Chapter 3
Forced Oscillations and Resonance

Abstract In this chapter, we study a mechanical system forced to oscillate by the application of an external force varying harmonically with time. The amplitude of the oscillations, which is shown to depend on the frequency of the external force, reaches its peak value when the frequency of the applied force is close to the natural frequency of the system, a phenomena called resonance. However, details depend on the energy loss in the system, a property described by a quality factor Q, and the phase difference is described by so-called phasors. Emphasis is placed on how the system behaves when the external force starts and vanishes. Numerical calculations facilitate the analysis. At the end, some relevant details concerning the physiology of the human ear are briefly mentioned.

3.1 Introductory Remarks

The words "resonance" and "resound" are derived from the Latin root *resonare* (to sound again). If we sing with the correct pitch, we can make a cavity to sing along and, to somehow, augment the sound we emitted. Nowadays, the word is used in diverse contexts, but it always has the connotation of an impulse causing reverberation in some medium. When we tune the radio to receive weak signals from a transmitter, we see to it that other, unwanted signals, also captured by the radio antenna at the same time, are suppressed. It may seem like pure magic. The physics behind such phenomena is straightforward when we limit ourselves to the simplest cases. If we dig a little deeper, we uncover details that make our inquiry much more demanding and exciting.

3.2 Forced Vibrations

The Foucault pendulum in the foyer of the Physics building at the University of Oslo oscillates with the same amplitude year after year, although it encounters air resistance, which, in principle, should have dampened its motion. This is because

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the bob at the end of the pendulum receives a small electromagnetic push each time it passes the lowest point. When that happens, a small red LED lights up. The push comes exactly at the time the bob is moving away from the equilibrium point. In this way, the time period is almost completely determined by the natural oscillation period of the pendulum itself (determined by the length of the pendulum and acceleration due to gravity).

In other contexts, "the pushes" come at a rate different from the natural rate of oscillation of the system. Electrons in an antenna, the diaphragm of the loudspeaker, the wobble of a boat when waves pass by, are all examples of systems being forced by a vibratory motion energized by an external force that varies in time independently of the system in motion. Under such circumstances, the system is said to be executing *forced oscillations*.

In principle, an external time-dependent force can vary in infinitely many ways. The simplest description is given by a harmonic time-varying force, i.e. as a sinusoid or cosinusoid. In the first part of the chapter, we assume that the harmonic force lasts for a "long time" (the meaning of the phrase will be explained later).

If we return to the mechanical pendulum examined earlier and confine ourselves to a simple friction term and a harmonic external force, the movement can be described analytically.

For a mechanical system, the starting point is Newton's second law (see Chap. 2): The sum of the forces equals the product of mass with acceleration:

$$F\cos(\omega_F t) - kz(t) - b\dot{z}(t) = m\ddot{z}(t)$$

where $F\cos(\omega_F t)$ is the external force that oscillates with its own angular frequency ω_F. If we put

$$\omega_0^2 = k/m \ ,$$

(angular frequency of the freely oscillating system), the equation can also be written as follows:

$$\ddot{z}(t) + (b/m)\dot{z}(t) + \omega_0^2 z(t) = (F/m)\cos(\omega_F t) \ . \qquad (3.1)$$

This is an inhomogeneous second-order differential equation, and its general solution may be written as:

$$z(t) = z_h(t) + z_p(t)$$

where z_h is the general solution of the corresponding homogeneous equation (with F replaced by zero) and z_p is a particular solution to the inhomogeneous equation itself.

We have already found in Chap. 2 the general solution of the corresponding homogeneous equation, so the challenge is to find a particular solution.

We know that the solution of the homogeneous equation decreases with time to zero. Therefore, after a long time from start, the movement will be dominated by the external periodic force.

It becomes natural then to investigate if a particular solution may have the form:

$$z_p(t) = A\cos(\omega_F t - \phi) \tag{3.2}$$

where A is real.

Here, we have to discuss the choice of the sign of the phase term ϕ. Assume ϕ to be positive. In that case, we have: If F is maximum at time $t = t_1$ (for example, $\omega_F t_1 = 2\pi$), the displacement $z_p(t)$ will reach its maximum value at a time $T = t_2$ (with $t_2 > t_1$), i.e. at a time later than when F was at its maximum.

We then say that the output $z_p(t)$ is *delayed* with respect to the applied force.

When the expressions for $z_p(t)$ and $F(t)$ are inserted into Eq. (3.1) and the terms are rearranged, the following result is obtained:

$$(\omega_0^2 - \omega_F^2)\cos(\omega_F t - \phi) - (b/m)\omega_F \sin(\omega_F t - \phi) = F/(Am)\cos(\omega_F t)\ .$$

If we use the trigonometric identities for the sines and cosines of difference of angles (see Rottmann), we find:

$$(\omega_0^2 - \omega_F^2)\{\cos(\omega_F t)\cos\phi + \sin(\omega_F t)\sin\phi\} - (b/m)\omega_F\{\sin(\omega_F t)\cos\phi - \cos(\omega_F t)\sin\phi\}$$
$$= F/(Am)\cos(\omega_F t)\ .$$

Upon collecting the terms with $\sin(\omega_F t)$ and $\cos(\omega_F t)$, we get:

$$\left[(\omega_0^2 - \omega_F^2)\cos\phi - F/(Am) + (\omega_F b/m)\sin\phi\right]\cos(\omega_F t)$$
$$+\left[(\omega_0^2 - \omega_F^2)\sin\phi - (\omega_F b/m)\cos\phi\right]\sin(\omega_F t) = 0\ .$$

Since $\sin(\omega_F t)$ and $\cos(\omega_F t)$ are linearly independent functions of t, the above equation can be satisfied only if each term within the square brackets vanishes separately. This conclusion gives us two equations which can be used for the determination of the two unknowns, namely A and ϕ.

Equating to zero the terms within the square brackets multiplying $\sin(\omega_F t)$, we find:

$$(\omega_0^2 - \omega_F^2)\sin\phi = (\omega_F b/m)\cos\phi\ .$$

The phase difference between the output and the applied force can be expressed as:

$$\cot \phi = \frac{\cos \phi}{\sin \phi} = \frac{\omega_0^2 - \omega_F^2}{\omega_F b/m} \ . \tag{3.3}$$

We see that when $\omega_F = \omega_0$, $\cot \phi = 0$, which means that $\phi = \pi/2$ or $3\pi/2$. Since $\cot \phi$ changes from a positive to negative value when ω_F passes ω_0 from below, only the choice $\phi = \pi/2$ is acceptable.

When we set the expression with the square brackets multiplying $\cos(\omega_F t)$ to zero, we get:

$$(\omega_0^2 - \omega_F^2) \cos \phi - F/(Am) - (b\omega_F/m) \sin \phi = 0 \ .$$

We use the expression $\sin x = \pm 1/\sqrt{1 + \cot^2 x}$ from Rottmann (and a corresponding expression of cos) together with Eq. (3.3).

After a few intermediate steps, we get the following expressions for the amplitude of the required oscillations:

$$A = \frac{F/m}{\sqrt{(\omega_0^2 - \omega_F^2)^2 + (b\omega_F/m)^2}} \ . \tag{3.4}$$

It is time now to sum up what we have done:

When a system obeying an inhomogeneous linear second-order ordinary differential equation is subjected to a harmonic force that lasts indefinitely, a particular solution (which applies "long after" the force is applied) is itself a harmonic oscillation *of the same frequency* that is phase shifted with respect to the original force, as given in Eq. (3.2). "Long after" refers to a time many time constants $1/\gamma$ long, where γ is proportional to the damping of the system. We refer to the exponential decaying term $e^{-\gamma t}$ in the solution of the homogeneous differential equation discussed in the previous chapter.

The amplitude of the oscillations is then given by Eq. (3.4), and the phase difference between the output and the applied force (or the input) is given by Eq. (3.3). Figure 3.1 shows schematically how the amplitude and phase vary with the frequency of the applied force. The frequency of the force is given relative to the frequency of the oscillations in the same system if there was no applied force or no friction/damping.

We see that the amplitude is greatest when the frequency of the applied force is nearly the same as the natural frequency of oscillation in the same system when the applied force and damping are *both absent*. We call this phenomenon *resonance*, and it will be discussed in more detail in the next section.

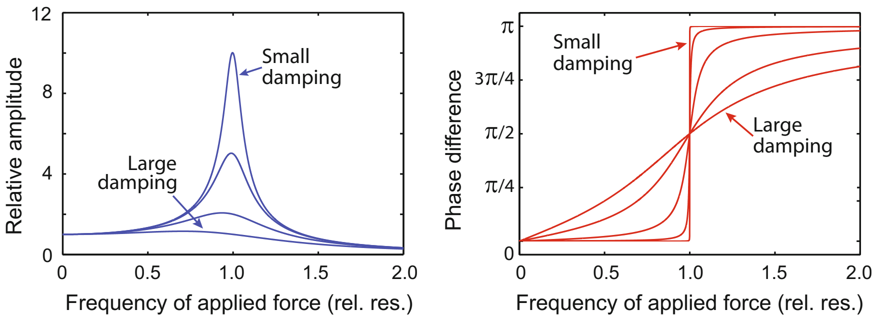

Fig. 3.1 The amplitude of a forced oscillation (*left*) and the phase difference between the output and the applied force (*right*) as a function of the frequency of the applied force

The phase ϕ appearing in Eq. (3.2) is approximately equal to $\pi/2$ at resonance; that is, the output is lagging behind (in phase) by about $\pi/2$ with respect to the applied force. For the spring oscillation, it means that the force is greatest in the upward direction when the pendulum has its highest speed and passes the equilibrium point on the way upwards.

Away from resonance, the phase difference is less than (greater than) $\pi/2$ when the applied frequency is lower than (higher than) the "natural" frequency. These relationships can be summarized so that the pendulum "is impatient" and tries to move faster when the applied force changes too slowly relative to resonance frequency ("natural frequency"). The movement of the pendulum depends more and more on the force when the force changes too quickly in relation to resonant frequency.
The phase difference is an important characteristic of forced fluctuations.

3.3 Resonance

One sees from Eq. (3.4) that the amplitude of the forced oscillations varies with the frequency of the applied force. When the frequency is such that the amplitude is greatest, the system is said to be at *resonance*.

It may be useful to reflect a little about what is needed to get the largest possible output, which corresponds to the highest possible energy for the system.

Let us start with the mechanical mass–spring oscillator again. We then have a mechanical force that works on a moving system. We remember from mechanics that the work done by the force is equal to the magnitude of the force multiplied by how far the system moves under the action of the force. For a constant force, the power delivered by the force equals the power multiplied by the velocity of the system experiencing the force. Force and velocity are vectorial forces, and it is their dot product that counts (Remember $P = \vec{F} \cdot \vec{v}$ from the mechanics course.).

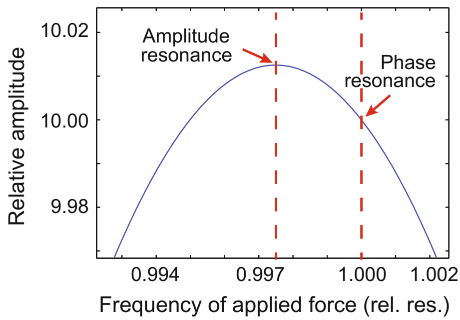

Fig. 3.2 A close-up view of the relative amplitude in a forced oscillation as a function of the frequency of the applied force. Note the numbers along the axes

In our case, the force will deliver the greatest possible power to the system if the power has the highest value while the pendulum bob has the highest possible velocity. Force and velocity must work in the same direction. This will happen if the force, for example, is the greatest, while the bob passes the equilibrium position on the way up. This corresponds to the position is phase shifted $\pi/2$ by force. To achieve such a state, the external force must swing with the *resonance frequency*.

So far, we have been somewhat imprecise when we have discussed resonance. Strictly speaking, we must differentiate between two nuances of the term resonance, namely *phase resonance* and *amplitude resonance*. The difference between the two is often in practice so small that we do not have to worry about it.

Phase resonance is said to occur when the phase difference between the applied force and the output equals $\pi/2$. This happens when the frequency of the applied force (input frequency) coincides with the natural frequency of the (undamped) system.

A close-up view of Fig. 3.1 shown in Fig. 3.2 shows that the amplitude is greatest at a slightly lower frequency than the natural frequency. The small but significant difference is due to a detail we mentioned when we discussed damped harmonic motion in the previous chapter. In the presence of damping, the oscillation frequency is slightly lower than the natural frequency. The frequency at which amplitude is greatest indicates *amplitude resonance* for the system. The two resonance frequencies are often quite close to each other, as already mentioned.

Let us find mathematical expressions for the two resonance frequencies.

The amplitude resonance frequency can be found by differentiating the expression for the amplitude given by Eq. (3.4) (a common procedure for finding extreme values). We calculate the ω_F angular frequency at which:

$$\frac{\mathrm{d}A}{\mathrm{d}\omega_F} = 0\,.$$

We find that

$$\omega_F = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}} \ .$$

If we want to state the frequency rather than the angular frequency, we use the expression:

The amplitude resonance frequency is:

$$f_{\text{amp.res.}} = \frac{1}{2\pi}\sqrt{\omega_0^2 - \frac{b^2}{2m^2}} \tag{3.5}$$

where $\omega_0 = \sqrt{k/m}$.

The phase resonance frequency is:

$$f_{\text{ph.res.}} = \frac{1}{2\pi}\omega_0 \ . \tag{3.6}$$

We observe that the two resonance frequencies coincide only when $b = 0$ (no damping).

3.3.1 Phasor Description

We will now consider forced oscillations in an electrical circuit. First, we will proceed in much the same manner as adopted in dealing with the mechanical system examined above, but eventually we will go over to an alternative description based on phasors. The system is a series RCL circuit with a harmonically varying voltage source $V_0 \cos(\omega_F t)$, as shown in Fig. 3.3. The differential equation for the system then becomes [compare with Eq. (2.27)]:

$$L\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C}Q = V_0\cos(\omega_F t) \ . \tag{3.7}$$

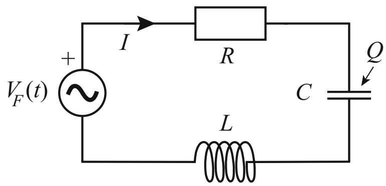

Fig. 3.3 A series RCL circuit driven by a harmonically varying applied voltage. The labels +, I, and Q indicate the signs chosen for our symbols

This is an inhomogeneous equation, whose solution is found in the same way as for its mechanical counterpart considered above. The solution consists of a sum of a particular solution and the general solution of the homogeneous equation (with $V_0 = 0$). The solution of the homogeneous equation is already known, and it only remains for us to find a particular solution. We try a similar solution as for the mechanical system, but adopt a complex representation:

$$Q_p(t) = A\mathrm{e}^{\mathrm{i}\omega_F t} \tag{3.8}$$

where A can be a complex number.

At the same time, a complex exponential form is chosen for the externally applied voltage:

$$V(t) = V_0 \cos(\omega_F t) \rightarrow V_0 \mathrm{e}^{\mathrm{i}\omega_F t} \ . \tag{3.9}$$

It goes without saying that the real part of the expressions are to be used for representing physical quantities.

Inserting the expressions for $Q_p(t)$ and $V(t)$ into Eq. (3.7), and cancelling the common factor $\mathrm{e}^{(\mathrm{i}\omega_F t)}$, we get:

$$-L\omega_F^2 A + \mathrm{i}R\omega_F A + \frac{1}{C}A = V_0 \ .$$

Solving the equation for A, we get:

$$A\left(-L\omega_F^2 + \mathrm{i}R\omega_F + \frac{1}{C}\right) = V_0$$

$$A = \frac{V_0}{\frac{1}{C} - L\omega_F^2 + \mathrm{i}R\omega_F} \ .$$

A again becomes a complex number (except when $R = 0$).

The instantaneous current in the RCL circuit is found by applying Ohm's law to the resistor:

$$I = \frac{V_R}{R} = \frac{\mathrm{d}Q}{\mathrm{d}t} \ .$$

If we wait long enough for the solution of the homogeneous equation to die out, only the particular solution remains, and the current is then given by the expression:

$$I = \frac{\mathrm{d}Q_p}{\mathrm{d}t} = A\mathrm{i}\omega_F \mathrm{e}^{\mathrm{i}\omega_F t}$$

Simple manipulations lead one to the following expression:

$$I(t) = \frac{V_0}{R + \mathrm{i}(L\omega_F - \frac{1}{C\omega_F})}\mathrm{e}^{\mathrm{i}\omega_F t} \ . \tag{3.10}$$

This expression should be compared with V_F, the voltage applied to the circuit, which in complex form is given by:

$$V_F(t) = V_0 e^{i\omega_F t} \ .$$

It follows from Eq. (3.10) that if $R = 0$ the current will be phase shifted 90° relative to applied voltage. If in addition $L = 0$, the current will *lead* the voltage by 90°. However, if $\omega_F L$ is much larger than $1/(\omega_F C)$ (C "shorted"), the current will be offset 90° *after* the voltage (In a calculation exercise at the end of the chapter you are asked to show this.).

If $R \neq 0$, but $L\omega_F - \frac{1}{C\omega_F} = 0$, the current and voltage will be in phase, and $I = V_0/R$. This corresponds to $\omega_F = \frac{1}{\sqrt{LC}}$, which was named phase resonance above.

> The connection between R, C, L, current and phase can be elegantly illustrated by means of phasors. We have already mentioned phasors, but now we extend the scope by drawing in multiple rotating vectors at the same time. Figure 3.4 shows an example.

Both currents and voltages are displayed in the same plot. We start with a vector that represents the current generated by the applied voltage. Then we draw vectors representing voltages across the resistor, capacitor and inductor resulting from the current flow. The vector which shows the voltage across the capacitor will then be 90° after the vector showing the current, the voltage across the resistor will have the same direction as the current and the voltage across the inductance will be 90° ahead of the current. The total voltage across the serial link of R, C and L should then be the vector sum of the three voltage phasors and correspond to the applied voltage. We see that the phase difference between current and voltage will be between +90° and −90°.

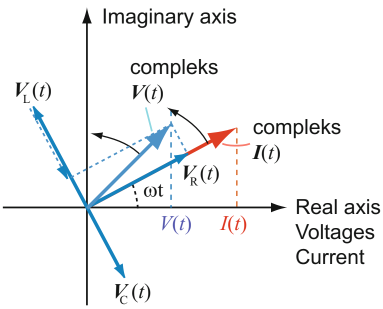

Fig. 3.4 Example of phasor description of an RCL circuit subjected to a harmonically varying voltage. The current at any time (anywhere in the circuit) is the x components of the vector $I(t)$, while the voltage across the various circuit components is given by the x component of the vectors $V_R(t)$, $V_C(t)$ and $V_L(t)$, and their sum is $V(t)$. See the text for details

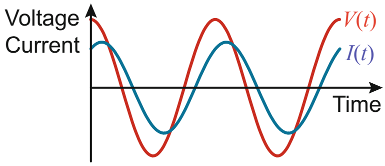

Fig. 3.5 A time plot in which the current slightly leads the applied voltage

Phasor diagrams can also be based on quantities other than those we have chosen here. One variant is to use complex impedances that are added vectorially. The strength of phasor diagrams is that we can easily understand, for example, how the phase differences change with frequency. The depiction in Fig. 3.4 applies only to a particular applied angular frequency ω_F. If the angular frequency increases, the voltage across the capacitor decreases while the voltage across the inductance will increase. Phase resonance occurs when the two voltage vectors are exactly the same size (but oppositely directed) so that their sum is zero.

Figure 3.5 shows the time development of voltage and current in a time plot. The current in the circuit is slightly leading the applied voltage. For a series RCL circuit with an applied voltage, this means that the applied frequency is lower than the resonant frequency of the circuit.

Note that phasors can be used only after the initial rather complicated oscillatory pattern is over, and we have a steady sinusoidal output corresponding to the particular solution of differential equation.

3.4 The Quality Factor Q

In the context of forced oscillations, it is customary to characterize oscillating systems with a *Q-factor* or *Q-value*, where the symbol Q, not to be confused this with the charge Q in an electrical circuit, stands for "quality", which is why the Q-factor is also called the quality factor. The factor tells us something about how easy it is to make the system oscillate, or how long the system will continue to oscillate after the driving force is removed. This is more or less equivalent to how small loss/friction is in the system.

The quality factor for a spring oscillator is given by:

$$Q = \frac{m\omega_0}{b} = \sqrt{\frac{mk}{b^2}} \,. \tag{3.11}$$

We see from this formula that the smaller the value of b, the larger is the quality factor Q.

Figure 3.6 shows how the oscillation amplitude varies with the frequency of the applied force for four different quality factors. A Q-value of 0.5, in this case, cor-

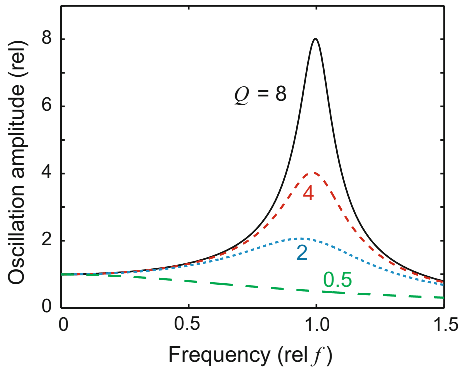

Fig. 3.6 When the frequency of the applied force changes relative to the system's own natural frequency, the amplitude will be greatest when the two frequencies are nearly equal. The higher the quality factor Q (i.e. smaller loss), the higher the resonance amplitude

responds to critical damping and we see no hint of any resonance for such a large damping.

There are two traditional ways of defining Q. The first is:

$$Q \equiv 2\pi \frac{\text{stored energy}}{\text{energy loss per period}} = 2\pi \frac{E}{E_{\text{loss-per-period}}} . \tag{3.12}$$

This definition implies a particular detail which few people are familiar with, but which is extremely important for forced oscillations in many contexts. Once we have achieved a steady state (when the applied force has been working for long and is still present), the loss of energy will be compensated by the work done on the system by the applied force. We see from Eq. (3.12) that a system with a high Q-value loses only a tiny part of the total energy per period.

Suppose now we turn off the applied force. Then the system will oscillate at the *amplitude resonance frequency* $\omega' = \sqrt{\omega_0^2 - (b/2m)^2} \approx \omega_0$", and the energy will eventually disappear. It will take the order of $Q/(2\pi)$ periods before the energy is used up and the oscillations end. Let us look a little more closely at this.

Loss of energy per period is a slightly unfamiliar quantity. Let us consider first P_{loss}, which is "energy loss per second" with the unit watt. We know that after the force has been removed, the loss will be given by:

$$P_{\text{loss}} = -\frac{\mathrm{d}E}{\mathrm{d}t} . \tag{3.13}$$

Then we can approximate the loss of energy over a period of time T with:

$$E_{\text{loss-per-period}} = P_{\text{loss}} \, T .$$

Using the definition given in Eq. (3.12), we get:

$$P_{\rm loss} = \frac{2\pi}{TQ} E \ . \tag{3.14}$$

Combining Eqs. (3.13) and (3.14) and the relation $\omega = 2\pi/T$, we get a differential equation governing the time development of the stored energy after the removal of the driving force. The equation is:

$$P_{\rm loss} \equiv -\frac{dE}{dt} = \frac{\omega_0}{Q} E \ .$$

The solution is:

$$E(t) = E_0 \mathrm{e}^{-\omega_0 t/Q} \ .$$

The energy falls to $1/\mathrm{e}$ of the initial energy after a time

$$\Delta t = \frac{Q}{\omega_0} = \frac{QT}{2\pi} \ . \tag{3.15}$$

We see that the amplitude of oscillation decreases in a neat exponential manner after the removal of an applied oscillatory force, with the time constant given in Eq. (3.15).

It can be shown that nearly the same time constant describes the *growth* of the output after the application of oscillating force. Obviously, the time course is not as simple because it depends, apart from other factors, on whether or not the frequency of the applied force equals the resonant frequency of the circuit (see Fig. 3.7). Nevertheless, if it takes an interval of the order of 10 ms for an oscillation to die out after an applied force is removed, it will also take nearly the same interval to build a steady amplitude after we switch on the applied force.

One might think that the time constant (and thus the Q-value) of the system could be found by referring to the thin red line in Fig. 3.7 and noting how long it takes from the moment the force is removed till the output falls to $1/\mathrm{e}$ of the value just before the power was turned off. It turns out, however, that the number so inferred is twice the expected value! The difference can be traced to the fact that the time constant deduced in Eq. (3.15) applies to how *energy* changes over time, whereas Fig. 3.7 shows amplitude and not energy. *The energy is proportional to the square of the amplitude.* Note that the stationary amplitude after the force has worked for a while is greatest at the resonance frequency!

The curves in Fig. 3.7 show that after an applied force is turned on, the amplitude of the oscillations increases, without becoming infinite. Sooner or later, the loss in energy is as large as the power applied through the oscillating force. After equilibrium

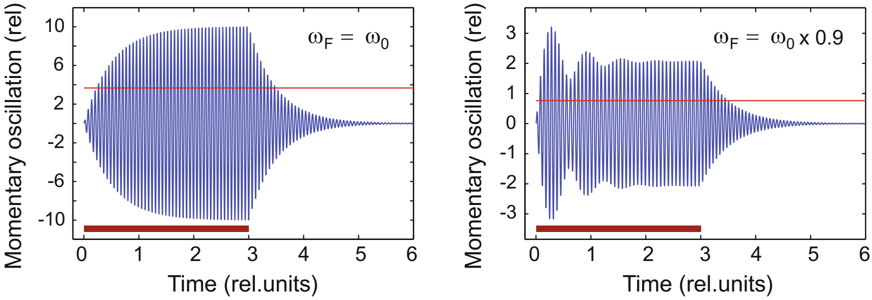

Fig. 3.7 Two examples of the build-up of oscillations in an oscillating system after an external sinusoidal force is coupled and subsequently removed (the force acts only during the interval indicated by a thick red line at the bottom). The frequency of the applied voltage is equal to the resonant frequency on the left and slightly lower on the right. While the force is present, the system oscillates with the frequency of the force. After the force has ceased, the circuit oscillates with its own resonance frequency. The thin red line marks the value $1/e$ times the maximum amplitude just before the applied force was removed. The Q-factor of the circuit is 25

a steady state is achieved, the amplitude of the oscillations will remain constant as long as the applied force has constant amplitude.

The mathematical solution of an inhomogeneous differential equation for an oscillating system subjected to an oscillatory force with given initial conditions is rather tedious. However, it is possible to find such a solution exactly using, for example, Maple or Mathematica. However, we have used numerical solutions in the preparation of Fig. 3.7; it is a rational approach since complex differential equations can often be solved numerically about as easily as simple differential equations. More about this in the next chapter.

In experimental context, a different and important definition of the Q-value is often used instead of that in Eq. (3.12). If we create a plot that shows *energy* (NOTE: *not* amplitude) in the oscillating system as a function of frequency (as in Fig. 3.8), the Q-value is defined as:

$$Q = \frac{f_0}{\Delta f} \tag{3.16}$$

where the half-width Δf, shown in the figure, compared to the resonance frequency f_0.

This relationship can be shown to be in accordance with the relationship given in Eq. (3.12), at least for high Q-values.

The definitions given in Eqs. (3.12) and (3.16) apply to all physical oscillating systems, not just the mechanical ones.

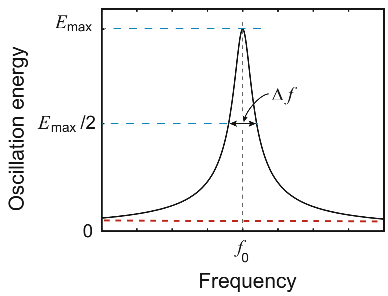

Fig. 3.8 The Q-value can also be defined from a graphical representation of *energy* stored in the oscillating system as a function of frequency. The Q-value is then given as the resonance rate f_0 divided by the half value Δf

For the most interested: It is now possible to make a remarkable observation: A resonant circuit responds significantly to frequencies within a frequency band of width

$$\Delta f = \frac{f_0}{Q} \, .$$

However, the circuit needs a certain amount of time

$$\Delta t = \frac{Q}{\omega}$$

to build-up the response if we start from zero. It takes about the same time also for a response that is already built to die out.

The product of Δf and Δt comes out to be:

$$\Delta t \Delta f = \frac{Q}{\omega}\frac{f_0}{Q}$$

$$\Delta t \Delta f = \frac{1}{2\pi} \, . \tag{3.17}$$

Multiplying this expression with Planck's constant h, and using the quantum postulate that the energy of a photon is equal to $E = hf$, we get:

$$\Delta t \Delta E = \frac{h}{2\pi} \, . \tag{3.18}$$

This expression is almost identical to what is known as Heisenberg's uncertainty relationship for energy and time. There is a factor 1/2 in front of the term after the equality sign, but such a factor will depend on how we choose to define widths in frequency and time.

There are certain parallels between a macroscopically oscillating system and the relationships we know from quantum physics. In quantum physics, Heisenberg's uncertainty relationship is interpreted as an "uncertainty" in time and energy: we cannot "measure" the time of an event more accurately than what is implicit in the relationship

$$\Delta t = \frac{h}{2\pi \Delta E}$$

provided that we do not change the energy of a system by more than ΔE.

Our macroscopic variant applies irrespective of whether we do measurements or not, but measurements will of course reflect the relationship that exists. We will return to this relationship later in the book, but in the form of Eq. (3.17) instead of (3.18).

"Inertia" in a circuit is important for what we can do with measurements. For a high Q oscillation cavity in the microwave region (called a "cavity"), we can easily achieve Q-values of 10,000 or more. If such a cavity is used in pulsed microwave spectroscopy, it will take of the order of 60,000 periods to significantly change the energy in the cavity. If the microwave frequency is 10 GHz (10^{10} Hz), the time constant for energy changes will be of the order of 6 μs. If we study relatively slow atomic processes, this may be acceptable, and the sensitivity of the system is usually proportional to the quality factor. However, if we want to investigate time intervals lasting only a few periods of the observed oscillations, we must use cavities with much lower Q-value. More will be said about this in the next chapter.

3.5 Oscillations Driven by a Limited-Duration Force

So far, we have considered a system that is influenced by an oscillating force lasting "infinitely long", or a force that has lasted for a long time and ends abruptly. In such a situation, we can determine a quality factor Q experimentally in terms of the frequency response of the system as shown in Fig. 3.8 and Eq. (3.16). Relative oscillation energy (relative amplitude squared) must be determined after the system has reached the stationary state, i.e. when the amplitude no longer changes with time.

How will such a system behave if the oscillatory force lasts only for a short time? We will now investigate this matter.

When we introduce a limited-duration force (a "temporary force"), we must choose how the force should be started, maintained and terminated. For a variety of reasons, we want to avoid sudden changes, and have chosen a force whose overall amplitude follows a Gaussian shape, but follows, on a finer scale, a cosinuosidal variation. Mathematically, we shall describe such a force by the function:

$$F(t) = F_0 \cos[\omega(t - t_0)] e^{-[(t-t_0)/\sigma]^2} \ . \tag{3.19}$$

where σ indicates the duration of the force (the time during which the amplitude falls to $1/e$ of its maximum value). ω is the angular frequency of the underlying cosine function, and t_0 is the time at which the force has the maximum amplitude (peak of the pulse occurs at time t_0). The oscillating system is assumed to be at rest before the force is applied.

Figure 3.9 shows two examples of temporary forces with different durations. Here, the force has a frequency equal to 100 Hz (period $T = 10$ ms). In the figure on the left, σ is equal to 25 ms, i.e. $2.5 \times T$, and successive peaks have been marked (from maximum onwards until the amplitude has decreased to $1/e$) to highlight the role played by the size of σ. In the figure to the right, $\sigma = 100$ ms, i.e. $10 \times T$; again, the markings give an indication of the relationship between ω (or rather the frequency or period) and σ.

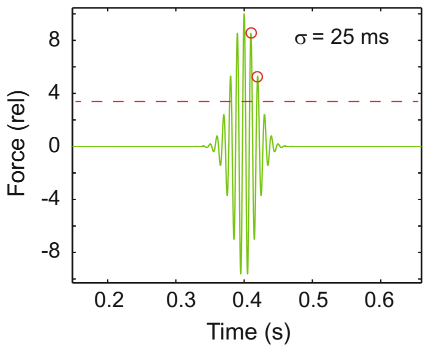

Fig. 3.9 The force $F(t)$ for centre frequency 100 Hz and pulse width σ equal to 0.025 and 0.10 s. See the text for further explanations

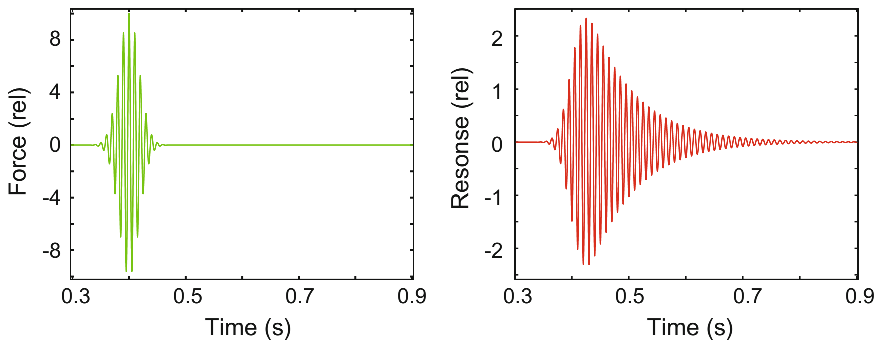

Fig. 3.10 The temporal response of the system (*right*) due to the applied force shown in the left part of the figure

We would now like to study how an oscillating system will behave when it is subjected to a temporary force. Based on Fig. 3.7, we expect the response to be quite complicated. Since it is not easy to make headway analytically, we have opted for numerical calculations instead.

Figure 3.10 shows the time course for one temporary force along with the response of the system. For simplicity, the frequency of the force has been made equal to the resonant frequency of the system, and according to the initial conditions chosen, the system is at rest before the force is applied.

Figure 3.10 shows some interesting features. The system attempts, but fails to keep pace with the force as it grows. We see that the peak of the response (amplitude) occurs a little later than the time at which the force reached its maximum value.

The force adds some energy to the system. When the force decreases as quickly as it does in this case, the system cannot get rid of the supplied energy at the same rate as that at which the force decreases. Left with surplus energy after the vanishing of the force, the system executes damped harmonic oscillations at its own rate. It may

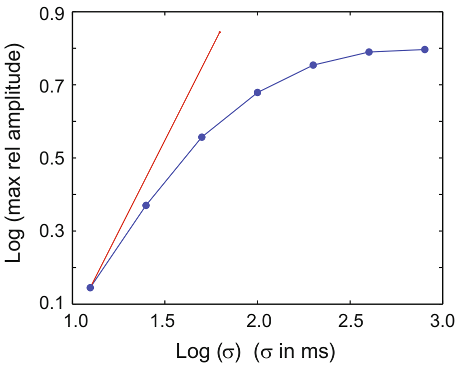

Fig. 3.11 Dependence of the maximum amplitude on the duration of the applied force (σ). Note the logarithmic scale on both axes

be mentioned that σ here is 25 ms and that the Q-factor of the oscillating system is chosen to be 25, which corresponds to a decay time for the energy for the oscillations of 40 ms.

It may be useful to point out some relationships between various parameters:

- How much energy can be delivered to the system within a given time depends on the force (proportionality?).
- The amount of energy that can be delivered, for a given input of force, will depend on how long the force works.
- The loss of energy is independent of the strength of the force after it has disappeared.
- The loss of energy is proportional to the amplitude of the oscillations.

As mentioned, we expect the amplitude to increase when the force lasts longer and longer, but the precise relationship is not self-evident. In Fig. 3.11 are shown calculated results for the maximum amplitude attained by the system for different σ values. ω always corresponds to the resonance frequency of the system. The figure has logarithmic axes to get a large enough range of σ. The straight line represents the case that the amplitude increases linearly with σ (duration of the force).

We see that for too small σ (the power lasting only a few oscillation periods), the maximum amplitude increases approximately proportionally with the duration of the force. When the force lasts longer, this does not apply anymore, and beyond a certain limit, the amplitude of the oscillation does not increase, however long the duration of the pulse may be. This is due to the fact that at the given amplitude, the loss is as large as the energy supplied by the power.

If the amplitude of the force is increased, the amplitude of the oscillations will also increase, but so will the loss. It is therefore found that the duration of the force required to obtain the maximum amplitude is approximately independent of the amplitude of the force.

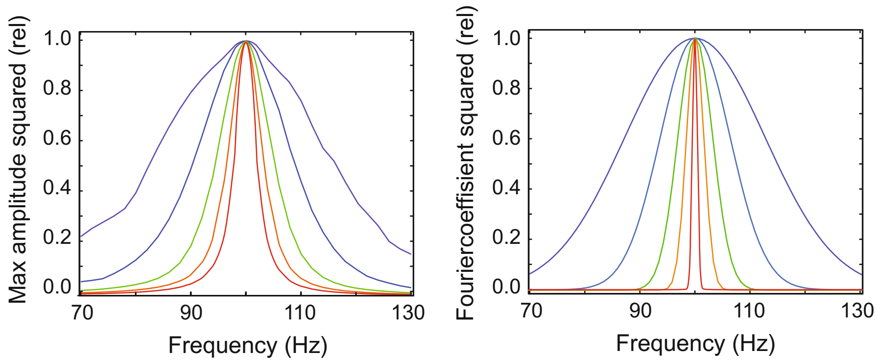

Fig. 3.12 The frequency response (actually only maximum amplitude) of the oscillating system for different durations (σ) of the force pulse (*left part*). The σ values used are respectively 25, 50, 100, 200, 400 and 800 ms (*from blue/widest to red/narrowest curves*). In the right part of the figure, corresponding frequency analyses of the force pulses themselves are shown. See the text for further explanations

3.6 Frequency Response of Systems Driven by Temporary Forces *

There is an unexpected consequence of using short-term "force pulses". We will address this topic already now,[1], but will return to it more than once in other parts of the book. Full understanding of the phenomenon under discussion is possible only after a review of Fourier analysis (see Chap. 5).

In Fig. 3.8, we showed how large an oscillation energy (proportional to amplitude squared) a system gets if it is exposed to a harmonic force with an "infinitely long" duration. The oscillation energy achieved was plotted as a function of the frequency of the applied force. A plot like this is usually called "*frequency response*" of the system, and the curve can be used to determine the Q-factor of the oscillating system from Eq. (3.16). The narrower the frequency response, the higher the Q-factor.

It is natural to determine the frequency response also for the case when the force lasts only a short time. The maximum energy system achieves as a result of the power is plotted as a function of the centre frequency of the power in a similar manner as in Fig. 3.8, and the result is given in the left part of Fig. 3.12. Relative energy is proportional to the square of the amplitude of the oscillations.

It turns out (left part of Fig. 3.12) that the frequency response of the system becomes different with temporary "force pulses" than with a harmonic force of infinitely long duration (as shown in Fig. 3.8). The frequency response becomes

[1]This sub-chapter is for the most interested readers only.

wider and wider (spreading over ever greater frequency range on both sides of the resonant frequency) as duration of the force pulse becomes shorter and shorter.

If, on the other hand, we apply longer and longer "force pulses", the frequency response of the system will reach a limiting value. There is a lower limit for the width of the curve, and thus a maximum limit for the calculated Q-factor. In general, the term Q-factor is used only for this limiting value. For shorter power pulses, the frequency response is specified rather than the Q-value.

However, it is possible to make a frequency analysis of the *temporary force pulse itself*. We will find out how this is done in Chap. 5 when we come to review Fourier analysis. To provide already now a rough idea of what a frequency analysis entails, it will be enough to say that such an analysis yields information about the frequency content of a signal, and tell us "which frequencies will be needed to reproduce the signal at hand".

The right part of Fig. 3.12 shows the frequency analysis of the "*force* as a function of time" for the same σ values as in the left part of the figure. The figure actually shows a classical analogy to Heisenberg's uncertainty relationship also known as the time-bandwidth product . We already found this in Eq. (3.17), and we will return to this in Chap. 5.

The two halves of Fig. 3.12 can be condensed into a single plot, and the result will then be as shown in Fig. 3.13.

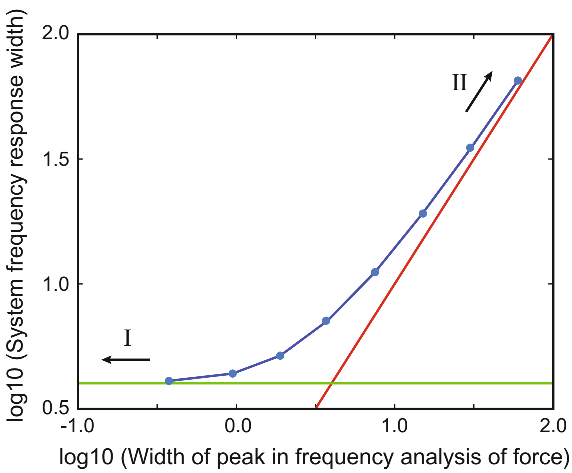

Fig. 3.13 The correlation between the frequency response of a system and the frequency of the driving force when the duration of the force changes. There are two border cases. In one case (I) the force lasts so long that the response depends only on the system itself (how much loss it is, and thus which Q-value it has). In the other case (II), the system's loss is so low in relation to the working time of the influence that the response to the force depends only on the force itself (how short time it lasts). The system's features have the least to say for the response

Based on these observations, we can say that:

- The quality factor is a parameter/quantity which *characterizes the oscillating system*. The smaller the loss in the system, the higher the Q-factor and the narrower frequency response, well and mark for harmonic forces that last long.
- When the force lasts for a short time (few oscillations) the frequency of the force is poorly defined. When an oscillating system is subjected to such a force, the frequency response is dominated by the *frequency characteristic of the power itself* and, to a lesser extent, the system itself.

Figure 3.13 is of some interest in the debate about whether Heisenberg's uncertainty relationship is primarily due to the perturbing influence of measurement on a system, or to the system itself. We do not delve into this issue here, but the result suggests that each point of view has some merit.

3.7 Example: Hearing

Finally in this chapter, we will say a little about our hearing and the mechanisms behind the process. Forced oscillations occupy the centre state in the present section, while other aspects associated with hearing will be treated in Chap. 7.

In our ears (see Figs. 3.14, 3.15 and 3.16), sound waves in the air cause oscillations at different frequencies in the auditory canal, tympanic membrane (eardrum), auditory ossicles (three tiny bones in the middle ear that conduct sound from the tympanic membrane to the inner ear), and the cochlea ("snailhouse")—a system of fluid-filled ducts which makes up the inner ear.

It is the inner ear that is of particular interest for us here, since it exemplifies resonance phenomena and demonstrates how ingenious our hearing sense is. Figure 3.15

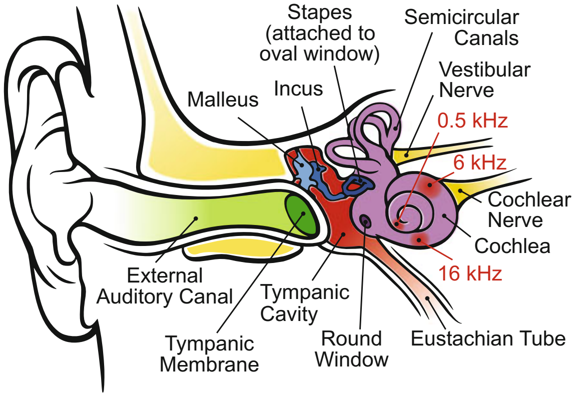

Fig. 3.14 Anatomical structures of the human ear. Inductiveload, CC BY 2.5, [1]

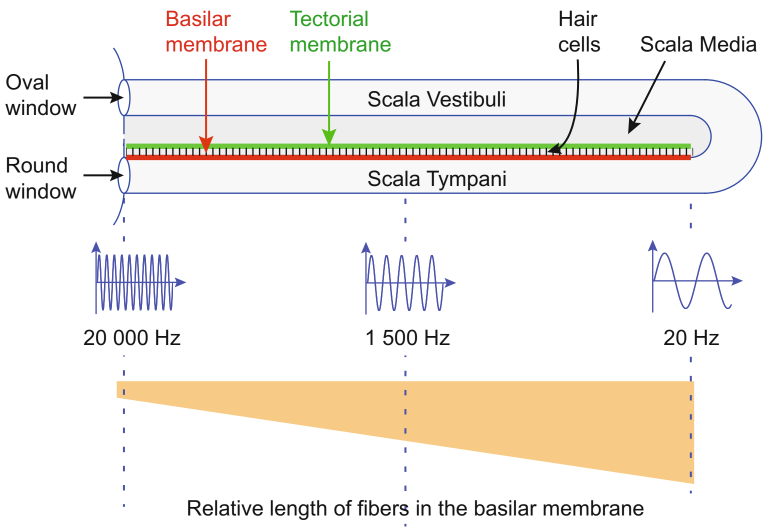

Fig. 3.15 The inner ear has a three-channel structure that stretches almost three rounds from bottom to top. This figure indicates how this would look like if we stretched out the insides of the cochlea. See the text for details

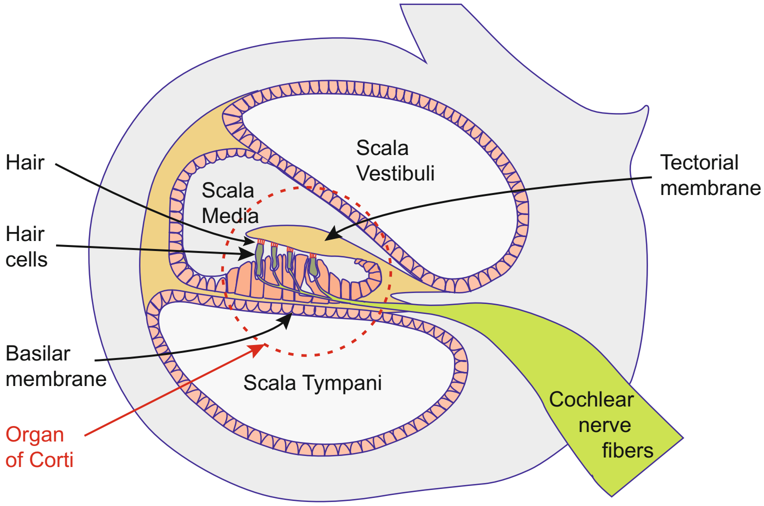

Fig. 3.16 Details on the anatomical structure of the basilar and tectorial membrane and their close connection through the organ of Corti. Note the hair cells that translate mechanical strain to electric signals. The organ of Corti structures are found along the full length of the basilar membrane with the result that it is an impressive number of nerve cells going from each ear to the brain

illustrate a "stretched out" cochlea with the fluid-filled ducts scala vestibuli from the oval window to the top of the cochlea and scala tympani from the top back to the round window (which is facing the air filled space of the middle ear).

One wall of the scala tympani has a particular structure called the basilar membrane, and weakly connected to the wall along the scala vestibuli we find the tectorial membrane. These membranes will oscillate when the ear picks up a sound signal.

Between the basilar and tectorial membranes, we find "hair cells" that respond to pressure. The amplitude of the oscillations is picked up by these hair cells, and the information is transmitted through the nerves to the brain (via different signal processing centres along the way).

It is a fascinating structure of cells named Organ of Corti (see Fig. 3.16) that translate pressure changes into electrical signals in nerves. Figure 3.16 also indicates how the third duct inside the cochlea, the air filled scala media, is a part of the total structure.

From our perspective, the important part is the basilar membrane. Earlier in this chapter, forced oscillations have been analysed. By way of a trial, that analysis will be applied to oscillations in the basilar membrane, which extends diametrically across the conical cavity of the cochlea in the inner ear (see Figs. 3.15 and 3.16).

The membrane can vibrate, just like the belly (top plate) of a violin, in unison with the pressure variations generated by the sound. The membrane, however, changes character from the outer to the inner parts of cochlea. The relative length of some fibres in the basilar membrane varies from the outer to the inner part as indicated in Fig. 3.15. As a result, if we hear a dark sound (low frequency), only the inner part of the basilar membrane will vibrate. If we hear a light sound (high frequency), only the outer part will vibrate. This is a fabulous design that allows us to hear many different frequencies at the same time as separate audio impressions. We can hear both a bass sound and a disk rhythm simultaneously, because the two sound stimuli excite different parts of the basilar membrane. The hair cells and nerve endings pick up vibrations from different parts of the membrane in parallel.

It was the biophysicist Georg von Békésy from Budapest who found out how the basilar membrane works as a "position-frequency map". He received the Nobel Prize in Physiology and Medicine for this work in 1961.

The basilar membrane is a mechanical oscillation system that behaves in a manner similar to the externally driven mass–spring oscillator and RCL circuit. Different parts of the membrane have properties that make them responsive to different frequency ranges. We can assign different Q-values to different parts of the basilar membrane.

Based on what we have learned in this chapter, we should expect that even if we hear a sound that delivers a harmonic force with a well-defined frequency on the eardrum, the basilar membrane will vibrate not at one position only along the basilar membrane, but over a somewhat wider area. Since we have "parallel processing" of the signals from the hair cells, the brain can still "calculate" a fairly well-defined centre frequency.

If, however, we listen to shorter and shorter sound pulses, we expect that wider and wider parts of the basilar membrane will be excited. This would make it harder for the brain to determine which centre frequency the sound pulse had. This means that it is harder to determine the pitch of a sound when the sound lasts very shortly.

When musicians play fast passages on, for example, a violin they can falter *a little* with the pitch without the error coming to the notice of a listener. If they stumbled as much with longer lasting tones, their slips will not escape the attention of the audience.

When the sound pulse lasts only one period (and this period, for example, corresponds to 1000 Hz), we only hear a “click”. It is impossible to tell which frequency was used to create the sound image itself.

On the other hand, it is easier to perceive the direction of the audio source of a click than the source of a sustained sound. The ability to determine the time fairly precisely when a sound occurs, along with the fact that we have two ears, is very important in order to determine the direction the incoming sound (Nevertheless, it should be mentioned that there are other mechanisms to determine where a sound comes from.).

According to Darwin, our ears are the result of millions of years of natural selection that was beneficial for the survival of our species. The ear has become a system where there is an optimal relationship between the ability to distinguish between different frequencies and the ability to follow fairly quick changes over time. Resonance, time response and frequency response are very important details to understand our hearing.

An interesting detail with regard to hearing relies on phase sensitivity. Nerve impulses (they are digital!) cannot be transmitted over nerve fibres with a repetition rate much higher than about 1000 Hz. It is therefore impossible for the ear to send signals to the brain with a better time resolution than about 1 ms. This means that the ear cannot, in principle, provide information about the phase of a sound vibration for frequencies higher than a few hundred hertz (Some disagree and claim that we can follow phases up to 2000 Hz.). The prevalent view is that sound impression become indifferent to the phase of the various frequency components of a sound signal.

3.8 Learning Objectives

After working through this chapter you should be able to:

- Set up the differential equation for a system subject to forced harmonic oscillations and find an analytical solution for this when the friction term is linear.
- Find a numerical solution of the aforementioned differential equation also for nonlinear friction terms and for nonharmonic forces (after having been through Chap. 4).

- Derive mathematical expressions for resonance frequency, phase shift and quality factor for a single mechanical oscillating system or an electrical oscillating circuit.
- Set up a phasor diagram to explain typical features of an RCL circuit for different frequencies of an applied voltage.
- Know the time course of the oscillations in a circuit, as an externally applied force begins and when it ends and how the time course is affected by the Q-factor.
- Know how the response to an oscillating system changes when the force lasts for a limited period of time.
- Know the coarse features of the anatomy of the ear well enough to explain how we can hear many pitches all at the same time.
- Know that in a mechanical system we cannot get both high frequency-selectivity and high time resolution simultaneously.

Both for the mechanical and electrical oscillating system examined so far, we end up with an equation where the second derivative of a quantity along with the quantity itself is included. It may lead to the opinion that all oscillations must be described by a second-degree differential equation.

However, there are also oscillations that are normally described by two or more coupled first-order differential equation and a significant time delay between the "force" and "the response" in the differential equations. In biology, such relationships are not uncommon.

3.9 Exercises

Suggested concepts for student active learning activities: Forced oscillation, resonance, phasor, phase difference, quality factor, initial and terminal transient behaviour, frequency response, simultaneous multiple frequency detection, basilar membrane, cochlea, inner ear.

Comprehension/discussion questions

1. For a mass–spring oscillator, the phase difference between the applied force and the amplitude of the bob change with the frequency of the applied force. How is the phase difference at the resonance frequency and at frequencies well below and well above it?
2. How does the phase difference between the applied force and the *velocity* vary for a mass–spring oscillator exposed to a harmonic force?
3. It is often easier to achieve a high Q-value in a oscillating system with a high resonance frequency than with a low one. Can you explain why?
4. If our hearing (through natural selection could distinguish much better between sound at nearby frequencies than we are able to achieve, what would the disadvantage have been?

5. We operate with two almost equal resonant frequencies. What are their characteristics? Is it possible for these frequencies to coincide?
6. What would happen to an oscillating system *without damping* if it was exposed to a harmonic applied force at the resonant frequency? What would happen if the applied force had a frequency slightly different from the resonance frequency?
7. In several laboratories attempting to detect gravity waves, oscillating systems with suitable resonance frequencies and Q-values are used as detectors. For example, a resonance frequency of about 2–4 kHz is chosen when one wants to detect gravity waves due to instability in rotating neutron stars. What is the motivation behind using an oscillating system as a detector for this purpose?
8. For a mechanical system, the phase shift $\pi/2$ between the amplitude and the applied force was explained by the fact that such a phase shift corresponds to the force supplying the maximum power to the system (maximum force applied over the longest possible way). Explain in a similar manner the phase shift also for the electrical RCL circuit with a harmonically varying applied voltage.
9. Attempt to explain the phase shift for the RCL series circuit with applied voltage in case the frequency is far less and far greater than the resonant frequency of the circuit alone. Based on how the impedance of a capacitor and the impedance of an inductance change with frequency.
10. How can the oscillations that led to the collapse of the Tacoma Narrows Bridge in Washington, USA, in 1940 be explained as a forced oscillation? Do you think the Q-value was big or small? (May be relevant to watch one of the movies featured on YouTube.)
11. An AC voltage $V(t) = V_1 \cos(\omega_F t)$ is applied to an electrical oscillating circuit, ω_F is equal to the resonance (angular) frequency of the circuit. After a long time, the oscillations in the circuit stabilize and the amplitude of the current fluctuations is I_1. An interval of duration t_1 elapses between the connection of the AC voltage to the circuit and the current reaching the value $0.9 \times I_1$. We then remove the voltage and let the circuit come to rest. We then reconnect to the AC voltage, but now with twice the amplitude: $V(t) = 2V_1 \cos(\omega_F t)$.
(a) How large is the current in the circuit (relative to I_1) a long time after the AC voltage was reconnected?
(b) How long does it take for the amplitude of the current in the circuit to reach 90% of the limiting, long-time value?
(c) What do we mean by the expression "long-time value" in this context?

Problems

12. In the case of old-fashioned radio reception in the medium wave range, we used circuitry consisting of an inductance (coil) and capacitance (capacitor) to discriminate between two radio stations. The radio stations occupied 9 kHz on the frequency band, and two radio stations could be as close as 9 kHz. In order for us to choose one radio station from another, the receiver had to have a variable resonant circuit that suited one radio station, but not another. The frequency of the Stavanger transmitter was 1313 kHz. Which Q-factor did the radio receiver's

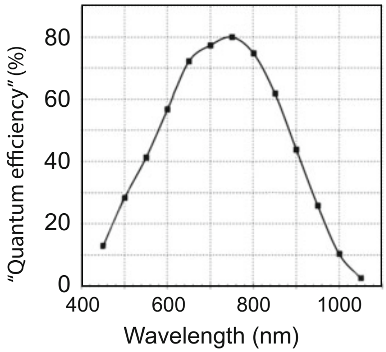

Fig. 3.17 Sensitivity curve of a "single-photon detector"

resonant circuit need? [These considerations are still applicable in our modern times, although digital technology makes certain changes.]

13. Figure 3.17 shows "sensitivity curve" for a "single-photon detector". Let us consider this curve as a sort of resonance curve, and try to estimate how long a continuous electromagnetic wave (light) will have to illuminate the detector to achieve maximum/stationary response in the detector? (Imagine a similar response as in Fig. 3.7.) The frequency of the light can be calculated from the relationship $\lambda f = c$ where λ is the wavelength, f the frequency and c the velocity of light.
14. Search the web and find at least ten different forms of resonance in physics. Enter a web address, where we can read a little about each of these forms of resonance.
15. Derive the expressions given in Eq. (3.11) from Eq. (3.12) and other expressions for an oscillating mass–spring oscillator.
16. The Q-value for an oscillating circuit is an important physical parameter.
    (a) Give at least three examples of how the Q-value influences the function/behaviour of a circuit.
    (b) Describe at least two procedures as to how the Q-value can be determined experimentally.
    (c) If we use a temporary force, it is more difficult to determine the Q-value experimentally. Explain why.
17. A series RCL circuit consists of a resistance R of $1.0\,\Omega$, a capacitor C of 100 nF, and an inductance L of $25\,\mu$H.
    (a) Comparing Eq. (3.7) (slightly modified) with Eq. (3.1), we realize that these equations are completely analogous. Just by replacing a few variables related to the mechanical mass–spring oscillator, we get the equation for an electrical series RCL circuit. Using this analogy, we can easily reshape the expressions for phase shift [Eq. (3.3)], amplitude [Eq. (3.4)], Q-value [Eq. (3.11)] and the expressions for phase resonance and amplitude resonance for the mass–spring oscillator, to corresponding formulas for a series RCL circuit. Determine all these terms for a series RCL circuit.
    (b) Calculate the resonant frequencies (both for phase and amplitude resonance)

of the circuit (based on amplitudes of charge oscillations, not current oscillations).

(c) Calculate the Q-value of the circuit.

(d) What is the difference in phase between the applied voltage and current in the circuit at phase resonance and at a frequency corresponding to $\omega_0 + \Delta\omega/2$ in Eq. (3.16)?

(e) How wide is the frequency response of the circuit for a "long-lasting" applied voltage?

(f) How "long" must the applied voltage actually last for the circuit to reach an almost stationary state (that amplitude no longer changes appreciably with time)?

(g) Assume that the circuit is subjected to a force pulse with centre frequency equal to the resonance frequency and that the force pulse has a Gaussian amplitude envelope function [Eq. (3.19)] where σ has a value equal to twice the time period corresponding to the centre frequency of the circuit. Estimate the width of the frequency response to the circuit with this force pulse.

Reference

1. Inductiveload, https://commons.wikimedia.org/wiki/File:Anatomy_of_Human_Ear_with_Cochlear_Frequency_Mapping.svg. Accessed April 2018