Chapter 2
Free and Damped Oscillations

Abstract This chapter introduces several equivalent mathematical expressions for the oscillation of a physical system and shows how one expression can be transformed into another. The expressions involve the following concepts: amplitude, frequency and phase. The motion of a mass attached to one end of a spring is described by Newton's laws. The resulting second-order homogeneous differential equation has three solutions, depending on the extent of energy loss (damping). The difference between a general and a particular solution is discussed, as well as superposition of solutions for linear and nonlinear equations. Oscillation in an electrical RCL circuit is discussed, and energy conservation in an oscillating system which has no energy dissipation is examined.

2.1 Introductory Remarks

Oscillations and vibrations are a more central part of physics than many people realize. The regular movement of a pendulum is the best-known example of this kind of motion. However, oscillations also permeate all wave phenomena. Our vision, our hearing, even nerve conduction in the body are closely related to oscillations, not to mention almost all communication via technological aids. In this chapter, we will look at the simplest mathematical descriptions of oscillations. Their simplicity should not tempt you into underestimating them. Small details, even if they appear to be insignificant, are important for understanding the more complex phenomena we will encounter later in the book.

2.2 Kinematics

In mechanics, we distinguish between kinematics and dynamics, and the distinction remains relevant when we consider oscillations. Within kinematics, the focus is primarily on *describing* motion. The description is usually the *solution* of differential

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equations or experimental measurements. The underlying physical laws are not taken into consideration.

In dynamics, on the other hand, we set up the differential equations of motion based on known physical laws. The equations are solved either by analytical or numerical methods, and we study how the solutions depend on the physical models we started with. If we seek physical understanding, dynamic considerations are of greater interest, but the kinematics can also be useful for acquiring familiarity with the relevant mathematical description and the quantities that are included.

How do we describe an oscillation? Let us take an example: A mass attached to one end of a spring oscillates vertically up and down. The top of the spring is affixed to a stationary point.

The kinematic description may go like this: The mass oscillates uniformly about an equilibrium point with a definite frequency. The maximum displacement A relative to the equilibrium point is called the *amplitude* of oscillation. The time taken by the mass to complete one oscillation is called *time period* T. The *oscillation frequency* f is the inverse of the time period, i.e. $f \equiv 1/T$, and is measured in reciprocal seconds or hertz (Hz).

Suppose we use a suitably chosen mass and a limited amplitude of displacement for the spring. By that we mean that the amplitude is such that the spring is always stretched, and never so much as to suffer deformation. We will be able to observe that the position of the mass in the vertical direction $z(t)$ will almost follow a mathematical sine/cosine function:

$$z(t) = A\cos(2\pi t/T) \ .$$

However, such a description is not complete. There is no absolute position or absolute time in physics. Therefore, when we specify a position z (along a line), we must also specify the point with respect to which the measurement is made. In our case, this reference point is the position of the mass when it is at rest.

Similarly, we must specify the reference point relative to which the progress of time is measured. In our case, the origin of time is chosen so that the position has a maximum value at the reference time $t = 0$. If there is a mismatch, we must compensate by introducing an *initial phase* ϕ, and use the expression

$$z(t) = A\cos(2\pi t/T + \phi) \ .$$

Since the quantity $2\pi/T$ occurs in many descriptions of oscillatory movements, it proves advantageous to define an *angular frequency* of ω as follows:

$$\omega \equiv 2\pi/T = 2\pi f$$

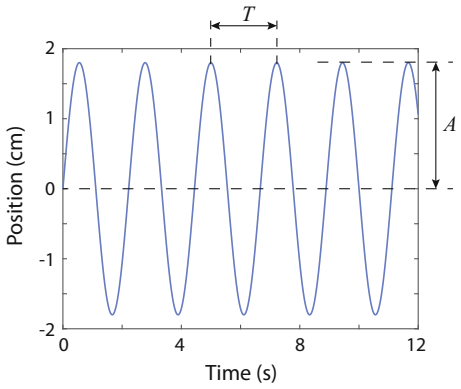

Fig. 2.1 A harmonic oscillation is characterized by amplitude, frequency and phase; see the text

where f is the frequency of oscillation. This is a fairly common way to describe an oscillation (Fig. 2.1).
However, a "simple harmonic oscillation" can be described in many ways. The most common mathematically equivalent ways are:

$$z(t) = A\cos\omega t + B\sin\omega t \tag{2.1}$$
$$= C\cos(\omega t + \phi) \tag{2.2}$$
$$= \Re\left\{\mathscr{D}\mathrm{e}^{\mathrm{i}\omega t}\right\} \tag{2.3}$$
$$= \Re\left\{E\mathrm{e}^{\mathrm{i}(\omega t+\phi)}\right\} \tag{2.4}$$

$\Re\{\}$ indicates that we take the the real part of the complex expression within the braces, and $\mathscr{D}$ is a complex number.

Euler's formula for the exponential function (complex form) has been used in the last two expressions. According to Euler's formula:

$$\mathrm{e}^{\mathrm{i}\alpha} = \cos\alpha + \mathrm{i}\sin\alpha\ .$$

This formula forms the basis for a graphical representation of a harmonic motion: First, imagine that we draw a vector of unit length in a plane. The starting point of the vector is placed at the origin and the vector forms an angle α with the x-axis. The vector can then be written as follows:

$$\hat{x}\cos\alpha + \hat{y}\sin\alpha$$

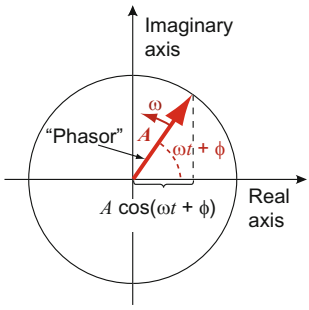

Fig. 2.2 A phasor is a vector of a given length. The phasor rotates at a given *angular frequency* and with a definite initial phase. The figure shows the position of the phasor at one point in time. See the text

where $\hat{x}$ and $\hat{y}$ are unit vectors along the x- and y- direction, respectively. The similarity to the previous expression is striking, assuming that the real part of the expression is taken to be the component along the x-direction and the imaginary part as the y-component.

This graphical vector representation can be extended immediately to represent a harmonic oscillation. We then use a vector with a length corresponding to the amplitude of the harmonic motion. The vector rotates with a fixed angular frequency of ω about the origin. The angle between the vector and the x axis is always $\omega t + \phi$. Then the x-component of the vector at any given time indicates the instantaneous amplitude of the harmonic oscillation. Such a graphical description is illustrated in Fig. 2.2 and is called an *phasor* description of the motion.

Phasors are very useful when multiple contributions to a motion or signal of the same frequency are to be summed up. The sum of all contributions can be found by vector addition. Especially in AC power, when voltages over different circuit components are summed, phasors are of great help. We will come back to their uses later. Phasors are useful also in other contexts, but mostly when all contributions in a sum have the same angular frequency.

It is important to learn all the mathematical expressions (2.1)–(2.4) for simple oscillatory motion so that they can be instantly recognized when they appear. It is also important to be able to convert quickly from one form to another. This book is full of such expressions!

2.3 Going from One Expression to Another

Phasors are of immense aid. As mentioned, a phasor is a vector that rotates in the complex plane as time passes (see Fig. 2.3). The vector rotates at an angular velocity equal to ω. The component of this vector along the real axis represents the physical value of our interest, and it is this component that can be expressed in more than four equivalent ways.

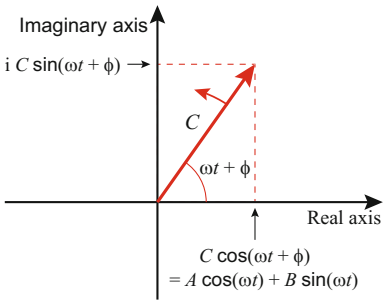

Fig. 2.3 Sketch of a phasor that rotates around the origin with an angular velocity ω.

2.3.1 First Conversion

Let us first show the transition from Eqs. (2.2) to (2.1). We use Rottmann's compilation of mathematical formula (an important tool when working with this book!), and use the trigonometric addition formula for cosines to get:

$$\begin{aligned} z(t) &= C\cos(\omega t+\phi) \\ &= C\left\{\cos\omega t\cos\phi-\sin\omega t\sin\phi\right\} \\ &= [C\cos\phi]\cos\omega t+[-C\sin\phi]\sin\omega t. \end{aligned}$$

This expression is formally identical to Eq. (2.1), from which it follows that:

$$C\cos(\omega t+\phi)=A\cos\omega t+B\sin\omega t \quad \text{if we set} \quad A=C\cos\phi \quad \text{and} \quad B=-C\sin\phi\,. \tag{2.5}$$

2.3.2 Second Conversion

We can go the opposite way by utilizing the details given in Eq. (2.5):

$$A^2+B^2=(C\cos\phi)^2+(C\sin\phi)^2=C^2(\sin^2\phi+\cos^2\phi)=C^2$$

$$C=\pm\sqrt{A^2+B^2}\,.$$

And, by dividing the last two relations in Eq. (2.5), we get:

$$\frac{B}{A}=\frac{-C\sin\phi}{C\cos\phi}=-\tan\phi$$

This is a fraction whose numerator is the y-component and the denominator the x-component of the phasor at $t = 0$. Then, it follows that

$$\phi = -\arctan\frac{B}{A} .$$

It should be noted here that both the tan and arctan have a periodicity of π, and one has to be careful about which of the two possible solutions one chooses. What quadrant ϕ is in depends on the sign of A and B separately. We must keep this in mind to make sure we choose the correct ϕ!

If a computer is used for calculating arctan, the *atan2*(B, A) variant is recommended for both Matlab and Python. Then the angle comes out in the correct quadrant.

With these reservations, we have shown:

$$A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t + \phi) \quad \text{where} \quad C = \sqrt{A^2 + B^2} \quad \text{and} \quad \phi = -\arctan\frac{B}{A} . \tag{2.6}$$

2.3.3 Third Conversion

The transition from Eqs. (2.4) to (2.2) is very simple if we use Euler's formula:

$$\mathrm{e}^{\mathrm{i}\alpha} = \cos\alpha + \mathrm{i}\sin\alpha .$$

From this, it follows that:

$$\Re\left\{E\mathrm{e}^{\mathrm{i}(\omega t+\phi)}\right\} = \Re\left\{E\left[\cos(\omega t + \phi) + \mathrm{i}\sin(\omega t + \phi)\right]\right\} = E\cos(\omega t + \phi) .$$

If this is equal to $C\cos(\omega t + \phi)$ one must have:

$$\Re\left\{E\mathrm{e}^{\mathrm{i}(\omega t+\phi)}\right\} = C\cos(\omega t + \phi) \quad \text{if} \quad C = E . \tag{2.7}$$

This simple relation holds equally well both ways (from Eqs. (2.4) to (2.2) or the opposite way).

2.3.4 Fourth Conversion

The last rendering to be considered here is also based on Euler's formula. It is the conversion of Eqs. (2.3) to (2.1). It is crucial to note that $\mathscr{D}$ is complex. We write this number as a sum of a real and an imaginary part:

$$\mathscr{D} = D_{\text{re}} + \text{i}D_{\text{im}}$$

where D_{re} and D_{im} are both real. This leads (once again through Euler's formula):

$$\Re\left\{\mathscr{D}\text{e}^{\text{i}\omega t}\right\} = \Re\left\{(D_{\text{re}} + \text{i}D_{\text{im}})(\cos\omega t + \text{i}\sin\omega t)\right\}$$

$$= \Re\left\{D_{\text{re}}\cos\omega t + \text{i}D_{\text{re}}\sin\omega t + \text{i}D_{\text{im}}\cos\omega t + \text{i}^2 D_{\text{im}}\sin\omega t\right\}$$

$$= D_{\text{re}}\cos\omega t - D_{\text{im}}\sin\omega t \ .$$

When this is compared with

$$A\cos\omega t + B\sin\omega t \ ,$$

one is led to the simple relation:

$$\Re\left\{\mathscr{D}\text{e}^{\text{i}\omega t}\right\} = A\cos(\omega t) + B\sin(\omega t) \quad \text{if} \quad \mathscr{D} = A - \text{i}B \ . \tag{2.8}$$

This simple relationship also works both ways (from Eqs. (2.3) to (2.1) or the converse).

We could also look at the expression $z(t) = C\sin(\omega t + \phi)$ instead of $z(t) = C\cos(\omega t + \phi)$, but with the procedures outlined above it should be easy to navigate from one form to the next.

When we come to treat waves in later chapters, we will often start with harmonic waves. The expressions then become almost identical to those we have in Eqs. (2.1)–(2.4). It is important to be familiar with these expressions.

2.4 Dynamical Description of a Mechanical System

Let us come back now to physics. A spring often follows Hooke's law: the deviation from the equilibrium point is proportional to the restoring force exerted by the spring.

Suppose that the suspension hangs vertically without any mass at the end. It has a length of L_0. If a mass m is attached to the free end, and we wait until the system has settled, the spring will have a new length, say L_1, that satisfies the equation

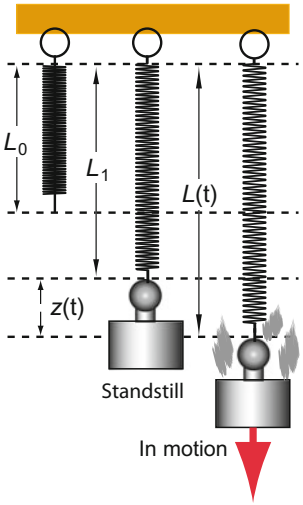

Fig. 2.4 Definition of different lengths of the spring with and without an attached mass; see the text

$$k(L_1 - L_0) = mg$$

where the experimentally determined k is called the spring constant, and g, the acceleration due to gravity, is considered constant (disregarding the variation of g with the height) (Fig. 2.4).

If the mass is pulled down slightly and released, the force acting on the mass will always be

$$F(t) = k[L(t) - L_0] - mg$$

where $L(t)$ is the instantaneous length of the spring. Upon combining the last two equations, one gets

$$\begin{aligned} F(t) &= k[L(t) - L_0] - k(L_1 - L_0) \\ &= k[L(t) - L_1] \, . \end{aligned}$$

Important: The elongation of the spring from length L_0 to L_1 is a consequence of the force of gravity. Therefore, in later expressions, neither L_0 nor g, the acceleration due to gravity, will enter.

The displacement from the equilibrium point, i.e. $L(t) - L_1$ is renamed to $-z(t)$. The force that acts on the mass will then be

$$F(t) = -kz(t) \, .$$

The negative sign indicates that the restoring force is in the opposite direction with respect to the displacement.

According to Newton's law, the sum of the forces acting on the mass is equal to the product of the mass and the instantaneous acceleration:

$$F(t) = m\ddot{z}(t) = -kz(t) \ .$$

Note once more that the gravitational force is *not* directly included in this expression. This is because the restoring force due to the spring and the gravitational pull counterbalance each other when $z = 0$.

$\ddot{z}$ is the double derivative of z with respect to time, i.e. acceleration in the vertical direction:

$$\ddot{z} \equiv \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} \ .$$

The equation of motion can then be written as:

$$\ddot{z}(t) = -\frac{k}{m} z(t) \ . \tag{2.9}$$

This is a second-order homogeneous differential equation with constant coefficients, and we know its general solution to be

$$z(t) = B \sin\left(\sqrt{\frac{k}{m}}t\right) + C \cos\left(\sqrt{\frac{k}{m}}t\right)$$

where B and C are two constants (with dimensions of length). We can identify this solution as Eq. (2.1) if we set the angular frequency ω in the latter equation to

$$\omega = \sqrt{\frac{k}{m}} \ .$$

The constants B and C are found by imposing the initial conditions, and the particular solution for the oscillatory motion is thereby determined with one particular amplitude and one particular phase.

The angular frequency ω is convenient to use in mathematical expressions. However, when we observe an oscillating system, it is expedient to use frequency f and period T. Their interrelationship is stated below:

$$f = \frac{\omega}{2\pi} \ ,$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \ .$$

For the mechanical mass–spring oscillator one gets:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\,,$$

$$T = 2\pi\sqrt{\frac{m}{k}}\,.$$

What have we learned in this section? Well, we have seen that a mass, attached to a spring and experiencing the forces exerted by the spring and gravity, will oscillate up and down, executing a simple harmonic motion with a certain amplitude and time period. We have managed to "explain" the oscillatory motion by combining Hooke's law and Newton's second law.

The kinematic description gave in Sect. 2.1 is identical to the *solution* of the dynamic equation we set up in this section based on Newton's law.

2.5 Damped Oscillations

No macroscopic oscillations last ceaselessly without the addition of energy. The reason is that there are always forces that oppose the movement. We call these frictional forces.

Frictional forces are often difficult to relate to, because they arise from complicated physical phenomena occurring in the borderland between atomic and macroscopic dimensions. A basic understanding of friction has *begun* to grow during the last decades, because grappling with this part of physics requires extensive modelling by means of computers.

Air friction is complex and we need at least two terms to describe it:

$$F_f = -bv - Dv^2$$

where v is the velocity (with direction), and b and D are positive constants, which will be called friction coefficients.

An expression that also indicates the correct sign and direction is:

$$\overrightarrow{\boldsymbol{F}_f} = -b\,\vec{\boldsymbol{v}} - Dv^2\frac{\vec{\boldsymbol{v}}}{v} = -b\,\vec{\boldsymbol{v}} - D\,|v|\,\vec{\boldsymbol{v}}\,. \tag{2.10}$$

In other words, the friction force $\overrightarrow{\boldsymbol{F}_f}$ works in a direction opposite to that of the velocity $\vec{\boldsymbol{v}}$.

If we start with a system executing harmonic motion without friction, and we add friction as given in Eq. (2.10), it is not possible to find a general solution using analytical mathematics alone. If the problem is simplified by setting the frictional force to $-bv$ only, it is possible to use analytical methods. The solution is useful for slow motion in air. For small speeds, the term Dv^2 will be less than the term bv in Eq. (2.10) so that the v^2 term can be neglected.

Remarks: $-Dv^2$ is a nonlinear term that is often associated with turbulence, one of the difficult areas of physics, often associated with chaotic systems. Friction of this type depends on a number of parameters that can be partially included into the so-called Reynolds number. In some calculations, the quantity D must be replaced by a function $D(v)$ if Eq. (2.10) is to be used. Alternatively, the Navier–Stokes equation can be used as a starting point. Reasonably accurate calculations of the friction of a ball, plane or rocket can be accomplished only by using numerical methods (Those interested will be able to find more material in Wikipedia under the headings "Reynolds number" and "Navier–Stokes equation".).

Since no great skill is needed for solving the *simplified* differential equation, we accept the challenge! The solution method will consolidate our familiarity with complex exponents and will show the elegance of the formalism. Moreover, this is standard classical physics widely covered in textbooks, and the results are useful in many contexts. The mathematical approach itself finds applications in many other parts of physics.

The starting point is, as before, Newton's second law, and we use it for a mass that oscillates up and down at the end of a spring in air. The equations can now be written:

$$\sum F = ma \equiv m\ddot{z}$$

$$-kz(t) - b\dot{z}(t) = m\ddot{z}(t)$$

$$\ddot{z}(t) + \frac{b}{m}\dot{z}(t) + \frac{k}{m}z(t) = 0\,. \tag{2.11}$$

This is a homogeneous second-order differential equation, and we choose a trial solution of the type:

$$z(t) = A\mathrm{e}^{\alpha t}\,. \tag{2.12}$$

Remark: Here, both A and α are assumed to be complex numbers.

Differentiation of the exponential function (2.12), insertion into (2.11) and finally the abbreviation of exponential terms and the factor A gives the characteristic polynomial

$$\alpha^2 + \frac{b}{m}\alpha + \frac{k}{m} = 0\,.$$

We rename the fractions to get a tidier expression:

$$\frac{b}{m} \equiv 2\gamma \tag{2.13}$$

$$\frac{k}{m} \equiv \omega^2 \,. \tag{2.14}$$

The equation now becomes:

$$\alpha^2 + 2\gamma\alpha + \omega^2 = 0 \,.$$

This is a quadratic equation whose roots can be written as:

$$\alpha_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega^2} \,. \tag{2.15}$$

There arise three different types of solutions, depending on the discriminant:

- $\gamma > \omega$: **Supercritical damping, overdamping**

  If the frictional force becomes large, we get what is called overdamping. The criterion of overdamping $\gamma > \omega$ is mathematically equivalent to $b > 2\sqrt{km}$. In this case, both A and α in Eq. (2.12) are real numbers, and the general solution can be written as:

  $$z(t) = A_1 \mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega^2}\right)t} + A_2 \mathrm{e}^{\left(-\gamma-\sqrt{\gamma^2-\omega^2}\right)t} \,. \tag{2.16}$$

  where A_1 and A_2, determined by the initial conditions, involve the initial values of velocity and displacement.

- This is a sum of two exponentially decaying functions, one of which goes to zero faster than the other. There is no trace of oscillatory motion here.
  Note that, for certain initial conditions, A_1 and A_2 may have different signs, and the time course of the displacement may hold surprises!
- $\gamma = \omega$: **Critical damping**
  The frictional force and the effective spring force now match each other in such a way that the movement becomes particularly simple. Based on Eqs. (2.12) and (2.15), we find one solution: It can be described as a simple exponential function:

  $$z(t) = A\mathrm{e}^{-\gamma t} \,.$$

It is known from the theory of differential equations that the general solution of a second-order differential equation must have *two* arbitrary constants, so that one may satisfy two initial conditions. This means that we have yet to find the full solution. To find the missing solution, we will use a simple trial solution of the type:

$$z(t) = f(t)\mathrm{e}^{-\gamma t} \ .$$

If this trial solution is substituted into our differential equation (2.11) with $\gamma = \omega$, we find easily that $\ddot{f}$ must be equal to 0. After two integrations with respect to t, we find $f(t) = A + Bt$.

Thus the general solution of Eq. (2.11) for critical damping is then:

$$z(t) = A\mathrm{e}^{-\gamma t} + Bt\mathrm{e}^{-\gamma t} \ . \tag{2.17}$$

Critical damping in many cases corresponds to the fastest damping of a system and is the one sought for, for example, in vehicle shock absorbers.

- $\gamma < \omega$: **Sub-critical damping; underdamping**
  In this case, α in Eq. (2.15) becomes complex, which means that the solution will contain both an exponential decreasing factor and an oscillating sinusoidal term. From Eq. (2.15), we get then:

$$\alpha_{\pm} = -\gamma \pm \sqrt{\gamma^2 - \omega^2} \tag{2.18}$$

$$= -\gamma \pm \mathrm{i}\omega' \ . \tag{2.19}$$

where $\omega' \equiv \sqrt{\omega^2 - \gamma^2}$ is a real number. The general solution then becomes:

$$z(t) = \mathrm{e}^{-\gamma t}\Re\left\{\mathscr{A}\mathrm{e}^{\mathrm{i}\omega' t} + \mathscr{B}\mathrm{e}^{-\mathrm{i}\omega' t}\right\}$$

where $\mathscr{A}$ and $\mathscr{B}$ are complex numbers, and $\Re$ means that we take the real part of the expression.

The solution for sub-critical damping can be put in a simpler form:

$$z(t) = \mathrm{e}^{-\gamma t} A\cos(\omega' t + \phi) \ . \tag{2.20}$$

Here the constant A and ϕ must be assigned such values as to make the particular solution conform to a given physical system. The mass will oscillate on both sides

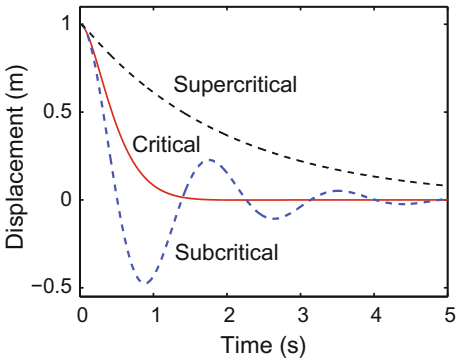

Fig. 2.5 Examples of overcritical, critical and sub-critical damping of an oscillation that would be simple harmonic in the absence of friction. The friction is increased by a factor of four from one curve to another: sub-critical, critical and overcritical damping

of the equilibrium point while the amplitude decreases to zero. The oscillation frequency is lower than when there is no damping (something that is to be expected since the friction acts to slow down all movement).

It is common in textbooks to present a figure that typically shows the time course for a damped harmonic motion, and Fig. 2.5 perpetuates the tradition. However, it should be noted that such figures can be very misleading, because they often assume that the initial velocity is zero (as in our figure). In a task last in this chapter, we ask you to investigate how an overdamped harmonic motion looks under some other initial conditions. If you solve that task, you will see that the solution is more diverse than the traditional figures indicate!

2.6 Superposition and Nonlinear Equations

When we tried to figure out how a damped oscillation changes with time, we assumed the validity of the differential equation:

$$\ddot{z}(t) + \frac{b}{m}\dot{z}(t) + \frac{k}{m}z(t) = 0 \tag{2.21}$$

and found a general solution that consisted of two parts. For overcritical damping, the solution looks like this:

$$z_A(t) = A_1 \mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega^2}\right)t} + A_2\,\mathrm{e}^{\left(-\gamma-\sqrt{\gamma^2-\omega^2}\right)t}$$

where γ and ω are defined in Eqs. (2.13) and (2.14) above.

In the interests of simplicity, we set:

$$f_1(t) = \mathrm{e}^{\left(-\gamma+\sqrt{\gamma^2-\omega^2}\right)t}$$

and

$$f_2(t) = \mathrm{e}^{\left(-\gamma-\sqrt{\gamma^2-\omega^2}\right)t} .$$

One solution can then be written as:

$$z_A(t) = A_1 f_1(t) + A_2 f_2(t) .$$

Another solution of the differential equation could be:

$$z_B(t) = B_1 f_1(t) + B_2 f_2(t) .$$

It is easy then to see that

$$z_{AB}(t) = [A_1 f_1(t) + A_2 f_2(t)] + [B_1 f_1(t) + B_2 f_2(t)]$$

$$z_{AB}(t) = (A_1 + B_1) f_1(t) + (A_2 + B_2) f_2(t)$$

will also be a solution of the differential equation. This is due to the fact that the differential equation (2.21) is a *linear equation*.

This is called the "superposition principle". This principle pervades many parts of physics (and notably also in quantum mechanics).

Previously, many people considered superposition principles to be a fundamental property of nature, but it is not. The reason for the misunderstanding is perhaps that most physicists of those days worked only with linear systems where the superposition principle holds. Today, thanks to computers and numerical methods, we can tackle physical systems that were previously inaccessible. This means that there has been an "explosion" in physics in the last few decades, and the development is far from over.

Let us see what differences arise when nonlinear descriptions are used. By nonlinear description, for example, we mean that forces describing a system showing a nonlinear dependence on position or speed. For example, when we described damped oscillations, we found that friction must often be modelled with at least two terms:

$$F = -bv - Dv^2 .$$

The second term on the right-hand side makes a nonlinear contribution to the force.

The differential equation would then become:

$$\ddot{z}(t) + \frac{b}{m}\ddot{z}(t) + \frac{D}{m}[\ddot{z}(t)]^2 + \frac{k}{m}z(t) = 0 \tag{2.22}$$

In this case, we can prove the following:

If $f_A(t)$ is one solution of this equation, and $f_B(t)$ is another solution, it is in general *not* true that the function $f_A(t) + f_B(t)$ is a solution of Eq. (2.22).

In other words, when we include a second-order term to complete the friction description, we see that the superposition principle no longer applies! Even if we find a possible solution for such an oscillating system, and then another solution, the sum of these individual solutions will not necessarily be a solution of the differential equation.

The term Dv^2 is a nonlinear term, and when the physics is such that nonlinear terms play a nonnegligible role, the superposition principle does not apply.

Take a look at the "list of nonlinear partial differential equations" on the Wikipedia to get an impression of how important nonlinear processes have now become within, for example, various areas of physics. The overview indirectly shows how many more issues we can study today compared to what was possible a few decades ago. Despite this, we still have a regrettable tendency to use formalism and interpret phenomena, in both classical and quantum physics, as if the world was strictly linear. I dare say, physicists will have, within a few decades, such a rich store of experience to build on that the general attitude will change. Time will show!

2.7 Electrical Oscillations

Before we proceed with forced oscillations, we will derive the equation of oscillatory motion for an electrical circuit. The purpose is to show that the mathematics here is completely analogous to that used in mechanical system.

In electromagnetism, there are three principal circuit elements: Resistors, inductors (coils) and capacitors. Their behaviours in an electrical circuit are given by the following relationships (where Q stands for the charge, $I = dQ/dt$ is electric current, V is voltage, R is resistance, L inductance and C capacitance):

$$V_R = RI \tag{2.23}$$

$$V_C = Q/C \tag{2.24}$$

$$V_L = L\,\mathrm{d}I/\mathrm{d}t = L\,\mathrm{d}^2Q/\mathrm{d}t^2\ . \tag{2.25}$$

If the circuit elements are connected in a closed loop, the total voltage change will be zero when we go around the loop from any point to the same point (Kirchhoff's law). For example, we connect a (charged) capacitor to a resistor (by closing the switch in Fig. 2.6), the voltage across the capacitor will always be the opposite of the voltage across the resistor. Thus, it follows that

$$RI = -Q/C$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{1}{RC}Q\,.$$

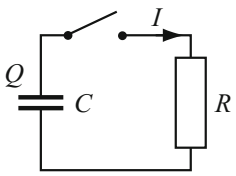

Fig. 2.6 The voltage across a charged capacitor will decrease exponentially to zero after the capacitor is connected to a resistor

$$RI = -Q/C$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{1}{RC}Q\,.$$

If the charge on the capacitor was Q_0 at time $t = 0$, the solution of this differential equation is:

$$Q = Q_0 e^{-t/RC}\,.$$

The charge on the capacitor thus decreases exponentially and goes to zero (The reader is supposed to be familiar with this.).

In the context of "oscillations and waves", we will concentrate on oscillating electrical circuits. An oscillating electrical circuit usually consists of at least one capacitor and an inductor. If the two elements are connected in series so as to form a closed loop, Kirchhoff's law gives:

$$\frac{Q}{C} = -L\frac{\mathrm{d}I}{\mathrm{d}t} = -L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}$$

$$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = -\frac{1}{LC}Q\,.$$

We can write this in the same way as was done for the mechanical system:

$$\ddot{Q}(t) = -\frac{1}{LC}Q(t)\,. \tag{2.26}$$

If we compare Eq. (2.26) with Eq. (2.9), we see that they are completely analogous. The coefficient on the right-hand side is k/m for the mechanical system, and $1/LC$ in the electrical analogue, but they are both positive constants.

This is oscillation once more, and we know that the overall solution is:

$$Q = Q_0 \cos(\omega t + \phi)$$

where $\omega = 1/\sqrt{LC}$. Q_0 and ϕ are two constants whose values are fixed on the basis of the initial state ($t = 0$) of the system.

It may be worth reflecting on why there must be two initial conditions to obtain a specific solution for the LC circuit as compared to the RC circuit. In the RC circuit, the current is uniquely given if the charge is given. We can then decide, by means of a snapshot, either the charge or the voltage, will vary with time (assuming that R and C are known). For the LC circuit, this is not the case. There we must know, for example, both charge and current at a particular instant, or the charge at two adjacent times, to determine the further development. The reason is that we can not deduce power from one charge (or voltage) alone. The difference in physical descriptions for the RC and (R)CL circuit is reflected mathematically by the difference between a first-order and a second-order differential equations.

An electrical circuit in practice contains some kind of loss/resistance. Let us take the simplest example, namely that the loss is due to a constant series resistance R in the closed loop. If Kirchhoff's law is used again, we get the following differential equation:

$$\frac{Q}{C} = -RI - L\frac{\mathrm{d}I}{\mathrm{d}t} = -R\frac{\mathrm{d}Q}{\mathrm{d}t} - L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}$$

or

$$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{LC}Q = 0\,. \tag{2.27}$$

This is a homogeneous second-order differential equation that can be solved using the characteristic polynomial:

$$a^2 + \frac{R}{L}a + \frac{1}{LC} = 0$$

whose solution is:

$$a_\pm = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\,.$$

The general solution to the differential equation is:

$$Q = Q_{0,1}\mathrm{e}^{-\frac{R}{2L}t+\left(\sqrt{(\frac{R}{2L})^2-\frac{1}{LC}}\right)t} + Q_{0,2}\mathrm{e}^{-\frac{R}{2L}t-\left(\sqrt{(\frac{R}{2L})^2-\frac{1}{LC}}\right)t}\,. \tag{2.28}$$

We note that for $R = 0$, we recover Eq. (2.26), whose solution is

$$\begin{aligned} Q &= Q_{0,1}\mathrm{e}^{(\sqrt{-1/LC})t} + Q_{0,2}\mathrm{e}^{-(\sqrt{-1/LC})t} \\ &= Q_{0,1}\mathrm{e}^{i(\sqrt{1/LC})t} + Q_{0,2}\mathrm{e}^{-i(\sqrt{1/LC})t} \\ &= Q_0 \cos(\omega t + \phi)\,. \end{aligned}$$

where $\omega = 1/\sqrt{LC}$. We see again that there are two constants to be determined by means of the initial conditions.

When $R \neq 0$, we get an exponentially decreasing term $e^{-(R/2L)t}$ multiplied by either an oscillating term or a second exponentially decreasing term, depending on whether $(R/2L)^2$ is less or greater than $1/LC$. When $(R/2L)^2 = 1/LC$, the term under the radical in Eq. (2.28) becomes zero, which corresponds to what we have seen previously with two coincident roots. In such a case, the overall solution turns out to of the same form as Eq. (2.17). Again, it is natural to talk about sub-critical, critical and supercritical damping, similar to a mechanical pendulum.

We have seen that electrical circuits are described by equations completely analogous to those for a mechanical pendulum. Other physical phenomena show similar oscillating behaviour.

Common to all the systems examined above is the *equation for oscillatory motion*, which can be stated, in its simplest form, as

$$\frac{d^2 f}{dt^2} + c_1 \frac{df}{dt} + c_2 f = 0$$

where c_1 and c_2 are positive constants.

2.8 Energy Considerations

Let us calculate the energy and its time development in electrical circuits. We limit ourselves to a loss-less oscillating system, that is, we take $R = 0$. The solution of the differential equation is then:

$$Q = Q_0 \cos(\omega t + \phi)$$

where $\omega = \frac{1}{\sqrt{LC}}$. Q_0 and ϕ are two constants whose values are determined by using the initial conditions ($t = 0$) of the system.

The energy stored in the capacitor at any particular time is given by:

$$E_C = \frac{1}{2} QV = \frac{1}{2} \frac{Q^2}{C} .$$

The instantaneous energy is thus:

$$E_C(t) = \frac{1}{2}\frac{[Q_0\cos(\omega t+\phi)]^2}{C} = \frac{1}{2}\frac{{Q_0}^2}{C}\cos^2(\omega t+\phi)\ .$$

From electromagnetism we know that the energy stored in an inductor is given by the expression:

$$E_L = \frac{1}{2}LI^2 = \frac{1}{2}L\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^2\ .$$

Substituting the expression for Q from the general solution, the instantaneous energy in the inductance is found to be

$$E_L(t) = \frac{1}{2}L\left[\frac{\mathrm{d}[Q_0\cos(\omega t+\phi)]}{\mathrm{d}t}\right]^2 = \frac{1}{2}L{Q_0}^2\omega^2\sin^2(\omega t+\phi)\ .$$

Since $\omega = \dfrac{1}{\sqrt{LC}}$, the expression can also be written as:

$$E_L(t) = \frac{1}{2}\frac{{Q_0}^2}{C}\sin^2(\omega t+\phi)\ .$$

The total energy, found by summing the two contributions, is thus:

$$E_{\mathrm{tot}}(t) = E_C(t)+E_L(t) = \frac{1}{2}\frac{{Q_0}^2}{C}\left[\cos^2(\omega t+\phi)+\sin^2(\omega t+\phi)\right]$$

$$E_{\mathrm{tot}}(t) = \frac{1}{2}\frac{{Q_0}^2}{C}\ .$$

We notice that the total energy remains constant, i.e. time-independent. Although the energy of the capacitor and inductor varies from zero to a maximum value and back in an oscillatory fashion, these variations are time shifted

by a quarter period, making the sum independent of time. The energy "flows" back and forth between the capacitor and inductor. A time shift between two energy forms seems to be a characteristic feature of all oscillations. Simple oscillations are often solutions of second-order differential equation, but oscillations may also originate from phenomena that have to be expressed mathematically in different way.

For the mechanical system, potential energy (from the conservative spring force) and kinetic energy are the two energy forms. You are recommended to perform a similar calculation as we have done in this section for the mechanical system to see that the result is indeed analogous to what we found for the electrical system (This is the theme for a calculation task in the end of this chapter.).

The energy calculations we have just completed apply only if there is no loss in the system. If loss due to resistance (the equivalent of friction) is preset, the energy will of course decrease over time. The energy loss per unit time pattern will depend on the extent of damping (supercritical, critical or sub-critical), but in general, the energy loss will follow an exponential decline.

2.9 Learning Objectives

The title of the book is "Physics of Oscillation and Waves", but just about all basic theory of *oscillations* is presented already in this chapter and Chap. 3. Nevertheless, the basic ideas from these two chapters will resurface many times when we refer to waves. We therefore think that a thorough study of this chapter and Chap. 3 will pay handsome dividends when the reader moves to later chapters.

After working through this chapter you should be able to

- Know that a harmonic oscillatory motion can be expressed mathematically in a variety of ways, both with sines and/or cosine functions, or in complex form (using Euler's formula). One goal is to recognize the different forms and to be able to go mathematically from any of these representations to another.
- Know that oscillations may occur in systems affected by a force that tries to bring the system back to equilibrium. Mathematically, this can be described easily in simple cases:

$$\ddot{z} = -kz$$

  where x is the displacement from the equilibrium position and k is a real, positive number.

- Know that any oscillation must contain the two terms given in the equation in the previous paragraph, but that other terms may also be included.
- Know how physical laws/relationships are combined by deriving the second-order differential equation for both a mechanical and an electrical system.
- Know that in order to find a unique solution to the above-mentioned equation, two independent initial conditions must be imposed and suggest at least a few different choices of initial conditions.
- Be able to derive and solve the equation of oscillatory motion both for free and damped oscillation with linear damping. This means that you must be able to distinguish between supercritical, critical and sub-critical damping, and to outline graphically typical features for different initial conditions.
- Be able to deduce the equation for oscillatory motion also for a nonlinearly damped system and find the solution numerically (after studying Chap. 4).
- Be able to explain why the superposition principle does not apply when nonlinear terms are included in the equation of motion.

2.10 Exercises

Remark:
For each of the remaining chapters, we suggest concepts to be used for student active learning activities. Working in groups of two to four students, improved learning may be achieved if the students discuss these concepts vocally together.

The purpose of the comprehension/discussion tasks is to challenge the student's understanding of phenomena or formalism. Even for these tasks, it may be beneficial for learning that students discuss the tasks vocally in small groups.

The "problems" are more traditional physics problems. However, our apperception is that the correct answer alone is not considered a satisfactory solution. Full marks are awarded only if the correct answer is supplemented with sound arguments, underlying assumptions, and approaches used for arriving at the answer.

Suggested concepts for student active learning activities: Kinematics, dynamics, amplitude, phase, frequency, harmonic, second-order differential equation, general solution, particular solution, initial conditions, phasor, damping, characteristic polynomial, supercritical/critical/sub-critical damping, superposition, linear equation.

Comprehension/discussion questions

1. Make a sketch similar to Fig. 1.1, which shows a time plot for one oscillation, but also draw the time course for another oscillation with the same amplitude and initial phase term, but a different frequency compared to the first one. Repeat the same for the case where the amplitudes are different, while the phase and

frequency are the same. Finally, present the third variant of such sketches (Find out what is meant by this.).

2. What demands must we make for a force to be able to form the basis for oscillations?
3. If a spring is cut in the middle, what will be the spring constant for each part compared to that for the original spring? How large is the time period for a mass at the end of the half-spring compared with the period of the mass in the original spring?
4. Suppose we have a mass in a spring that oscillates up and down with a certain time period here on earth, and that the spring and the mass are brought to the moon. Will the time period change?
5. Suppose we do as in the previous task, but take a pendulum instead of a mass and spring. Will the time period change?
6. A good bouncing ball can bounce up and down many times against a hard horizontal surface. Is this a harmonic motion (as we have used the word)?
7. In the text, a rather vague statement is made about a judicious choice of mass and maximum extension of the spring to achieve an approximately harmonic oscillatory motion. Can you give examples of what conditions will be unfavourable for a harmonic motion?

Problems

8. Show mathematically that the total energy of an oscillating mass–spring system (executing up and down movement only) is constant in time if there is no friction present (Remember that changes in potential energy in the gravitational field disappear if you take the equilibrium position of the plot as the starting point for the calculations.).
9. It is sometimes advantageous to describe dynamics by plotting velocity versus position, instead of position versus time, as we have done so far. Create such a plot for a mass that swings up and down at the end of a spring (plot in *phase plane*). What is the shape of the plot?
10. Make a plot in the phase plane (see previous task) for the movement of a bouncing ball that bounces vertically up and down on a hard surface (practically without loss). What is the shape of the plot? Comment on similarities/differences between the plots in this and the previous task.
11. A spring hangs vertically in a stand. Without any mass, the spring is 30 cm long. We attach a 100 g ball at the lower end, stretch the spring by pulling the mass (and then releasing it) and find, after the ball has come to rest, that the spring has become 48 cm long. We then pull the ball 8.0 cm vertically downwards, keep the ball steady, and then let go. Find the oscillation period of the ball. Write a mathematical expression that can describe the oscillatory movement. Find the maximum and minimum force between the ball and the spring.
12. An oscillating mass in a spring moves at a frequency of 0.40 Hz. At time $t = 2.0\,\text{s}$, its position is $+2.4\,\text{cm}$ above the equilibrium position and the velocity of the mass is $-16\,\text{cm/s}$. Find the acceleration of the mass at time $t = 2.0\,\text{s}$. Find a mathematical description appropriate to the movement.

13. A mass m hangs in a massless spring with spring constant k. The amplitude is A. How big is the displacement relative to the equilibrium point when the kinetic energy is equal to half of the potential energy?
14. An oscillatory motion can be described by the equation $z(t) = A\cos(\omega t + \phi)$ where $A = 1.2\,\text{m}$, the frequency $f = \omega/(2\pi) = 3.0\,\text{Hz}$, and $\phi = 30°$. Find out how this oscillatory motion can be formally specified when we (a) do not use the phase term, but only a combination of sine and cosine terms, and (b) when using a complex description based on Euler's formula.
15. Another oscillatory motion is given at $y(t) = \Re\{(-5.8 + 2.2\text{i})\text{e}^{\text{i}\omega t}\}$. Convert the equation to the same form as Eq. (2.1) and convert further until it has the same form as Eq. (2.1).
16. Show that the period of a mathematical pendulum with small amplitude is given by $T = 2\pi\sqrt{L/g}$ where L is the length of the pendulum and g is the acceleration due to gravity. Hint: Use the relation $\tau = I\alpha$ where τ is the torque, I the moment of inertia (mL^2) and α is the angular acceleration, to show that the equation of motion is $\ddot{\theta}(t) = (g/L)\sin\theta$ and then use the usual approach for sines at small angles.
17. A mass weighing 1.00 N is hung at the end of a light spring with spring constant 1.50 N/m. If we let the mass swing up and down, the period is T. If instead we let the mass settle down and pull it to the side and release it, the resulting movement will have a period of $2T$ (the amplitude in the second case is very small). What is the length of the spring without the mass? (You may need the expression in the previous assignment.)
    Note: We recommend strongly that you make a real mass/spring system with a length so that the period of the sidewise pendulum oscillation is twice the period for the vertical mass–spring pendulum. Start the movement of the system by a pure vertical displacement of the mass, and release it from rest at this position. Watch the movement. You may be surprised! What you witness is an example of a so-called parametric oscillator.
18. Show that the energy loss for a damped pendulum where the frictional force is $F_f = -bv$ is given by $\text{d}E/\text{d}t = -bv^2$. Here, b is a positive number (friction coefficient) and v is the velocity (Start from the mechanical energy of the system, $E = E_{\text{potential}} + E_{\text{kinetic}}$.).
19. An object of $m = 2.0\,\text{kg}$ hangs at the end of a spring with the spring constant $k = 50\,\text{N/m}$. We ignore the mass of the spring. The system is set in oscillations and is damped. When the velocity of the mass is 0.50 m/s, the damping force is 8.0 N.
    (a) what is the system's natural oscillation frequency f (i.e. if the damping was not present)?
    (b) Determine the frequency of the damped oscillations.
    (c) How long does it take before the amplitude is reduced to 1% of the original value?