Chapter 5
Fourier Analysis

Abstract In this chapter, the first major challenge is to understand the difference between two descriptions of a signal: one in the time domain and another in the frequency domain. We initially use a gradual increase in complexity to help the reader grasp the difference. We then use phasors in order to introduce positive and negative frequencies, a detail that is encountered later. The formal mathematical Fourier transform and inverse transform are then introduced as well as Fourier series. The remainder of the chapter is devoted to discrete Fourier transform in the form of fast Fourier transform (FFT). All exact details on intervals in time and frequency are stated with great care. Important details like aliasing/folding and sampling theorem are given. We also analyse a time-limited oscillating signal and get our first encounter with the bandwidth theorem, and a theme we will recur to in several later chapters of this book.

5.1 Introductory Examples

5.1.1 A Historical Remark

Fourier transformation and Fourier analysis bear close resemblance to the medieval use of epicycles for calculating how planets and the sun moved relative to each other. That gives us an inkling of how powerful Fourier analysis is, but at the same time it reminds us that Fourier analysis can sometimes hinder a deeper understanding of the phenomena around us. Several later chapters in this book are based on a good understanding of Fourier transformation, including the awareness of the danger to think and argue almost in the same manner as in the Middle Ages.

5.1.2 A Harmonic Function

Before delving into the details about Fourier transformation, it will be useful to take a look at Chap. 2. We saw that a harmonic function can be written in several different

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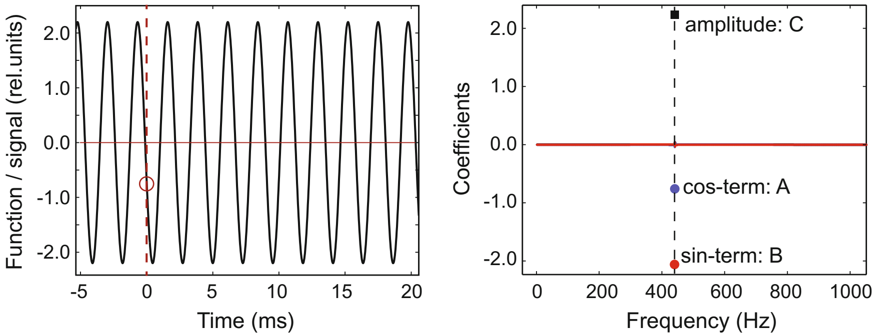

Fig. 5.1 Section of a harmonic function plotted, in the left part, as a function of time ("time domain") and, in the right part, as a function of frequency ("frequency domain"). See text for other details

ways:

$$z(t) = C\cos(\omega t + \phi) = A\cos(\omega t) + B\sin(\omega t) = \Re\left\{\mathscr{D}e^{i\omega t}\right\}. \tag{5.1}$$

$\Re\{\}$ means that we take the real part of the complex expression within the braces, and $\mathscr{D}$ is a complex number.

In the left part of Fig. 5.1, we have plotted a section of an arbitrary harmonic function of time. Amplitude C is 2.2 in some unspecified units and the frequency $f = 440$ Hz, which corresponds to the period $T \approx 2.27$ ms $\approx 1/440$ s. We chose the phase shift $\Phi = 110°$. This means that the value of the function is neither zero nor at the maximum at time $t = 0$.

The three parameters C, $\omega = 2\pi f$ and ϕ specify the function $z(t) = C\cos(\omega t + \phi)$ unambiguously. Using the identities in Chap. 2, this function can also be expressed as $A\cos(\omega t) + B\sin(\omega t)$. In that case, $A = C\cos\phi \approx -0.76$ and $B = -C\sin\phi \approx 2.06$. The three parameters that specify the function completely are A, B and ω.

Usually we plot a function of time as has been done in the left part of Fig. 5.1. However, we can also display the function graphically in an altogether different way, which is done in the right part of the figure. Here we have *frequency* along the x-axis and the coefficients A and B along the y-axis, and colour coding has been used to distinguish A from B. Since we have *time* along the x-axis in the left part of Fig. 5.1, we call this a "time-domain" representation of the function. For the right part, the frequency is along the x-axis, and we therefore call this a "frequency-domain" representation. Both representations contain (under certain assumptions) the same information.

In the frequency-domain picture, we have also displayed C. Occasionally we are interested only in amplitudes and not phases. Then $C = \sqrt{A^2 + B^2}$ is useful, and C is always positive (or zero). However, C and ω alone are not sufficient to determine the function unambiguously—phase information is missing.

If we use the last expression in Eq. (5.1), we can also specify the function as follows:

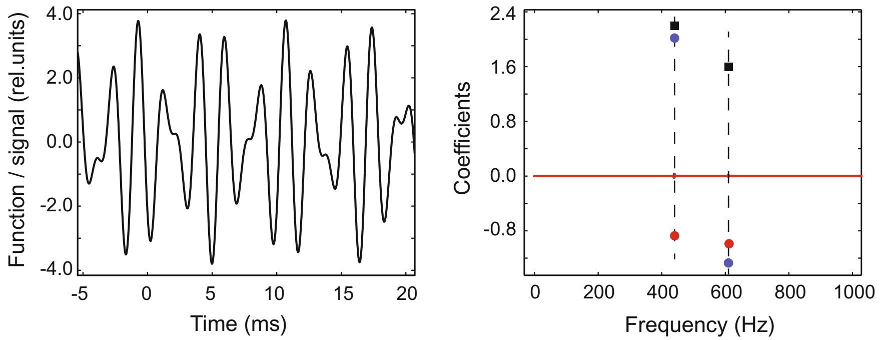

Fig. 5.2 A segment of a function that is a sum of two harmonic functions with frequencies 440 and 610 Hz plotted, one the left, as a function of time ("time-domain picture") and on the right as a function of frequency ("frequency-domain picture"). The colour coding is the same as in the previous figure. See text for other details

$$z(t) = \Re\left\{\mathscr{D}e^{i\omega t}\right\}. \tag{5.2}$$

It is important to remember that $\mathscr{D}$ is a complex number, and that $\mathscr{D} = A - iB$ so that $\mathscr{D}$ is the detail in Eq. (5.2) that contains the information about the phase of the harmonic function. The amplitude C is the absolute value of the complex number $\mathscr{D}$.

If you do not remember all the details in Chap. 2 which are used in transforming one version to another in Eq. (5.1), it is recommended that you revise that section now. In the rest of this chapter, we will use the rendering given in Eq. (5.2), and it is very important to fully understand this expression.

At present we need to refer only to the mathematics in Chap. 2. We will show that, by using a so-called Fourier transform, we can generate the plot in the right part of Fig. 5.1 completely automatically. The prime purpose of this introductory part is to find out what are meant by the terms "time-domain picture" and "frequency-domain picture".

5.1.3 Two Harmonic Functions

Let us see now what happens when we have a sum of two harmonic functions. The time-domain picture is given in the left part of Fig. 5.2. Since we have generated this function ourselves, we know that it is described by

$$z(t) = C_1\cos(\omega_1 t + \phi_1) + C_2\cos(\omega_2 t + \phi_2) \tag{5.3}$$

where all the six parameters appearing above are known.

We can also use the alternative form:

$$z(t) = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \tag{5.4}$$

where A_1, A_2, B_1 are B_2 are to be found by using C_1, ϕ_1, C_2 and ϕ_2, and, since the frequencies ω_1 and ω_1 are known, we can make a frequency plot corresponding to this function. Such a plot is shown in the right part of the figure.

Someone who did not know how the function was generated, and obliged to evaluate it only from the time plot in the left part of Fig. 5.2, would find it difficult to say with certainty that this a sum of only two harmonic signals. It would be quite a challenge to determine the amplitudes and phases.

However, with the help of Fourier transformation, which is the subject of this chapter, we can use the time plot to calculate, automatically, A_1, A_2, B_1, B_2, ω_1, and ω_2 and we can confirm that there are no other contributions to the signal. You may now appreciate how useful Fourier analysis can be!

We recall the rendering based on Euler's formula and complex coefficients. For two harmonic functions, this takes the form:

$$z(t) = \Re \left\{ \mathscr{D}_1 e^{i\omega_1 t} + \mathscr{D}_2 e^{i\omega_2 t} \right\} . \tag{5.5}$$

It is important to realize that all three form of writing in Eqs. (5.3), (5.4) and (5.5) are equivalent.

Since the coefficients $\mathscr{D}_1$ and $\mathscr{D}_2$ can be determined by Fourier transformation, they are commonly called *Fourier coefficients* of the $z(t)$ function.

5.1.4 Periodic, Nonharmonic Functions

In the last example, the signal was nonperiodic. In many parts of physics, we deal with periodic functions. An example is shown in Fig. 5.3. Looking at this feature in the time-domain picture, it is hard to understand that such a signal can be described in a relatively simple way.

Since we have generated the signal ourselves, we know how it was constructed. The signal is made as a sum of six harmonic functions, each of which is described by a set of $[A_i, B_i, \omega_i]$-values. In order to get a periodic signal, each ω_i was taken as $n\omega_0$, an integral multiple of the lowest value ω_0, called "the fundamental frequency". In our case, $\omega_0 = 610\,\text{Hz}$ and $n = 1, 2, \ldots 6$. The right part of Fig. 5.3 shows how the frequency-domain picture in this case looks like.

It is pleasing to note that even in this case we succeeded, thanks to a Fourier transformation, in analysing the $z(t)$ signal directly, and in finding how the signal was composed. It would be almost impossible to extract these details without Fourier transformation, as there are 18 different parameters to be determined. We will come back to the details later.

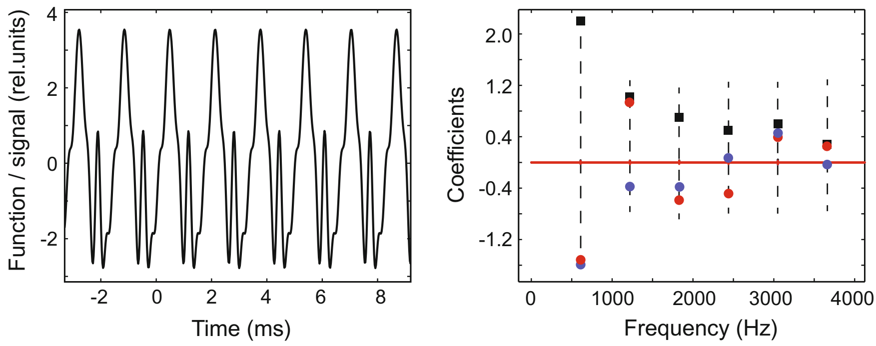

Fig. 5.3 Time-domain picture on the left shows a section of a periodic, nonharmonic function and on the right is shown the corresponding frequency-domain picture. See text for other details

It turns out that the more a periodic signal differs from a pure sinusoid, the more harmonic functions (higher n values) are needed for describing it.

We remind the reader that if we choose Euler's formula and complex coefficients, a *periodic* function would look like this:

$$z(t) = \Re \left\{ \sum_{n=1}^{N} \mathscr{D}_n \mathrm{e}^{\mathrm{i}n\omega_0 t} \right\} .$$

In our case $N = 6$.

5.1.5 Nonharmonic, Nonperiodic Functions

In the end, we look at something rather odd. We have seen in the three previous examples that it is possible to make many different signals by combining harmonic functions with different amplitudes and phases. As we shall see immediately, an *arbitrary* function, including nonharmonic and nonperiodic functions, can be written as a sum of harmonic functions as follows:

$$z(t) = \sum_{n=1}^{N} C_n \cos(\omega_n t + \phi_n) = \Re \left\{ \sum_{n=1}^{N} \mathscr{D}_n \mathrm{e}^{\mathrm{i}\omega_n t} \right\} \tag{5.6}$$

for some large N. Occasionally, we have to use a very large number of frequencies in the description of a function. We can then replace the summation by an integral with a continuous function $\mathscr{D}(\omega)$ that specifies the coefficients:

$$z(t) = \Re \left\{ \int_{\omega=0}^{+\infty} \mathscr{D}(\omega) \mathrm{e}^{\mathrm{i}\omega t} \right\} \tag{5.7}$$

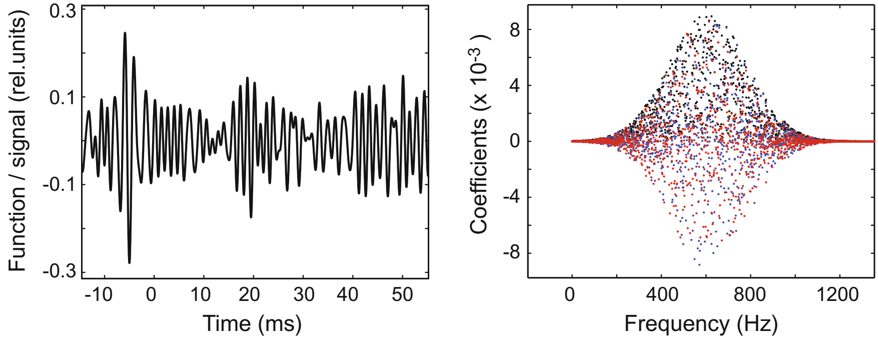

Fig. 5.4 Left part is a "time-domain picture" of a nonperiodic, nonharmonic function, and on the right is the "frequency-domain picture" of the same function. See text for other details

In Fig. 5.4, we have created a signal that is built by adding more than 3000 harmonic functions with frequencies lying in a wide band centred around 610 Hz. The amplitude varies randomly, but the largest amplitudes occur only for frequencies in the broad region near 610 Hz. The phases are random. The sum signal is then both nonharmonic and nonperiodic, as indicated in the time plot on the left. An analysis similar to that we have done in the previous examples gives the coefficients (and amplitudes) indicated in the right part of the figure.

5.2 Real Values, Negative Frequencies

It is a little tiresome that when we use the functional form given in Eq. (5.2), we always have to find the real value $\Re$ of the complex expression inside the braces on the right. There is a useful trick to get around this problem.

The basic element is this equation is the exponential term $e^{i\omega t}$ and Euler's formula $e^{i\omega t} = \cos(\omega t) + \mathrm{i}\sin(\omega t)$. This relation is often illustrated through phasors.

The function $z(t) = C\cos(\omega t + \phi)$ can be described by a phasor which at time t has an orientation as shown in Fig. 5.5. The phasor rotates in a positive direction (anticlockwise) with the angular frequency ω, and it is always the component along the x-axis (the real axis) that indicates the value of $z(t)$.

If we now create a vector of the same length C, but always reflected about the x-axis relative to the previous one, rotating in the *negative* direction (clockwise), the sum of this phasor and the previous will always be along the x-axis. There will be no imaginary contribution!

The maximum value of the sum of the two vectors will be equal to $2C$, so we need to enter a factor of 1/2 to correct for this. The maximum of the sum vector occurs every time $\omega t + \phi$ is an integer multiple of 2π.

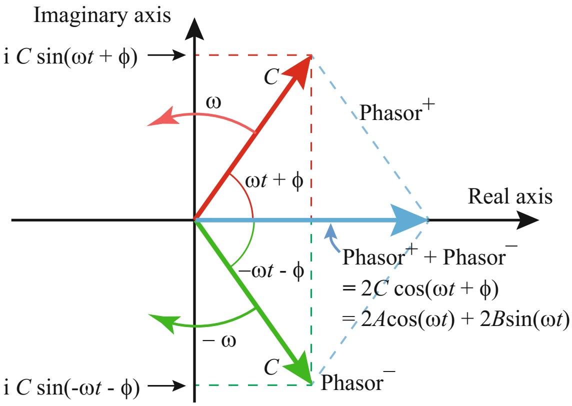

Fig. 5.5 Common phasor description (*in red*) of a harmonic function $C\cos(\omega t + \phi)$ at time t. A second phasor is also drawn (*in green*), which is the reflection of the original phasor about the x-axis, and rotates therefore the opposite way. Adding the two vectors, we get a resultant (*blue*) that always lies along the real axis, but has twice the length we are interested in

We have now put sufficient pictorial flesh on algebraic bones to make the following formula palatable:

$$C\cos(\omega t + \phi) = \frac{1}{2}\left\{\mathscr{D}e^{i\omega t} + \mathscr{D}^* e^{-i\omega t}\right\} = \frac{1}{2}\left\{\mathscr{D}e^{i\omega t} + c.c.\right\} \tag{5.8}$$

where the asterisk in $\mathscr{D}^*$ and "c.c." stands for "complex conjugate".

We see that by introducing "negative frequencies", we can avoid having to take the real value of the complex function $\mathscr{D}e^{i\omega t}$.

Fourier analysis uses the connection given in Eq. (5.8), which means that what was said in the introductory examples was not the whole truth. If we actually do a Fourier analysis of the first harmonic function we examined, the frequency-domain picture will have the appearance shown in the right part of Fig. 5.6. We receive contributions from −440 to +440 Hz. The coefficients in front of the cosine term have the same value for positive and negative frequency, but only half of the coefficient A in Eq. (5.1). However, the coefficients in the sine term, which correspond to the imaginary axis of the phasor diagrams, have changed sign when we go from positive to negative frequency. Here too the factor 1/2 comes in. The same also applies to the C's since $C = \sqrt{A^2 + B^2}$.

All Fourier analysers of real signals have in principle this positive and negative division, where the coefficients are complex conjugate of each other. A little later, under the heading "folding", we will see that the negative frequencies appear in a rather odd way in the so-called fast Fourier transform (FFT).

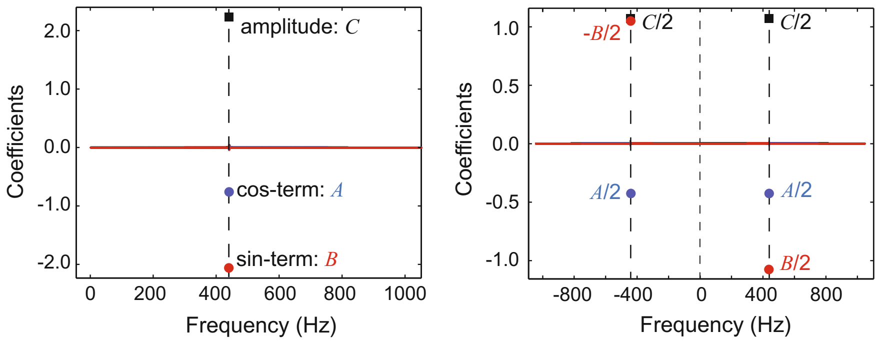

Fig. 5.6 Frequency-domain picture obtained when we work with only the positive frequencies on the left. In that case, we must ourselves extract the real part of the expression in Eq. (5.5) if we use this representation. With normal Fourier transform of real signals, half of the coefficients $\mathscr{D}(\omega)$ are apportioned to the frequency ω and the other half to the frequency $-\omega$; furthermore, the coefficient at a negative frequency is the complex conjugate of the corresponding coefficient at positive frequency

5.3 Fourier Transformation in Mathematics

So far in this chapter, we have seen several examples of how a continuous signal or function of time can be written as a sum (or integral) of harmonic functions. This actually applies in general, as was shown by the French mathematician and physicist Joseph Fourier (1768–1830).[1]

We would like to write Fourier's relation in the following manner:

Let $f(t)$ be an integrable function of t (usually time) as a continuous parameter. In physics, $f(t)$ is often a real function, but mathematically it may be complex. The function $f(t)$ can then be described as an integral of harmonic functions as the limiting value of a sum:

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega. \quad (5.9)$$

Here $F(\omega)$ corresponds to Fourier coefficients and is called the "Fourier transform of f". $F(\omega)$ forms the so-called frequency-domain picture of the function, while $f(t)$ represents the time-domain picture.

On comparing with Eqs. (5.6), (5.7) and (5.8), we see that we have now changed the notation to $z(t) \rightarrow f(T)$ and $\mathscr{D}(\omega) \rightarrow F(\omega)$ and we have availed ourselves of negative frequencies by allowing the integration to go from minus infinity to plus infinity. If $f(t)$ is a real function, $F(\omega) = F^*(-\omega)$.

[1]Fourier is also known to have demonstrated/explained the global warming effect in 1824.

The challenge now is to find $F(\omega)$, and this is where Fourier lends us a helping hand of giant proportions. He introduced Fourier transformation in analytical mathematics:

Given $f(t)$, a new function $F(\omega)$ (the Fourier transform of f) can be calculated as follows:

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}dt. \tag{5.10}$$

The parameter ω is the angular frequency if t represents time. Both t and ω are continuously variables.

You may have come across Fourier transformation in an earlier course in mathematics. In mathematics, the transformation is often linked to the inner product between two functions, and one defines a basis of sine and cosine functions and uses Gram–Schmidt process on a function to find its Fourier transform. Here, we choose a more practical approach in our context.

It may seem difficult to understand that Eq. (5.10) will work as we would like it to, but let us look at some basic properties in analytical mathematics.

The harmonic functions $\sin(\omega t)$ and $\cos(\omega t)$ together form a complete set of integrable functions that can describe any other integrable function. The functions $\sin(\omega_1 t)$ are orthogonal to $\sin(\omega t)$ when $\omega \neq \omega_1$, all $\sin(\omega t)$ are orthogonal to all $\cos(\omega t)$. This is embodied in the familiar expression of the delta function:

$$\delta(\omega_1 - \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(\omega_1-\omega)t}dt. \tag{5.11}$$

As an example, we now allow $f(t)$ to be the simple harmonic function in Eq. (5.1), but for the sake of simplicity, skip the details of finding the real value. We then write:

$$f(t) = \mathscr{D}\mathrm{e}^{\mathrm{i}\omega_1 t}.$$

Substitution in Eq. (5.10) gives:

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{D}\mathrm{e}^{\mathrm{i}\omega_1 t}\mathrm{e}^{-\mathrm{i}\omega t}dt,$$

$$F(\omega) = \mathscr{D} \times \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\omega_1-\omega)t}dt.$$

We recognize the last part as the delta function, and the result is that $F(\omega)$ is zero everywhere except when $\omega_1 = \omega$ where $F(\omega_1) = \mathscr{D}$. We therefore see that, in this case, Eq. (5.10) does indeed work as desired.

Equation (5.10) gives what we call the Fourier transform of the function $f(t)$. In our context, it amounts to exchanging the time-domain description of a function with one in the frequency domain.

Equation (5.9) gives what we call an *inverse* Fourier transformation. It takes us from the frequency-domain representation of a function to a picture in the time domain.

Note that in a Fourier transform we integrate over time and the exponent has a minus sign in front. In the inverse transformation, we integrate over frequency and the exponent has a plus sign in front. Also note that the factor $1/(2\pi)$ is only used in one transformation, as we have chosen to express the two equations that, in part, belong together. Another choice is to use a $1/\sqrt{2\pi}$ in both Eqs. (5.10) and (5.11).

Remarks: Several reasons account for why Fourier transformation became popular in mathematics and physics. There are many simple mathematical relationships for harmonic functions. This means that if we have to deal with a troublesome function $f(t)$ and do not know how to handle it directly, we can use Fourier transformation as an intermediate step in the calculation. By Fourier transforming the awkward function, we obtain a linear sum (or integral) of harmonic functions. We can then perform mathematical operations on this alternative expression and use inverse Fourier transformation on the result to retrieve the result we actually wanted. Fourier transformation is therefore used extensively in analytical mathematics for, among other purposes, solving differential equations.

We know from mathematics that there are several complete sets of functions (e.g. polynomials), and in different parts of physics, we prefer to choose a basis set that is best adapted for the particular system under consideration. Fourier transformation utilizes probably the most widely used basis set of functions; unfortunately, it is also applied in situations where it is not particularly beneficial.

5.3.1 Fourier Series

A special case in Fourier transformation is of particular interest, especially when we study Chap. 7 to analyse sound from musical instruments. If $f(t)$ is *a periodic function* with period T, Fourier transformation can be made more efficient than through the general transformation in Eq. (5.10). The transformation can be specified by an infinite but *discrete* set of numbers, called Fourier coefficients, $\{c_k\}$, the index k being a natural number between minus and plus infinity(!).

The Fourier coefficients are calculated by integrating over a single period T:

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\mathrm{e}^{-\mathrm{i}k\omega_1 t}dt \tag{5.12}$$

where $\omega_1 = (2\pi/T)$, that is to say, the angular frequency corresponding to a function that has *exactly one period in the time interval* T, and k is an integer.

Since in this case $f(t)$ is periodic, the lower limit for integration (t_0) can be chosen freely in principle. It is supposed that $f(t)$ is piecewise smooth and continuous, and that $\int |f(t)|^2 dt < +\infty$ when the integration is over an interval of length T.

The inverse transformation is then given by the relation:

$$f(t) = \sum_{k=-\infty}^{+\infty} c_k \mathrm{e}^{\mathrm{i}k\omega_1 t} \tag{5.13}$$

where, once again, $\omega_1 \equiv 2\pi/T$ corresponds to a frequency that has precisely one sine period within the interval T.

Should $f(t)$ be real, it is easy to see that the symmetry properties of the sine and cosine functions lead to the relation

$$f(t) = a_0 + \sum_{k=1}^{\infty}\{a_k \cos(k\omega_1 t) + b_k \sin(k\omega_1 t)\} \tag{5.14}$$

where

$$a_k = c_k + c_{-k} = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(k\omega_1 t)dt, \tag{5.15}$$

$$b_k = \mathrm{i}(c_k - c_{-k}) = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(k\omega_1 t)dt. \tag{5.16}$$

Take note of the factor 2 in the last two expressions! The reason for this factor is the simple recognition that the mean of both $\sin^2$ and $\cos^2$ is 1/2 and another factor of 2 that was explained above when we mentioned the inclusion of negative frequencies.

Equation (5.14) along with the expressions (5.15) and (5.16) are as precious as gold! They show that any *periodic signal* with period T can be written as a sum of harmonic signals *having exactly integral number of cycles within the period* T.

5.4 Frequency Analysis

Hitherto there has been a lot of mathematics and little physics in this chapter. It is therefore high time to give a few examples of the practical use of Fourier transformation.

Fourier transformation is widely used for so-called frequency analysis where we determine which frequency components are present in a signal. We often call the frequency-domain picture a "frequency spectrum". The frequency spectrum is useful because it often gives a "fingerprint" of the physical processes that lie behind the signal under consideration.

The number of sunspots increases and decreases over time regularly with an approximately 11-year cycle, we are often told. What is the basis for such an assertion? We can plot the number of sunspots per year over a number of years. We then get a curve like the left part of Fig. 5.7 where the curve corresponds to the $f(t)$ function in the theory above. This is the so-called time picture.

In the right part of Fig. 5.7, an extract of the results is shown after a Fourier transformation of the data in the left part. Actually, the results after a Fourier transformation are complex numbers. However, if we are not interested in getting $A\cos(\omega t)$ and $B\sin(\omega t)$ separately for the different frequencies, but are rather interested in the amplitude $C = \sqrt{A^2 + B^2}$, we choose to plot the absolute value of the complex numbers. It is the absolute values that are plotted in the right part of Fig. 5.7.

The peaks near the middle of the figure correspond to a harmonic function with a frequency of 0.09 or 0.10 per year. Since a frequency of 0.09–0.10 per year corresponds to a period of approximately 10–11 years, we get a satisfactory confirmation that the sunspots in the 300 years analysed have a considerable periodicity at 10–11 years. At the same time, the noise in the plot shows that the indicated time period is more poorly defined than what we find for example in the movement of a shuttle!

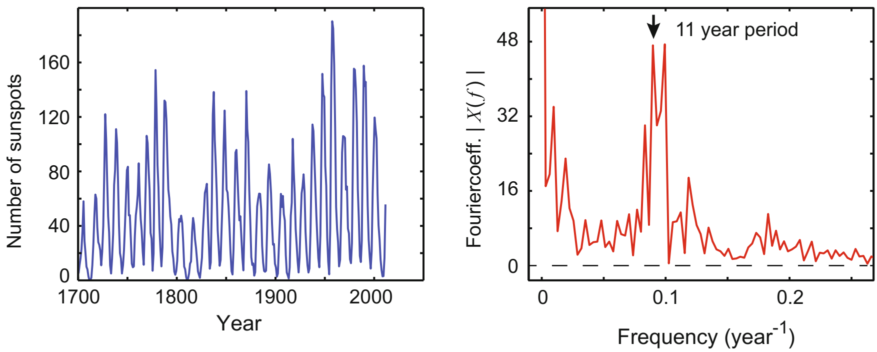

Fig. 5.7 Left part shows the number of sunspots that appeared annually over the past three hundred years. The right part shows an excerpt from the corresponding Fourier transformed functions (absolute values of $\{c_k\}$-s in Eq. (5.12)). The sunspots data were accessed on 30.1.2012 from http://sidc.be/silso/datafiles

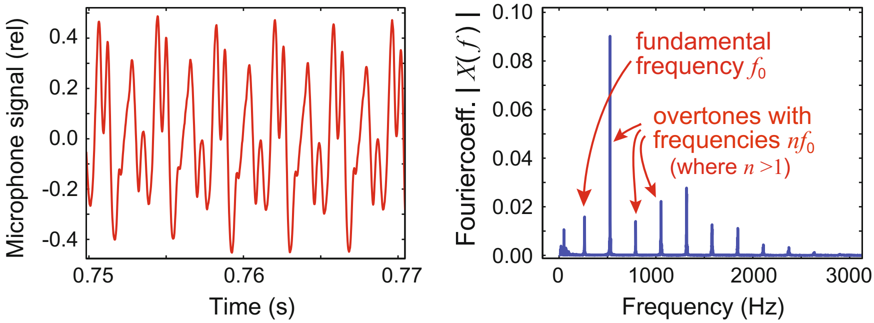

Fig. 5.8 An example of sound from a flute displayed both in the time domain and frequency domain. Amplitudes in the frequency domain are given as absolute values of $\{c_k\}$-s in Eq. (5.12))

In this book, we often use Fourier transformation to analyse sound. For example, Fig. 5.8 shows a time-domain picture and a frequency-domain picture for a audio signal from a transverse flute. The figure also shows relative amplitudes in the frequency spectrum. We then lose the phase information, but the "strength" of the different frequency components shows up well.

The spectrum consists mainly of a number of peaks with different heights. The peak positions have a certain regularity. There is a frequency f_0 (might have been called f_1), the so-called *fundamental tone*, such that the other members of a group of lines have approximately the frequencies kf_0, where k is an integer. We say that the frequencies kf_0 for $k > 1$ are *harmonics of the fundamental tone* and we refer to them as *"overtones"*.

The frequency spectrum shows that when we play a flute, the air will not vibrate in a harmonic manner (like a pure sine). The signal is periodic, but has a different time course (shape) than a pure sinusoid. A periodic signal that is not sinusoidal (harmonic) will automatically lead to overtones in the frequency range. It is a result of pure mathematics.

The reason that it does not become a pure sinusoid is that the physical process involved in the production of the sound is complicated and turbulence is involved. There is no reason why this process should end up in a mathematically perfect harmonic audio signal. For periodic fluctuations with a time course very different from a pure sinusoid, there are many overtones. The ear will perceive the vibrations as sound different from that which has fewer harmonics.

Different instruments can be characterized by the frequency spectrum of the sound they generate. Some instruments provide fewer overtones/harmonics, while others (e.g. oboe) provide many!

The frequency spectrum can be used as a starting point also for synthesis of sound: Since we know the intensity distribution in the frequency spectrum, we can start with this distribution and make an inverse Fourier transform to generate vibrations that sound like a flute.

It must be noted, however, that our sound impression is determined not only by the frequency spectrum of a sustained audio signal, but also by how the sound starts and fades. In this context, Fourier transformation is of little help. Wavelet transformation of this type of sound discussed later in the book is much more suitable for such an analysis.

A tiny detail at the end: In Fig. 5.8, we also see a peak at a frequency near zero. It is located at 50 Hz, which is the frequency of the mains supply. This signal has somehow sneaked in with the sound of the flute, perhaps because the electronics have picked up electrical or magnetic fields somewhere in the signal path.

It is important to be able to identify peaks in a frequency spectrum that corresponds to the fundamental frequency and its harmonics, and features which do not fit into such a line-up.

5.5 Discrete Fourier Transformation

A general Fourier transformation within analytical mathematics given by Eq. (5.10) is based on a continuous function $f(t)$ and a continuous Fourier coefficient function $F(\omega)$.

In our modern age, experimental and computer-generated data are only quasi-continuous. We sample a continuous function and end up with a function described only through a finite number of data points. Both the sunspot data and the audio data we just processed were based on a finite number of data points. Assume that N data points are registered ("sampled") sequentially with a fixed time difference Δt. The total time for data sampling is T, and the sampling rate is $f_s = 1/\Delta t$. Data points have values x_n where $n = 0, \ldots, N-1$. The times corresponding to these data points are then given as:

$$t_n = \frac{T}{N} n \quad \text{for} \quad n = 0, 1, \ldots (N-1).$$

Based on the N numbers we started with, we cannot generate more than N independent numbers through a Fourier transformation. The integral of Eqs. (5.10) and (5.9) must then be replaced by summation sign and the sum extends over a finite number of data points in both the time domain and the frequency domain.

A side effect of discrete Fourier transformation is that when we Fourier transform N data points x_n taken at times $t_0, t_1, \ldots, t_{N-1}$, the result in practice is the same as if we had one periodic signal which was defined from minus to plus infinity, with period T.

We have seen in the theory of Fourier series that for periodic signals only discrete frequencies are included in the description. These are:

$$\omega_k = \frac{2\pi}{T} k \quad \text{for} \quad k = \ldots, -2, -1, 0, 1, 2, \ldots.$$

When we record the function at only N instants, as mentioned above, the data cannot encompass a frequency range with infinitely many discrete frequencies. It is only possible to operate with N frequencies, namely

$$\omega_k = \frac{2\pi}{T} k \quad \text{for} \quad k = -\frac{N-1}{2}, -\frac{N-1}{2} + 1, \ldots, -2, -1, 0, 1, 2, \ldots, \frac{N-1}{2} - 1, \frac{N-1}{2}.$$

Note that the highest frequency included is

$$f_{max} = \frac{\omega_{max}}{2\pi} = \frac{1}{2}\frac{N-1}{T} = \frac{1}{2}\frac{N-1}{N} f_s \approx \frac{f_s}{2}$$

for a sufficiently large N. Here f_s is the sampling frequency.

In the original Fourier transformation, $e^{-i\omega t}$ entered as a factor in the integrand. For N discrete data points, this is replaced by the following expressions:

$$-i\omega t \rightarrow -i\omega_k t_n = -i\frac{2\pi}{T} k \times \frac{n}{N} T = -i\frac{2\pi kn}{N}. \tag{5.17}$$

The discrete Fourier transformation is thus given by the formula:

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-i\frac{2\pi}{N}kn} \tag{5.18}$$

for $k = 0, \ldots, N-1$. If the set x_n consists of values given in the time domain, X_k will be the corresponding set of values in the frequency domain.

Note that here we indicate that k runs from 0 to $N-1$, which corresponds to frequencies from 0 to $\frac{N-1}{N} f_s \approx f_s$, while earlier we let k be between $-(N-1)/2$ and $+(N-1)/2$, corresponding to frequencies from $\approx -f_s/2$ to $\approx +f_s/2$. Since we only operate with sine and cosine functions with an integral number of wavelengths, it does not matter whether we use one set or the other. We come back to this page when we mention folding or **aliasing**.

Further, take note of the factor $1/N$ in this expression. This factor is advantageous for the variant of Fourier transformation we will use, because then we get a simple correlation between Fourier coefficients and amplitudes, as in the introductory sections of the chapter.

Through the expression in Eq. (5.17), we have shown that the expression for the discrete Fourier transform in Eq. (5.18) is based squarely on the same expression as we had in the original Fourier transformation. The difference is that in the discrete case we operate with a function described at N points and that only N frequencies are included in the description.

The inverse discrete Fourier transformation naturally looks like this:

$$x_n = \sum_{k=0}^{N-1} X_k e^{i\frac{2\pi}{N}kn} \tag{5.19}$$

for $n = 0, \ldots, N-1$.

5.5.1 Fast Fourier Transform (FFT)

Discrete Fourier transformation will be our choice when we use Fourier transformation in this book. We could have written a program ourselves to complete the procedure given in Eqs. (5.18) and (5.19), but we will not do that. It would not be a particularly effective program if we used the expressions directly. There exists nowadays a highly effective algorithm for discrete Fourier transformation that utilizes the symmetry of the sine and cosine functions in a highly effective way to reduce the number of computational operations. Efficiency has contributed greatly to the fact that Fourier transformation is widely used in many subjects, not least physics.

The algorithm was apparently discovered already in 1805 by Carl Friedrich Gauss, but fell into oblivion (it was of little interest as long as we did not have computers). The algorithm was launched in 1965 by J. W. Cooley and J. Tukey, who worked at Princeton University. Their four-page article "An algorithm for the machine calculation of complex Fourier series" in *Math. Comput*. 19 (1965) 297–301, belongs to the "classic" articles that changed physics.

In Matlab and Python, we make use of Cooley and Tukey's algorithm when we apply *FFT* ("fast Fourier transform") or *IFFT* ("inverse fast Fourier transform"). With this method, it is advantageous that the number of points N is exactly one of the numbers 2^n where n is an integer. Then we will fully utilize the symmetry of the sine and cosine functions.

5.5.2 Aliasing/Folding

When using FFT, we need to take care of a particular detail. We previously saw that it was beneficial to introduce negative frequencies in Fourier transformation.

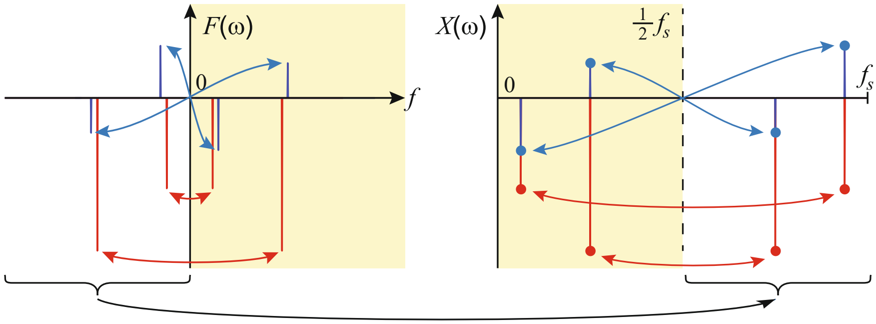

Fig. 5.9 **Left part**: A spectrum obtained by a continuous Fourier transformation of an infinite signal contains all frequencies between $-\infty$ and $+\infty$, but it is, in fact, a reflection and complex conjugation about the zero frequency (provided that the original signal was real). The real part of the Fourier transformed function is marked in red, the imaginary in blue (We have shifted the real ones relative to the imaginary points in the left part so that the sticks became distinct.). **Right part**: By discrete Fourier transformation of a signal, the information for negative frequencies (left part of the figure) is moved to the range *above* half the sampling frequency. Due to symmetries in sine and cosine functions, this also actually corresponds to signals with the frequencies $f_s - |f_{\text{negative}}|$. For this reason, FFT also receives a reflection/folding and complex conjugation in the analysis of real signals, but this time around half the sampling rate $f_s/2$. The part of the plots that have a light background colour contains all the information in the Fourier transformed signal of a real function since the other half is just the complex conjugate of the first

For a continuous Fourier transform of a real function $f(t)$, we saw that $F(\omega_0) = F^*(-\omega_0)$, that is, the Fourier transform at an angular frequency is the complex conjugate of the Fourier transform at the negative angular frequency. The same also applies to FFT. The data points after a Fourier transform with FFT are nevertheless arranged differently. The lower half of the frequency axis, which represents negative frequencies, is simply moved so that it is above (to the right of) the positive points along the frequency axis (see Fig. 5.9).

When we perform inverse Fourier transformation with IFFT, it is expected that the negative frequencies are positioned in the same way as they are after a simple FFT.

5.6 Important Concrete Details

5.6.1 Each Single Point

In Eq. (5.18), mathematically speaking, only a set of $\{x_n\}$ with N *numbers* can be transformed into a new set X_k with N *numbers* and back again. All the numbers are unlabelled.

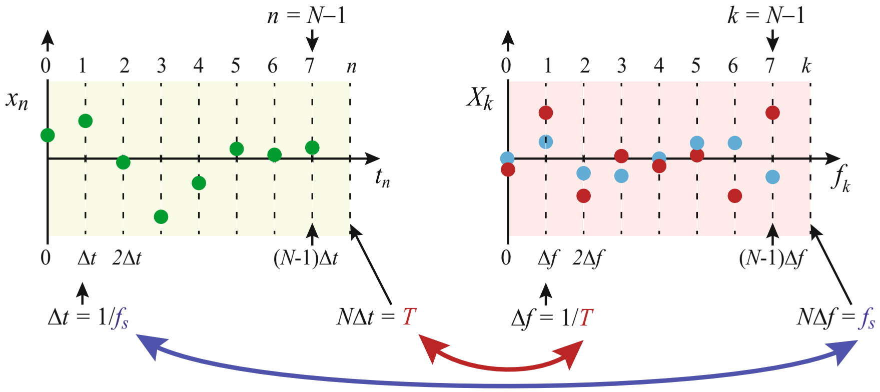

Fig. 5.10 A function sampled $N = 8$ times (*left*) along with the Fourier transform of the function (*right*) consisting of $N = 8$ complex numbers. The real values are given by red circles and the imaginary values by blue. Each point corresponds to a small time and frequency range (*left and right, respectively*). Note the relationship between the sampling rate f_s and Δt and in particular the relationship between T and Δf. In order to get a high resolution in the frequency range in the frequency range, we have to sample a signal for a sufficiently long time T

We, the users, must connect physics with the numbers. Let us explore what the indexes k, n and the number N represent.

We imagine that we make N observations of a physical quantity x_n over a limited time interval T (a single example is given in the left part of Fig. 5.10). If the observations are made at instants separated by an interval Δt, we say that *sampling rate* (or sampling frequency) is $f_s = 1/\Delta t$. The relationship between the quantities is as follows:

$$N = T f_s = T/\Delta t.$$

This is an important relationship that we should know by heart!

Note that each sampling corresponds to a very small time interval Δt. In our figure, the signal in the *beginning* of each time interval is recorded.

Fourier transformation in Eq. (5.18) gives us the frequency-domain picture (right part of Fig. 5.10). The frequency-domain picture consists of N complex numbers, and we *must* know what they represent in order to properly utilize Fourier transformation! *Here are the important details*:

- The first frequency component specifies the *mean* of all measurements (corresponding to frequency 0). The imaginary value is always zero (if f is real).

- The second frequency component indicates how much we have of a harmonic wave with a period of time T equal to the entire sampling time. The component is complex, which allows us to find amplitude and phase for this frequency component.
- Amplitudes calculated by using only the lower half of the frequency spectrum must be multiplied by 2 (due to the folding) to get the correct result. This does not apply to the first component (mean value, frequency zero).
- The next frequency components indicate contributions from harmonic waves with exactly 2, 3, 4,…periods within the total sampling time T.
- The previous points tell us that the difference in frequency from one point in a frequency spectrum to the neighbouring point is $\Delta f = 1/T$.
- Assuming that the number of samples N is even, the first component after the centre of all the components will be purely real. This is the component that corresponds to a harmonic oscillation of $N/2$ complete periods during the total sampling time T. This corresponds to a frequency equal to half of the sampling rate f_s mentioned above.
- All the remaining frequency components are complex conjugates of the lower frequency components (assuming that $f(t)$ is real). There is a "mirroring" around the point just above the middle of the numbers (mirroring about half the sampling rate). We do not get any new information from these numbers, and therefore we often drop them from the frequency spectrum.
- Since the mirroring occurs around the first point *after* the middle, the first point will not be mirrored (the point corresponding to the average value, the frequency 0).
- The last frequency in a frequency spectrum is $f_s(N-1)/N$ since the frequency ranges are half open.

Why, one may wonder, do we calculate the top $N/2 - 1$ frequency components when these correspond to "negative frequencies" in the original formalism (Eq. (5.10)). As long as f is real, these components are of little/no worth to us.

However, if f happens to be complex, as some users of Fourier transformation take it to be, these last, almost half of the components, are as significant as the others.

This is related to Euler's formula and phases. As long as we look at the real value of a phasor, it corresponds to the $\cos(\omega t + \phi)$ term, and it is identical regardless of whether ω is positive or negative. We can distinguish between positive and negative rotational speed of a phasor only if we take into account both the real and imaginary part of a complex number.

5.6.2 Sampling Theorem

As mentioned above, the top half of Fourier coefficients correspond to negative frequencies in the original formalism. However, we suggested that because of the

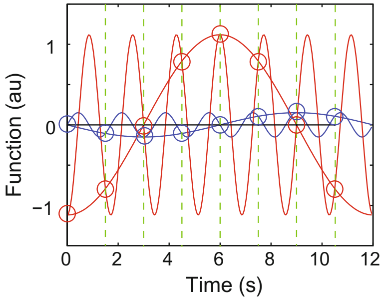

Fig. 5.11 Harmonic functions with frequencies f_k and f_{N-k} (here $k = 1$) have exactly the same value at the times for which the original function was defined (assuming that $k = 0, 1, \ldots, N-1$). Therefore, we cannot distinguish between the two for the sampling rate used. In order to distinguish functions with different frequencies, the rate must be at least twice as high as the highest frequency component. If you carefully consider the curves with the highest frequency in the figure, you will see that there are fewer than two samples per period for these

symmetry of the sine and cosine functions, it is also possible to consider these upper coefficients as coefficients of frequencies above half the sampling frequency (except that we get problems with the factor 1/2 mentioned earlier).

We can illustrate this by picking out two sets of Fourier coefficients from a Fourier transform of an arbitrary signal. We have chosen to include the relative coefficients for $k = 1$ (red curves) along with $k = N - 1$ and the imaginary coefficients for $k = 1$ and $k = N - 1$ (blue curves). The result is shown in Fig. 5.11.

The functions are drawn at "all" instants, but *the times where the original function is actually defined* is marked with vertical dotted lines. We then see that the functions of very different frequencies still have the exact same value at these times, although the values beyond these times are widely different. This is in accordance with equation

$$e^{i\frac{2\pi}{N}kn} = e^{-i\frac{2\pi}{N}k(N-n)} \tag{5.20}$$

for k and $n = 1, \ldots, N-1$ in the event that these indices generally range from 0 to $N - 1$.

The two functions $\cos(\omega_1 t)$ and $\cos[(N-1)\omega_1 t]$ are thus identical *at the discrete times* $t \in \{t_n\}$ *our description is valid* (ω_1 corresponds to one period during the time we have sampled the signal.). Similarly, for $\cos(2\omega_1 t)$ and $\cos[(N-2)\omega_1 t]$ and beyond for $\cos(3\omega_1 t)$ and $\cos[(N-3)\omega_1 t]$, etc. Then there is really no point in including the upper part of a Fourier spectrum, since all the information is actually in the lower half (Remember, this only applies when we transform a real function.).

Looking at the argument we see that at the given sampling rate, we would get exactly the same result when sampling continuous signal $\cos[(N-m)\omega_1 t]$ as if the continuous signal was $\cos(m\omega_1 t)$ (m is an integer). After the sampling, we cannot determine if the original signal was one or the other of these two possibilities—unless we have some additional information.

The additional information we need, we must supply ourselves through experimental design! We must simply ensure that there are no contributions with frequencies above half the sampling frequency of the signal we sampled. If so, we can be sure that the signal we sampled was $\cos(m\omega_1 t)$ and not $\cos[(N-m)\omega_1 t]$. This means that we must sample at least twice per period for the highest frequency that is present in the signal (see Fig. 5.11).

This is an example of a general principle:

If we want to represent a harmonic function in an unambiguous manner by a limited number of measurements, the target density (measurement frequency, sampling frequency) must be so large that we get at least two measurements within each period of the harmonic signal. The "Nyquist–Shannon Sampling Theorem" says this more succinctly:

The sampling frequency must be at least twice as high as the highest frequency component in a signal for the sampled signal to provide an unambiguous picture of the signal.

If the original signal happens to contain higher frequencies, these must be filtered by a low-pass filter before sampling to make the result unambiguous.

It is strongly recommended that you complete the second *problem* at the back of the chapter. Then you can explore how folding arises in practice, and how we can be utterly deceived if we are not sufficiently wary.

5.7 Fourier Transformation of Time-Limited Signals

It follows from Eq. (5.10) that a Fourier transform can be viewed as a sum (integral) of the product of the signal to be transformed with a pure sine or cosine:

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,dt.$$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt - \mathrm{i} \times \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt.$$

We assumed, without stating explicitly, that the signal we analysed lasted forever. Such signals do not exist in physics. It is therefore necessary to explore characteristic features of Fourier transformation when a signal lasts for a limited time.

We choose a signal that gradually becomes stronger, reaches a maximum value and then dies out again. Specifically, we choose that the amplitude change follows a so-called *Gaussian envelope*. Figure 5.12 shows two different signals (red curves), one lasting a very short time, and another that lasts considerably longer. Mathematically,

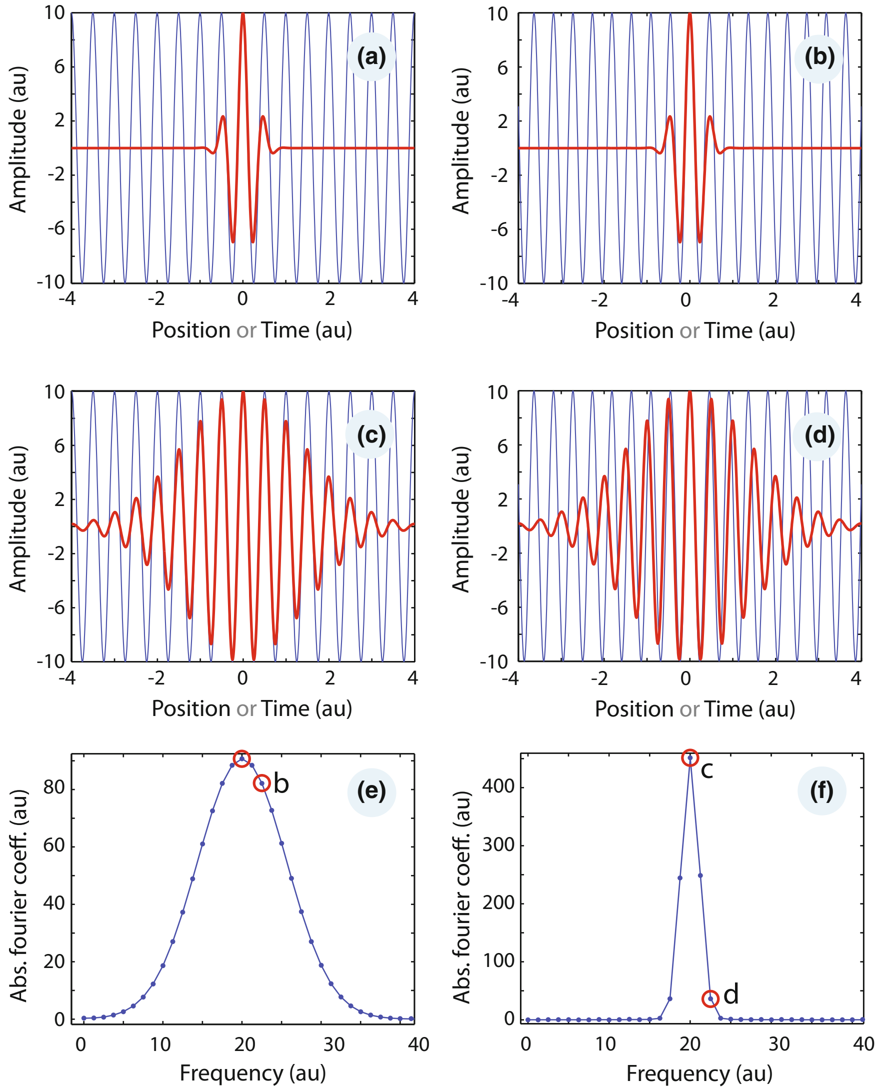

Fig. 5.12 Fourier transformation of a cosine signal multiplied with a Gaussian function. Only a small part of the total frequency range is shown. See the text for details

the signal is given as:

$$f(t) = C\cos[\omega(t - t_0)]\mathrm{e}^{-[(t-t_0)/\sigma]^2}$$

where σ gives the duration of the signal (the time after which the amplitude has decreased to $1/e$ of its maximum). ω is the angular frequency of the underlying cosine function, and t_0 is the time at which the signal has maximum amplitude (the peak of the signal occurs at time t_0).

In panels **a** and **b**, in Fig. 5.12 the signal is of short duration (small σ), but in panels **c** and **d** it lasts a little longer (σ five times as large as in **a** and **b**).

Panels **a** and **c** show, in addition to the signal pulse (in red), the cosine signal with a frequency equal to $\omega/2\pi$ (thinner blue line). In panels **b** and **d**, the cosine signal has 10% higher frequency, which explains why we will calculate X_k at two adjacent frequencies.

We see that the integral (sum) of the product between the red and blue curves in **a** and **b** will be about the same. On the other hand, we see that the corresponding integral of **d** must be significantly smaller than the integral of **c** since the signal we analyse and the cosine signal get out of phase a little bit away from the centre of the pulse in **d**. When the phases are opposite, the product becomes negative and the calculated integral (the Fourier coefficient) becomes smaller.

If we make a Fourier transform ("all frequencies") of the red curve itself in **a** (the short-duration signal) and take the absolute value of the Fourier coefficients, we get the result shown in **e**. The Fourier transform of the signal in **c** (the longer lasting signal) is displayed in the lower right corner of **f**. We can see that Fourier transformation captures the predictions we could make from visual examinations of **a**–**b**.

Note that the short-duration signal yielded a broad frequency spectrum, while the signal with several periods in the underlying cosine function gave a narrower frequency range. This is again a manifestation of the principle we have observed in the past, which has a clear resemblance to Heisenberg's uncertainty relationship. In classical physics, this is called *time-bandwidth theorem* or *time-bandwidth product*: The product of the width (duration) of a signal in the time domain and the width of the same signal in the frequency domain is a constant, whose precise value depends on the shape of the envelope of the signal.

$$\Delta t \ \Delta f \geq 1 .$$

The actual magnitude of the number on the right-hand side depends on how we define the widths Δt and Δf. We will later find in the chapter the same relationship with the number 1 replaced by 1/2, but then we use a different definition for the Δ's.

Figure 5.12 illustrates important features of Fourier analysis of a signal. More precisely, the following applies:

In a frequency analysis, we can distinguish between two signal contributions with frequencies f_1 and f_2 only if the *signals* last longer than the time $T = 1/(|f_1 - f_2|)$.

Even for signals that last a very long time, in experimental situations, we will have to *limit the observation of signal* for a time T. If we undertake an analysis of this signal, we will only be able to distinguish between frequency components that have a difference in frequency of at least $1/T$.

The difference we talk about means in both cases that there must be a difference of at least one period within the time we analyse (or the time the signal itself lasts) so that we can capture two different signal contributions in a Fourier transform. Suppose we have N_1 periods of one signal in time T and N_2 periods of the second signal. In order to be able to distinguish between the frequencies of the two signals, we must have $|N_1 - N_2| \geq 1$. [Easily derived from the relationship $T = 1/(|f_1 - f_2|)$.]

5.8 Food for Thought

The relationships in the time and frequency domains we see in Fig. 5.12 can easily lead to serious misinterpretations. In **a**, we see that the oscillation lasts only a very short time (a few periods). The rest of the time the amplitude is simply zero (or we could set it exactly to zero with no notable difference in the frequency spectrum).

What does Fourier transformation show? From the panel **e**, we can see that there are about 30 frequency components that are clearly different from zero. This means that we must have of the order of 30 different sine and cosine functions *which last all the time* (even when the signal is zero) to describe the original signal. We see this by writing the inverse Fourier transform in a way that should be familiar to us by now:

$$x_n = \sum_{k=0}^{N-1} \left[\Re(X_k)\cos(\omega_k t_n) - \Im(X_k)\sin(\omega_k t_n) \right] \tag{5.21}$$

for $n = 0, \ldots, N-1$. $\Re$ and $\Im$ stand, as before, for the real and imaginary parts, respectively.

There are some who conclude that the oscillation, when it appears to be zero, is not *really* zero but simply the sum of about 30 different sine and 30 different cosine functions throughout. This is nonsense!

It is true that we can describe the time-limited oscillation in panel **a** using all of these sine and cosine functions, but this is a pure mathematical view that has little to do with physics. Notwithstanding that, there is a good deal of physics and physical reality that goes hand in hand with the width of the frequency spectrum. However, there are other ways to make this point without invoking the presence of something physical when the amplitude is actually equal to zero. In Chap. 14, we will acquaint ourselves with the so-called wavelet transformation, and then this will become much clearer.

In my own field of research, quantum optics, we see how unfortunate this type of short circuit is. Some say that we must “use many different photons” to create a light pulse and that each photon must have the energy $E = hf$ where h is Planck’s constant and f frequency. Then a layer of physical reality is added to each Fourier coefficient, but one should focus more on what is physics and what is mathematics.

An important point here is that all time information about a signal disappears as soon as we take the absolute value of Fourier coefficients. As long as we retain complex Fourier coefficients, the time information remains intact, but is often very well hidden. The time information is scattered throughout the Fourier spectrum. Only a full inverse transformation (with complex Fourier coefficients!) from the frequency domain to the time domain can retrieve the temporal information. Fourier transformation, and in particular a frequency spectrum, has therefore limited value for signals that are zero during certain periods or completely change character otherwise during the sampling time.

Also in another context, a Fourier analysis can lead to unfortunate conclusions. Figure 5.13 shows the Fourier transform of a periodic motion. In essence, this figure resembles Fig. 5.8, which shows the frequency spectrum of sound from a transverse flute, with fundamental tone and harmonics. On that occasion, we said that the reason we get overtones is that the signal, though periodic, is not a pure sinusoid.

Some persons speak of the higher harmonics in another way. For example, they say “when we play a flute, air vibrates not only at one particular frequency, but at multiple frequencies simultaneously”. Though common, such phraseology is problematic.

If we say that “several frequencies are present simultaneously” in the motion that lies at the back of the Fourier spectrum in Fig. 5.13, the statement accords poorly with the underlying physics! The figure was made this way: we first calculated a planet’s path around the sun. The path was described by a set of coordinate as a function of

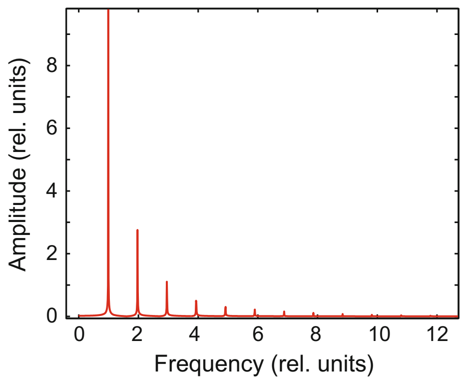

Fig. 5.13 Fourier transformation of a periodic motion. See the text for explanation

time $[x_i(t), y_i(t)]$. Figure 5.13 is simply the Fourier transform of $\{x_i(t)\}$ for a time that is much longer than the solar orbital period of the planet under consideration.

The reason we get a series of "harmonics" in this case is that planetary motion is periodic but not a pure sinusoid. We know that Fourier transformation is based on harmonic basis functions, and these correspond to circular motion. But if we think in terms of "several frequencies existing at the same time", it is tantamount to saying that the movement of the planet must be described with multiple circular movements occurring at the same time! In that case, we are back to the Middle Ages!

Some vicious tongues say that if computers, equipped with an arsenal of Fourier transform tools, had been around in Kepler's time, we would still have been working with the medieval *epicycles* (see Fig. 5.16). From our Fourier analysis in Fig. 5.13, we see that we can replace the ellipse with a series of circles with appropriate amplitudes (and phases). However, most people would agree that it makes more sense to use a description of planetary motion based on ellipses and not circles. I wish we were equally open to dropping mathematical formalism based on Fourier analysis also in some other contexts.

Fourier analysis can be performed for virtually all physical time variables, since the sine and cosine functions included in the analysis form a complete set of basis functions. Make sure you that you do *not* draw the conclusion that "when something is feasible, it is also beneficial". In the chapter on wavelet transformation, we will come back to this issue, since in wavelet analysis we can choose a set of basis functions totally different from everlasting sines and cosines. We can sum up in the following words:

> *Fourier transformation is a very good tool, but it has more or less the same basis as the medieval description of planetary movements. It is perfectly possible to describe planetary paths in terms of epicycles, but such an approach is not particularly fruitful. Similarly, a number of physical phenomena are described today by Fourier analysis where this formalism is not very suitable. It can lead to physical pictures that mislead more than they help us. Examples may be found in fields which include quantum optics.*

5.9 Programming Hints

5.9.1 Indices; Differences Between Matlab and Python

Strings such as $\{x_n\}$ and $\{X_k\}$ are described as arrays in the parlance of numerical analysis. It is important to remember that in Python, the indexes start with 0, while in Matlab they start with 1. In $\{X_k\}$, $k = 0$ and 1 correspond, respectively, to the frequency 0 (constant) and the frequency $1/T$. In Matlab, their counterparts are indices 1 and 2.

The expression for a discreet Fourier transform in Python will then be as follows:

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}kn} \tag{5.22}$$

for $k = 0, \ldots, N-1$.

On the other hand, the expression for a discreet Fourier transform in Matlab takes the following form:

$$X_k = \frac{1}{N} \sum_{n=1}^{N} x_n \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}(k-1)(n-1)} \tag{5.23}$$

for $k = 1, \ldots, N$.

For the inverse discrete Fourier transformation, similar remarks apply.

5.9.2 Fourier Transformation; Example of a Computer Program

```
% A simple example program which aim is to show how Fourier
% transform may be implemented in practice i Matlab. The
% example is a modification of an example program at a
% tutorial page at Matlab.

Fs = 1000;            % Sampling frequency
delta_t = 1/Fs;       % Time between each sampling
N = 1024;             % Number of samples
t = (0:N-1)*delta_t;  % Time description

% Create an artificial signal as a sum of a 50 Hz sine and a
% 120 Hz cosine signal, plus a random signal:
x = 0.7*sin(2*pi*50*t) + cos(2*pi*120*t);
x = x + 1.2*randn(size(t));

plot(Fs*t,x)         % Plot the time domain representation
title('The signal in time domain')
xlabel('time (millisec)')

X = fft(x,N)/N;      % Fast Fourier Transformation

freqv = (Fs/2)*linspace(0,1,N/2);  % The frequency range

% Plot the absolute value of the frequency components in the
```

```
% frequency domain representation. Plot only frequencies up to
% half the sampling frequency (drop the folded part).
figure;              % Avoids overwriting the previous plot
plot(freqv,2*abs(X(1:N/2))) % Plots half the frequency spectrum
title('Absolute value of the frequency domain representation')
xlabel('Frequency (Hz)')
ylabel('|X(freq)|')
```

5.10 Appendix: A Useful Point of View

There are big differences between how we physicists use and read the contents of mathematical expressions. In this appendix, I would like to give an example of a way of thinking that has been useful to me whenever I have wondered why some Fourier spectra look as they do.

We start with the mathematical expression shown below:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{5.24}$$

or the discrete variant of the same expression:

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-i\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{n=0}^{N-1} x_n \cos(\omega_k t_n) - i \times \frac{1}{N} \sum_{n=0}^{N-1} x_n \sin(\omega_k t_n) \tag{5.25}$$

where $\omega_k = \frac{2\pi}{T}k$, $t_n = \frac{T}{N}n$, and T is the total sampling time. Then we simply have a sum of single products $x_n \cos(\omega_k t_n)$ (or sines) with many n. The integral or the sum we get by adding a lot of such numbers (with a scaling that we need not discuss here).

If now $\{x_n\}$ is simply a cosine function with the same frequency and phase as $\cos(\omega_k t_n)$, the products of these two terms will always be equal to or greater than zero, being a $\cos^2$ function. Then the sum will be big and positive.

If $\{x_n\}$ is a cosine function with a frequency different from that of $\cos(\omega_k t_n)$, the two cosine functions will sometimes be in phase, yielding a positive product, but at other times with an opposite phase, resulting in a negative product.

Due to the factor $\frac{1}{N}$, the sum of all product terms will be close to zero if we get many periods of positive and negative contributions in all.

Based on this argument, we find that the Fourier transform of a single harmonic function when we have integration with limits plus minus infinity is considered a δ-function. But what will happen when the test function is simply zero everywhere except for a limited length of time T where the function is a simple harmonic function?

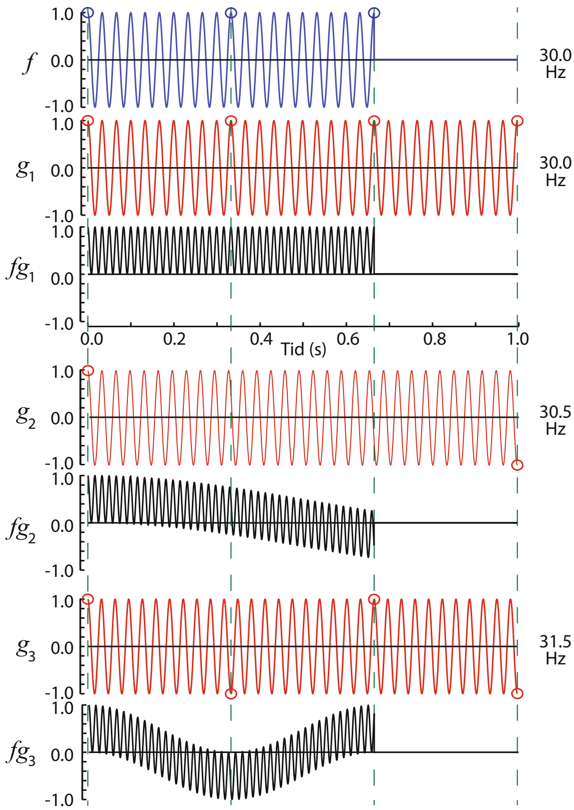

Fig. 5.14 Two functions included in the integral in Fourier integral Eq. (5.24). The f function to be analysed is shown in blue. It is different from zero only for a limited period of time. $g_k(t)$, which corresponds to $\Re\{e^{-i\omega_k t}\}$, is shown in red, and the product of the two functions in black. Three different frequencies ω_k are selected. See text for comments

Figure 5.14 shows a section of the function:

$$f(t) = \cos(\omega_a t) \qquad \text{for } t \in [0, T] \quad \text{and 0 otherwise.} \tag{5.26}$$

In the figure $T = 2/3$ s.

The time interval T in the figure is just sufficient to cover the entire window where $f(t)$ differs from zero. Also shown are plots $g(t) = \Re\{e^{-i\omega_k t}\} = \cos(\omega_k t)$ for three different choices of the analysing frequency ω_k, and the corresponding plots of the product functions $f(t)g(t)$.

The integral of the product function now receives contributions only in the time interval where f is different from zero. We get full contribution from the entire range when $\omega_k = \omega_a$. We see that the integral (sum of all values of product function) also becomes positive in the middle case where the difference between ω_k and ω_a is so small *in relation to the length of time interval* that the phase of f and the phase of $\cos(\omega_k)$ is always less than π.

In the bottom case, we have chosen an analysing frequency of ω_k which is such that

$$(\omega_k - \omega_a)T = 2\pi.$$

Because of the symmetry, we see that the integral here vanishes, but we realize that we would get a certain positive or negative value if we had chosen the frequency difference (in relation to T) as we did in this case.

What has this example shown us? In the first part of the chapter, we explained that when $f(t) = \cos(\omega_a t)$ for all t, the Fourier integral will be null in absolutely all cases where $\omega_k \neq \pm\omega_a$. In Fig. 5.14, we see that when the function we analyse lasts for a limited time T, the two frequencies may be slightly different and yet we may receive contributions to the Fourier integral. The contribution will be greatest when $(\omega_k - \omega_a)T < \pi$.

It should be noted that we can rename the quantities as follows: $(\omega_k - \omega_a) \equiv 2\pi\Delta f$ and $T \equiv \Delta t$. In that case, we get that the Fourier integral will have an appreciable value so long as

$$\Delta f \ \Delta t < 1/2.$$

This is again a relation analogous to the Heisenberg uncertainty relation.

We can repeat the same type of calculations of the $f\,g$–product function for many different ω_k relative to ω_a and add up positive and negative contributions over T, the interval we wish to integrate over. Examples of such calculations are shown in Fig. 5.15. When the two frequencies are identical, the area below the middle curve becomes the maximum, which corresponds to the peak value in the (real part) of the Fourier spectrum. The area may be positive or negative depending on whether the mean value of the $f\,g$–product function is above or below zero. Time intervals where the $f\,g$–product is positive is marked with blue background colour and intervals with negative product with red. The integral is just the sum of positive and negative areas in these plots. In case of **3** and **5**, the total area is equal to zero (as much positive as negative), while in case of **4** the total area is negative.

Deeper red or blue colour is used to mark the areas that are not balanced by corresponding area with the opposite sign. We see then that the deepest red-marked area in case **4** is greater in absolute value than the deepest blue area in case **6**, reflecting that the peak in the area near the **6** mark in the lower part of the figure is less than the (absolute) value of the peak in the area near the **4** mark.

Figure 5.15 indicates that the frequency spectrum of a portion of a harmonic function has a broad and sharp peak in the middle, and characteristic oscillations with smaller and smaller amplitude the farther away from the peak one moves.

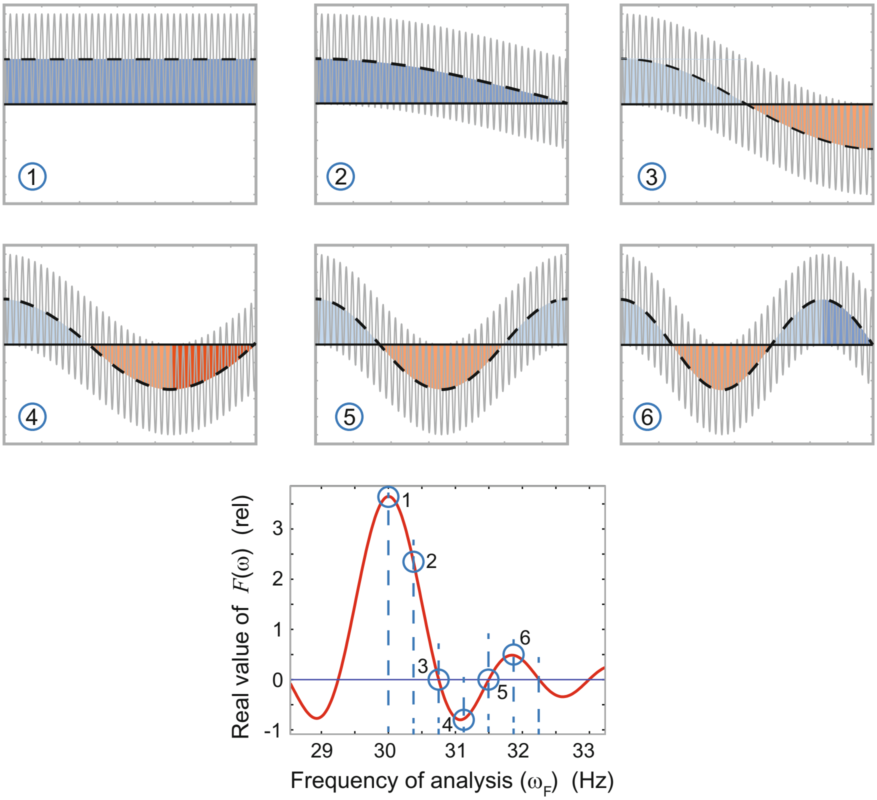

Fig. 5.15 Integrand (the $f\,g$–product) in the real part of Fourier calculations for different choices of the analysing frequency. The real part of a section of the Fourier spectrum of the function appearing in Eq. (5.26) is given at the bottom. See also the text for details

Remarks: In Chap. 13, we will see that the frequency spectrum in Fig. 5.15 appears again when we consider the diffraction image that emerges when we transmit a laser beam with visible light through a narrow gap. Within optics, there is a separate field called Fourier optics.

5.10.1 Program for Visualizing the Average of Sin–Cos Products

```
function sincosdemo
% Program to visualize the average of sine/cosine product

N = 2000;          % Number points in the description
T = 1.0;           % The time we describe the signal (1 sec)
t = linspace(0,T*(N-1)/N,N);  % Make the timeline t
```

```
freq1 = 100.0;  % One frequency is kept constant
freq2 = 100.0;  % 1) Try to vary this from 102 to e.g. 270
                % 2) Try also values where |freq2-freq1|< 2.0
omega1 = 2*pi*freq1; % Calculate angular frequencies
omega2 = 2*pi*freq2;
f = cos(omega1*t);  % Try also sin( )
g = cos(omega2*t);

plot(t,f.*g,'-b');  % Plot the product of f and g
xlabel('Time (s)');
ylabel('Signal (rel.units)');
null = zeros(N,1);
hold on;
plot(t,null,'-r');  % Draw also the zero line

integral = sum(f.*g);  % Drop the normalization 1/N
integral     % Write the "integral" (sum) to screen.
```

5.10.2 Program Snippets for Use in the Problems

Snippet 1: Here is a piece of code that shows how to read data from an audio file in Matlab:

```
    s = 'piccoloHigh.wav';  % File name (the file must be in
                            % the same folder as your program)
    N = 2^16;
    nstart = 1;   % First element number you want to use in
                  % your audio file
    nend = N;     % last element number you want to use
    [f,Fs] = audioread(s, [nstart nend]);
%    sound(f,Fs); % Play back the sound if you want
                  % (then remove %) for control purposes
    g = f(:,1);   % Pick a mono signal out of the stereo
                  % signal f
    X = (1.0/N)*fft(g);  % FastFourierTransform of the
                  % audio signal
    Xa = abs(X);  % Calculate the absolute value out of the
                  % freqency domain representation
```

Snippet 2: Recording a sound from the PC's microphone.

```
T = 2.0;     % Duration of sound track in seconds
Fs = 11025;  % Chosen sampling frequency (must be
             % supported by the system)
N = Fs*T;
t = linspace(0,T*(N-1)/N,N);  % For x-axix in plot
recObj = audiorecorder(Fs, 24, 1);
deadtime = 0.13;  % Delay. Trick due to Windows-problems
recordblocking(recObj, T+3*deadtime);
myRecording = getaudiodata(recObj);
stop(recObj);
Nstart = floor(Fs*deadtime);
Nend = Nstart + N -1;
y = myRecording(Nstart:Nend,1);
s = sum(y)/N;    % Removes the mean value
y = y-s;
plot(t,y,'-k');
title('Time domain representation');
xlabel('Time (sec)');
ylabel('Microphone signal (rel units)');
```

N.B. The code for sampling the sound does not work perfectly and sometimes leads to irreproducible results. This is because the sound card is also under control of Windows (or other operating system), and the result depends on other processes in the computer. Those who are particularly interested are referred to specially developed solutions via "PortAudio" (www.portaudio.com).

Snippet 3: One possible method to make an animation.

```
function waveanimation1
clear all;
k = 3;
omega = 8;
N = 1000;
x = linspace(0,20,N);
y = linspace(0,20,N);
p = plot(x,y,'-','EraseMode','xor');
axis([0 20 -2.5 2.5])
for i=1:200
    t = i*0.01;
    y = 2.0*sin(k*x-omega*t);
    set(p,'XData',x,'YData',y)
    drawnow
    pause(0.02);  % This is to slow down the animation
end
```

Pieces of code will be transferred from the *Problems* in several chapters to the "Snippet subsection" at the "Supplementary material" web page for this book available at http://www.physics.uio.no/pow.

5.11 Learning Objectives

After working through this chapter, you should know that:

- An integrable time-dependent continuous function can be transformed by continuous Fourier transformation into a "frequency-domain picture", which can then be uniquely transformed with an inverse Fourier transformation back to the starting point.
- A discrete function can be transformed by a discrete Fourier transform into a "frequency-domain picture", which can then be uniquely transformed with a discrete inverse Fourier transform back to the starting point.
- Only integers are included in a mathematical/numerical implementation of a Fourier transformation. We must manually keep track of the sampling times and the frequencies of the elements in the Fourier spectrum. We must also take account of normalization of the numerical values (e.g. whether or not we should divide/multiply the numbers after transformation by N), as different systems handle this differently.
- The frequency-domain picture in a discrete Fourier transformation consists of complex numbers, where the real part represents cosine contributions at the different frequencies, while the imaginary part represents the sine contributions. The absolute value of the complex numbers gives the amplitude of the contribution at the relevant frequency. The arctan of the ratio between imaginary and real parts indicates the phase of the frequency component (relative to a $\cos(\omega t + \phi)$ description).
- For a real signal, the last half of the Fourier coefficients are complex conjugate of the first half, and "mirroring" occurs. Therefore, we usually use only the first half of the frequency spectrum.
- In a discrete Fourier transform, the first element in the data string X_k corresponds to a constant (zero frequency), second element to the frequency $1/T$, third to frequency $2/T$, etc. Here T is the total time function/signal we start with is described above (total sampling time). It is necessary to sample for a long time if we are to get a high resolution in the frequency picture.
- If a signal is "sampled" with a sampling frequency f_s, we will only be able to process signals with frequencies below half the sampling frequency in an unambiguous manner.
- In order to avoid "folding" problems, a low-pass filter must be used to remove signal components that may have a frequency higher than half the sampling frequency. For numerical calculations, we have to make sure that the "sampling rate" is high enough for the signal we are processing.

- Fourier transformation is a great aid in studying stationary time-varying phenomena in much of physics. For example, Fourier transformation is extensively used in analysis and synthesis of sound.
- It is possible to implement Fourier transformation of (almost) any signal, but it does not mean that Fourier transformation is useful in every situation!
- Fourier transformation is (almost) suitable only for analysing signals that have more or less the same character throughout the sampling time. For transient signals that change character greatly during sampling time, a Fourier spectrum sometimes may be more misleading than useful.
- Normally when Fourier transformation is performed numerically, we use ready-made functions within the programming package we use. If we create the code ourselves, the calculations take an unduly long time (unless we code the actual "fast Fourier transform" algorithm). Calculations are most effective if the number of points in the description is 2^n.

5.12 Exercises

Suggested concepts for student active learning activities: Periodic/nonperiodic function, Fourier transformation, time domain, frequency domain, frequency analysis, fundamental frequency, harmonic frequencies, sampling, sampling frequency, folding, aliasing, sampling theorem, time-bandwidth product, classical analogue to Heisenberg's uncertainty relationship, high-pass/low-pass filters, stationary signal.

Comprehension/discussion questions

1. In a historical remark first in this chapter, we claimed the Fourier transformation and Fourier analysis bear close resemblance to the medieval use of epicycles for calculating how planets and the sun moved relative to each other (see Fig. 5.16). Discuss this claim and how Fourier analysis may lead to unwanted conclusions if it is used in an uncritical manner.
2. How can we make a synthetic sound by starting from a frequency spectrum? Would such sound simulate in a good way the output of a proper instrument?
3. For CD sound, the sampling rate is 44.1 kHz. In the case of sound recording, we must have a low-pass filter between the microphone amplifiers and sampling circuits that remove all frequencies above 22 kHz. What could happen to the sound during playback if we did not take this rule seriously?
4. After a fast Fourier transform (FFT), we often plot only a part of all the data produced. Mention examples of what may influence our choices.
5. Suppose that you Fourier analyse sound from a CD recording of an instrument and find that the fundamental tone has a frequency of 440 Hz. Where do you find the folded frequency?

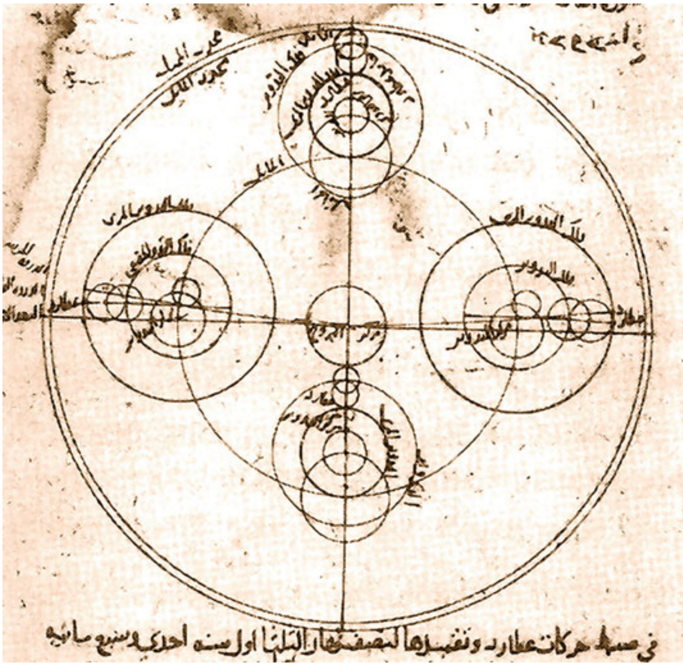

Fig. 5.16 A drawing of epicycles in an old Arabic document written by Ibn_al-Shatir [1], Public Domain

6. What are the resemblances between Fourier series and a discrete Fourier transform? Discuss the difference between periodic and nonperiodic signals.
7. Describe in your own words *why* the Fourier transform of a cosine function that lasts for a limited time T is different than if the cosine function had lasted from minus to plus infinity.
8. Consider Fig. 5.17 and tell us what it means to you.

Problems

Remember: A "Supplementary material" web page for this book is available at http://www.physics.uio.no/pow.

9. Show both mathematically and in a separate programming example that the first point in a digital Fourier transform of a signal is equal to the average value of the signal we started with.
10. Use the computer program provided in "computer software example" on page xxx to explore how folding works in practice. Let the signal be:

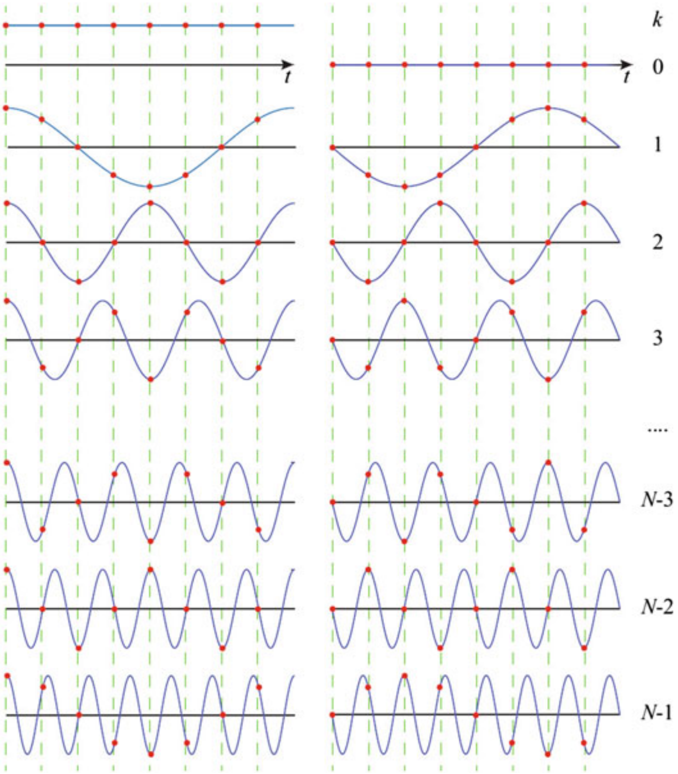

Fig. 5.17 Use this figure in Problem 8

```
freq = 100.0;  % Frequency in hertz
x = 0.8 * cos(2*pi*freq*t);  % Signal is a simple cosine
```

and run the program. Be sure to zoom in so much that you can check that the frequency in the frequency spectrum is correct.

Then run the program, setting the frequency (one by one) equal to 200, 400, 700, 950, 1300 (Hz). Do you find a pattern in where the peaks come out in the frequency spectrum?

11. Some people claim that the moon phases influence everything from the weather to the mood of us humans. Check if you can find indications that the temperature (maximum and/or minimum daily temperature) varies slightly with the moon phases (in addition to all other factors)?
The data for the place (and period) you are interested in can be downloaded from api.met.no. Alternatively, you can use an already downloaded and slightly simplified file *tempblindern10aar.txt* from the web pages providing supplementary material for our book. The file gives the temperature of Blindern, Oslo, Norway in the period 1 January 2003 through 31 December 2012. The fourth column in the file provides minimum temperatures, while the fifth column provides the

maximum values.
Explain carefully how you can draw a conclusion as to whether or not the moon phase affects the temperature.
Below you will find some lines of Matlab code that shows how data may be read from our file into a Matlab program (the data file has five columns, and we use the last two of them):

```
filename = 'tempBlindern10years.txt';
fileID = fopen(filename, 'r');
A = fscanf(fileID, '%d %d %f %f %f',[5,inf]);
minT = A(4,:);
maxT = A(5,:);
plot(minT,'-r');
hold on;
plot(maxT,'-b');
```

12. Collect sunspot data from the Web and create an updated figure similar to our Fig. 5.7. Pay particular attention to getting correct values along the axes of the frequency-domain representation. Is there a correspondence between the peak heights in the time-domain picture and the amplitude of the frequency spectrum? Below are some lines of Matlab code showing how data can be read into a Matlab program (two columns):

```
filename='sundata.txt';
fileID = fopen(filename, 'r');
A = fscanf(fileID,'%f %f', [2,inf]);
plot(A(1,:),A(2,:),'-b');
```

13. Pick up a short audio signal from a CD, a wav file or record sound from a microphone (use, for example, one of the program snippets a few pages before this one). The sampling rate is 44.1 kHz. Save $2^{14} = 16384$ data points (pairs of points if it is a stereo signal, but use only one of the channels). Perform a "fast Fourier transformation" and end up with 16384 new data points representing the frequency spectrum. How do you change your program from point number to frequency in Hz along the x-axis when the frequency spectrum is to be plotted?
14. What is the resolution along the x-axis of the plot in the previous task? In other words, how much change in frequency do we get by moving from one point in the frequency range to the next? Would the resolution be the same even if we had used only 1024 points as the starting point for Fourier transformation?
15. Write a program in Python or Matlab (or any other programming language) that creates a harmonic signal with exactly 13 periods within 512 points. Use the built-in FFT function to calculate a frequency range. Will this be what you expected? Feel free to let the signal be a pure sinusoidal signal or a sum of sine and cosine signals.
16. Modify the program so that the signal now has 13.2 periods within the 512 points. How does the frequency spectrum look now? Describe as well as you can!

17. Modify the program to get 16 full periods of *FIRKANTsignal* within $2^{14} = 16384$ points. How does the frequency spectrum look now? Find on the Internet how the amplitude of different frequency components should be for a square signal and verify that you get nearly the same output from your numerical calculations.
18. Modify the program so that you get 16 full *sagtenner* (triangular signal) within the 1024 points. Also describe this frequency spectrum!
19. In an example in Chap. 4, we calculated the angular amplitude of a physical pendulum executing large displacements. Perform these calculations for 3–4 different angular amplitudes and carry out a Fourier analysis of the motion in each case. Comment on the results.
20. AM radio (AM: Amplitude Modulated). Calculate how the signal sent from an AM transmitter looks like and find the frequency spectrum of the signal. It is easiest to do this for a radio signal on the long wave band (153–297 kHz). Let the carrier have the frequency $f_b = 200$ kHz and choose the speech signal to be a simple sine with frequency (in turn) $f_t = 440$ Hz and 4400 Hz. The signal should be sampled at a sampling rate of $f_s = 3.2$ MHz, and it may be appropriate to use $N = 2^{16} = 65536$ points. The AM signal is given by:

$$f(t) = (1 + A \sin(2\pi f_s t)) \times \sin(2\pi f_b t)$$

where A is the normalized amplitude of the audio signal (the loudest sound that can be sent without distortion is $A = 1.0$). Use a slightly smaller value, but please test how the signal is affected by A).

Plot the AM signal in both the time domain and the frequency domain. Select appropriate segments from the full data set to focus on what you want to display. Remember to set correct timing along the x-axis of the time-domain plot and correct frequency scale along the x-axis of the frequency spectrum.

Each radio station on the medium and long wave may extend over only a 9 kHz frequency band. What are the consequences for the quality of the sound being transmitted?
21. FM radio (FM: Frequency Modulated). Calculate how the signal sent from an FM transmitter looks like and find the frequency spectrum of the signal. Use the same parameters as in the previous task (although in practice, no long wave FM is used). The FM signal can be given as follows:

```
f(t) = sin(phase(t)); % Principally (!)
```

where the phase is integrated by means of the following loop:

```
phase(1) = 0.0;
for i=1:(N-1)
   phase(i+1)=phase(i) + \cdots
   omega_b*delta_t*(1.0 + A*sin(omega_t*t(i)));
end;
```

where “omega_b” and “omega_t” are the angular frequencies of the carrier and the speech signal, respectively. The time string “t(i)” is assumed to be calculated in advance (distance between the points is “delta_t”, which is determined by the sampling frequency).

A is again a standard amplitude for the audio signal, which also includes the so-called degree of modulation. You can choose, for example, $A = 0.2$ and 0.7 (in turn), and see how this affects both the time-domain picture and the frequency-domain picture.

Plot the FM signal in both the time domain and the frequency domain according to the same guidelines as in the previous task (Hint: It may be easiest to plot the case where the voice frequency is 4400 Hz and that $A = 0.7$.).

Are there any clear differences in how the frequency-domain picture appears for FM signals compared to AM signals?

22. Use inverse Fourier transformation to generate a simple sinusoid and play the sound on your computer. Use the inbuilt *sound* or *wavplay* function (program snippet 1 a few pages ahead of this on indicates how). Specifically, the following is recommended: Use the CD sampling rate $f_s = 44100$ Hz and $2^{16} = 65536$ points. The values of the signal f must not exceed the interval $[-1, +1]$. Attempt to make sound with frequencies 100, 440, 1000 and 3000 Hz. You may want to make a signal consisting of several simultaneous sinusoids too? Remember to scale the total signal before using *wavplay* or *sound*.

23. Read the audio file “transient.wav” and perform Fourier transformation to obtain the frequency spectrum. The audio file is available from the web pages providing supplementary material for our book, the sampling rate is $f_s = 44100$ Hz. Use 2^{17} points in the analysis. You may use program snippet 1 a few pages ahead of this one for reading the file.

    If you listen to the sound and then consider the frequency-domain picture, I hope that you would pause and reflect on what you have done. Fourier analysis is sometimes misused. What is the problem with the analysis performed on the current audio signal?

24. **(a)** Perform a frequency analysis of sound from a tuba and from a piccolo flute (audio files available from the course’s web pages). The sampling frequency is 44100 Hz. Use, e.g., 2^{16} points in the analysis. Plot the absolute value of the frequency spectrum (see program below). Determine the pitch of the tone on a tempered scale using the Fig. 5.18. Remember to get correct values along the frequency axis when plotting the frequency spectrum and zoom in to get a fairly accurate reading of the fundamental tone frequency.

    (b) The frequency spectrum shows varying degrees of harmonics as described in this chapter (we will return to this in later chapters). Zoom into the time signal so much that you get a few periods. Does the signal look like a harmonic signal, or is it far more irregular than a sinus? (Comparison must be done only when considering 3–8 periods in the audio signal.) Does there appear to be some kind of connection between how close the time signal is to a pure sinusoid and the number of harmonics in the frequency spectrum?

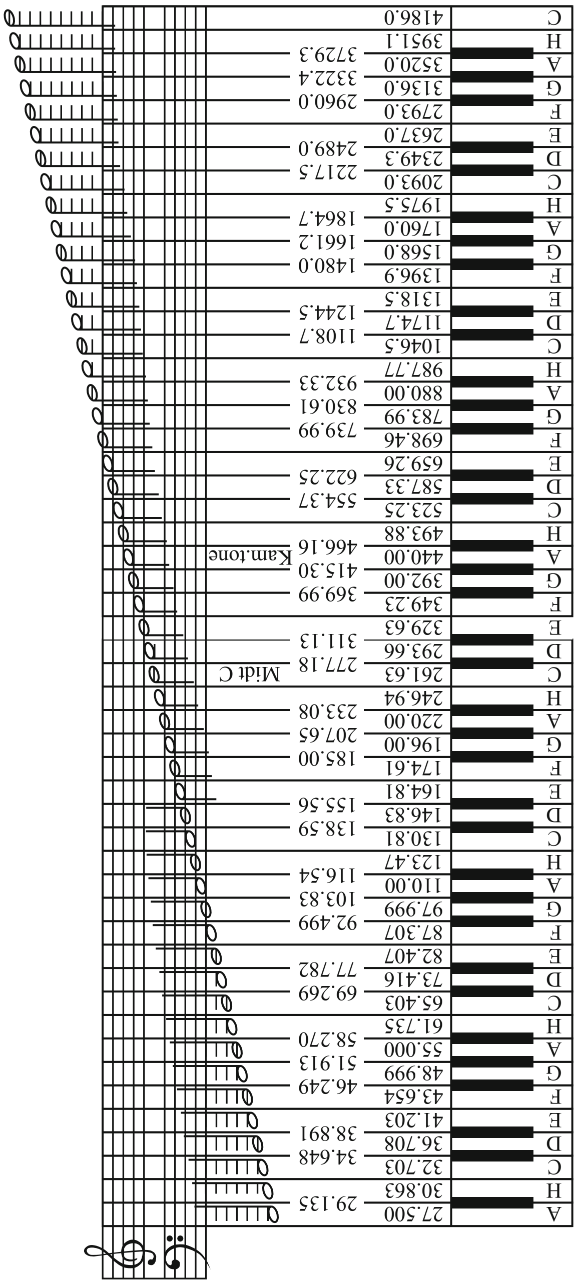

Fig. 5.18 Tone scale for a tempered scale as we find it on a piano. Frequencies for the tones are given. The figure is inspired from [2], but completely redrawn

(c) Attempt to include data only for such a small time interval that there is only room for one period in the signal. Carry out the Fourier transform of this small portion of the signal (need not have 2^n data points). Do you find a connection between the Fourier spectrum here compared to the Fourier spectrum when you used a long time string containing many periods in the audio signal?
(d) For one of the audio files, you are asked to test that an inverse Fourier transform of the Fourier transform brings us back to the original signal. Remember that we must keep the Fourier transform as complex numbers when the inverse transform is carried out. Plot the results.
(e) Perform an inverse Fourier transform on the *absolute value* of the Fourier transform signal. Describe the difference between the inverse of the complex Fourier transform and the one you found now. Try to give the reason for the difference.

25. "Open task" (i.e. very few guidelines and hints are given): Fourier transformation can be used in digital filtering. Explain the principle and how this can be done in practice. Create a small program that performs self-selected digital filtering of a real audio file, where it is possible to listen to the sound both before and after filtering (Be scrupulous in describing the details of what you do!).

References

1. Ibn_al-Shatir. https://en.wikipedia.org/wiki/Ibn_al-Shatir, Accessed April 2018
2. Unknown. http://amath.colorado.edu/outreach/demos/music/MathMusicSlides.pdf. Accessed Feb 18 2012