes this?
7. Is polarization a property of all electromagnetic waves, not just light waves? Can sound waves have a polarization? By the way: What do we mean by "polarization"?
8. An electromagnetic wave (e.g. strong light) may have an electric field of about 1000 V/m. Could it lead to electric shock if one is exposed to this powerful light?
9. The magnetic field in intense laser light can be up to 100 times as powerful as the earth's magnet field. What will happen if we shine with this laser light on the needle of a compass?
10. Poynting vector indicates the power in an electromagnetic wave. Can we use the Poynting vector to calculate the power that springs from a power line to residents nearby? Explain your answer.
11. If you flash with the light from an electric torch, would you experience a recoil similar to that one gets on firing a gun? Discuss your answer.
12. In any physical system/phenomenon, one may identify a length scale and a timescale. What is meant by such a statement when we consider electromagnetic waves?
13. A person measures the electric field E and the magnetic field B in vacuum for the same frequency f and position, but finds that $E/c \gg B$. Is this an indication of malfunction for one of the two instruments used in the measurements?
14. In several equations in this chapter, the relative electrical permittivity ε_r is included.
    (a) The speed of light is linked to this quantity. How?
    (b) The relative permittivity tells us something about what physical processes take place when light travels through glass. What processes are we thinking about?
    (c) Many think that it makes sense that light slows down on going from air or vacuum to glass, but they find it hard to understand that light regains the original speed upon leaving the glass. What, in your opinion, accounts for their difficulty?

Problems

15. Show that a plane electromagnetic wave in vacuum satisfies all four Maxwell's equations.
16. Write down Maxwell's equations in integral form, and state the correct names for them. Give a detailed derivation of Ampére's law in differential form.
17. The derivation of the wave equation from Maxwell's equations follows about the same tricks whether one uses them to arrive at the wave equation for the electric field or for the magnetic field. Make a list showing which steps/tricks are used (a relatively short account based on essential points without going into detail will suffice).
18. Find the frequency of yellow light of wavelength 580 nm. Do the same with wavelength of about 1 nm. The fastest oscilloscopes available now have a sampling rate in the range of 10–100 GHz. Can we use this kind of oscilloscope to see the oscillations in electric fields in the X-ray waves? What about yellow light?
19. An electromagnetic wave has an electric field given by $\vec{E}(y,t) = E_0 \cos(ky - \omega t)\vec{k}$. $E_0 = 6.3 \times 10^4$ V/m, and $\omega = 4.33 \times 10^{13}$ rad/s. Determine the wavelength of the wave. In which direction does the wave move? Determine $\vec{B}$ (vector). If you make any particular assumptions in the calculations, these must be stated.
20. An electromagnetic wave of frequency 65.0 Hz passes through an insulating material with a relative permittivity of 3.64 and relative permeability of 5.18 for this frequency. The electric field has an amplitude of 7.20×10^{-3} V/m. What is the wave speed in this medium? What is the wavelength in the medium? What is the amplitude of the magnetic field? What is the intensity of the wave? Are the calculations you have made really valid? Explain your answer.
21. An intense light source radiates light equally in all directions. At a distance of 5.0 m from the source, the radiation pressure on a surface that absorbs the light is approximately 9.0×10^{-9} Pa. What is the power of the emitted light?
22. A ground surface measurement shows that the intensity of sunlight is 0.78 kW/m^2. Estimate the power the radiation pressure will exert on a 1 m^2 large solar panel? State the assumptions you make. As a matter of interest, we may mention that the atmospheric pressure is about 101,325 Pa (about 10^5 Pa).
23. For an electromagnetic wave, it is assumed that the electric field at one point is aligned in the x-direction and magnetic field in the $-z$-direction. What is the direction of propagation of the wave? What if the fields were in the $-z$- and y-direction, respectively? Did you make any assumption for finding the answers?
24. An ordinary helium–neon laser in the laboratory has a power of 12 mW, and the beam has a diameter of 2.0 mm. Suppose the intensity is uniform over the cross section (which is completely wrong, but it can simplify the calculations). What are the amplitudes of the electric and magnetic fields in the beam? What is the average energy density of the electric field in the beam? What about the energy density in the magnetic field? How much energy do we have in a 1.0 m long section of the beam?

25. Measurements made a few hundred metres from a base station indicated an electric field of 1.9 V/m and a magnetic field of 1.2 mA/m (both at about 900 MHz). A knowledgeable person concluded that the measurements were not mutually consistent. What do you think was the reason for this conclusion?
26. Measurements at the ground just a few tens of metres from a power line registered an electric field of 1.2 kV/m and a "magnetic field" of 2.6 μT (microtesla) (both at 50 Hz). In practice, it is often magnetic flux density reported at low frequencies, but we can convert from B to H, and then find that 2.6 μT corresponds to the magnetic field value 2.1 A/m. Is there correspondence between electric field and magnetic field in this case? Comment on similarities/differences between the situations in the previous task and in this task.
27. One day, the electric and magnetic fields are measured at the same location near the power line as in the previous task, and the values are found to be 1.2 kV/m and 0.04 A/m. Can we conclude that there is something wrong with one of the measuring instruments in this case?
28. According to Radiation Protection Info 10–11: Radio Frequency Fields in our Environment (Norwegian Radiation Protection Agency) (http://www.nrpa.no/filer/5c7f10ca06.pdf, available 10 May 2018), the "radiation" from base stations, wireless networks, radio, etc., is less than 0.01 W/m^2 across our country. Calculate the electric field and magnetic field equivalent to 0.01 W/m^2 if we think that the radiation is dominated by mobile phone communications from a base station at 1800 MHz.
29. When we use a mobile phone somewhere where the coverage is poor so that the phone gives maximum power, the mobile phone supplies about 0.7–1.0 W power while communicating. Calculate the intensity 5 cm from the mobile phone if you assume an isotropic intensity around the phone. Compare the value with measured intensities from base stations, wireless networks, etc., given in the previous task.
30. It is not customary to report the "radiation" from a mobile phone in terms of power density (intensity) measured in W/m^2, but in Specific Absorption Rate (SAR).
    (a) Search the Web to find out about SAR. State the URL for the source you are using.
    (b) Explain what SAR implies and what is the SAR unit?
    (c) What do you think is the reason why such a unit has been adopted in this case, even though we use power density from base stations and suchlike, with about the same frequency as the mobile phone?
31. Let us consider interplanetary dust in our solar system. Suppose the dust is spherical and has a radius of r and a density of ρ. Suppose all radiation that hits the dust grain is absorbed. The sun has a total radiated power of P_0 and a mass M. The gravity constant is G. The distance from the sun is R. Derive an expression that indicates the relationship between the power exerted by the radiation pressure from the sun rays to the dust grain and the gravitational force between the sun and the dust grain. Determine the radius of the dust when the two forces are equal as we insert realistic values for the quantities that are

involved. ($\rho = 2.5 \times 10^3$ kg/m, $P_0 = 3.9 \times 10^{26}$ W, $M = 1.99 \times 10^{30}$ kg, $G = 6.67 \times 10^{-11}$ Nm^2/kg^2).

32. Relate the gravitational force between the earth and the sun, and the force on the earth due to the radiation pressure from the sun. The earth's mass is 5.98×10^{24} kg. You can estimate the radius of the earth by recalling that the distance between a pole and the equator is about 10,000 km.

Reference

1. PD, https://commons.wikimedia.org/wiki/File:James_Clerk_Maxwell_big.jpg. Accessed April 2018