Chapter 10
Reflection, Transmission and Polarization

Abstract In this chapter, Maxwell's equations are used for deducing laws of reflection/transmission of an electromagnetic wave entering an idealized plane boundary between two insulators, e.g. air (or vacuum) and glass. The expression for the Brewster angle is derived and Fresnel's equations are presented. Snel's law is derived using the principle of minimum time. Emphasis in the last part of the chapter is put on polarization and how it may be changed by the use of birefringent material like calcite or polarization filters. Use of polarization in polariometry as well as in stereoscopy is mentioned, and a brief comment on evanescent waves is given.

10.1 Introduction

In Chap. 9, we found that a plane electromagnetic wave with the phase velocity

$$c = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_r \mu_0 \mu_r}}$$

$$= \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \frac{1}{\sqrt{\varepsilon_r \mu_r}} = \frac{c_0}{\sqrt{\varepsilon_r \mu_r}}$$

is a possible solution of Maxwell's equations in an infinite homogeneous medium containing no "free charges". The symbols have their usual meanings.

The speed of light in a medium (without free charges) is the quotient of the speed of light in vacuum c_0 and the refractive index n for the medium:

$$c \equiv \frac{c_0}{n} \ .$$

The vast majority of media we are going to consider are diamagnetic or paramagnetic. This applies, for example, to optical glass, for which $\mu_r \approx 1.00$. As a result, we may write:

$$n \approx \sqrt{\varepsilon_r} \ .$$

A. I. Vistnes, *Physics of Oscillations and Waves*, Undergraduate Texts in Physics,
https://doi.org/10.1007/978-3-319-72314-3_10

In other words, the index of refraction is, in a manner of speaking, directly related to the "polarization susceptibility" of the medium, and the relative permittivity is a measure of this. The more easily an external electric field can distort the electron cloud around the atoms from their equilibrium positions, the slower is the speed of light in that medium.

For substances whose atoms are arranged in a regular and special way, as in a calcite crystal, it is easier to displace the electron clouds away from equilibrium when the electric field has one particular direction relative to the crystal than other directions. This causes light to travel more slowly (through the crystal) for one orientation of the crystal relative to the direction of light polarization than for other orientations. Calcite crystals, which have this property, are said to be birefringent. Doubly refracting materials are widely used in modern optics.

Other substances have the property that they only transmit light with the electric field in a particular orientation. Such substances can be used as so-called polarization filters, which are used in photography, material characterization, viewing 3D movies, and in astronomy.

We will treat birefringence and polarization filters in this chapter, but we *start* by analyzing how waves are partially reflected and partially transmitted when they strike an interface between two different media (in contact with each other). Again, Maxwell's equations are central to the calculations.

A running topic throughout the chapter is polarization, but polarization appears in two quite different contexts. Be careful not to confuse them!

10.2 Electromagnetic Wave Normally Incident on An Interface

Generally, there are infinitely many different geometries and as many different solutions of Maxwell's equations when an electromagnetic wave reaches an interface between two media. We need to simplify enormously in order to extract regularities that can be described in a mathematically closed form.

In this section, we will use the Faraday–Henry law together with an energy balance sheet to find out what fraction of an electromagnetic wave is reflected and what is transmitted when the wave enters, for example, from air into glass. We assume that the electromagnetic wave is approximately plane and strikes normally a plane interface between two different homogeneous media without free charges. We make the following assumptions for the second medium and the interface:

1. Assume that the medium itself is homogeneous within a volume of λ^3 where λ is the wavelength.
2. Assume that the interface is flat over an area much greater than λ^2.
3. Assume that the thickness of the interface is much less than the wavelength λ.

As long as we consider light of wavelength in the 400–800 nm range travelling through glass, where the atoms are a few tenths of a nanometre apart, these three assumptions are reasonably well fulfilled. But the conditions are certainly not met in all common cases. When light goes through raindrops, the drops are often so large that we can almost use the formalism that will be derived presently. But when the drops are so small that the above conditions are not met, Maxwell's equations must be used directly. For drops that are small, we get the so-called Mie scattering, which produces not a regular rainbow but an almost colourless arc.

Also for electromagnetic waves in completely different wavelength ranges than light, it is difficult to satisfy the three assumptions. Take for example X-rays with wavelength around 0.1 nm. Then, the wavelength is about the same as the distance between the atoms. For radio waves as well, the assumptions cannot be easily satisfied. This means that the laws to be deduced in this chapter are often limited in practice to electromagnetic waves in the form of visible light, or in any case nearby wavelengths.

The purpose of the following mathematics in this chapter is to derive useful expressions, but also to point out clearly the assumptions we base the calculation on. This is important so that we can judge the validity of the formulas in different contexts. Within the rapid growing field of nanotechnology, it becomes clear that common expressions are not applicable everywhere.

So, let us study what happens when an electromagnetic wave meets an interface head on. Let us suppose that the three above assumptions are satisfied and that we send electromagnetic waves normally to the interface. Part of the wave will be reflected at the interface and travel back in the original medium, while the rest of the wave is transmitted into the next medium and continues there. In Fig. 10.1, the three waves are drawn in a manner that brings out their main features. The waves that are drawn in can be considered, for example, as one component of the electric field (in a given direction perpendicular to the normal to the interface). The index of refraction on the left side of the figure is n_1 and that on the right side n_2, and we have not yet said anything about which of these is the larger. For the same reason, we have not considered whether the reflected wave would have the opposite sign to the incoming

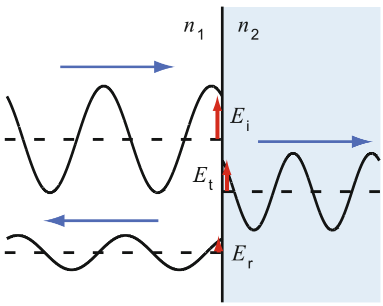


Fig. 10.1 An electromagnetic wave travelling perpendicular to another medium is partially reflected and partially transmitted. The waves are depicted separately in order to indicate instantaneous electric fields for each of them

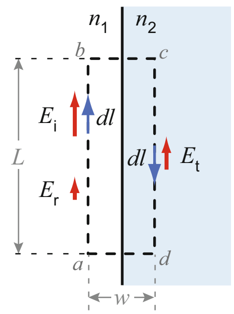


Fig. 10.2 Integration path (*blue arrows*) and electric field (*red arrows*) defining positive directions when applying Faraday's law to find relations between electric fields from different components. See the text for details

wave at the interface itself. We proceed tentatively, and calculate the signs shown in the figure, and we will discuss the details later.

First step: Faraday's law

We choose the rectangular integration path shown in Fig. 10.2 with a length L and width w. The integration path is oriented so that the long sides are parallel to the electric fields of the electromagnetic wave. We are ready to apply Faraday's law:

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = -\left(\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}\right)_{\text{inside}} . \tag{10.1}$$

We deal with the line integral first:

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \int_{ab} + \underbrace{\int_{bc}}_{=0} + \int_{cd} + \underbrace{\int_{da}}_{=0}$$
$$= (E_i + E_r)L - E_t L \; .$$

The integrals along bc and da contribute nothing because the paths are perpendicular to the electric fields; the first integral is positive, and the last negative, because, as shown in Fig. 10.2, the field is oppositely directed to the line element in the latter case.

As for the right-hand side of Eq. (10.1), our assumption that the interface is infinitely thin makes it permissible to choose w, and therefore the area $A = Lw$, to be arbitrarily small. Next, we express Φ_B as a surface integral, and get the simple result:

$$-\left(\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}\right)_{\text{inside}} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_A \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{A}} \approx 0$$

the last step being a consequence of the smallness of the area A.

The foregoing manipulations of Eq. (10.1) lead us to the result:

$$E_i L + E_r L - E_t L = 0$$

which implies that

$$E_i + E_r = E_t \ . \qquad (10.2)$$

We can apply a similar reasoning to the Ampére-Maxwell law to get

$$H_i + H_r = H_t \ .$$

Second step: Energy conservation

We can also set up an *energy balance sheet*: All energy incident per unit time on the interface must be equal to the energy that leaves the interface per unit time. We know from Chap. 9, that the intensity of an electromagnetic wave is given by:

$$I = cu_{\text{tot}} = \tfrac{1}{2}c\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{D}} = \tfrac{1}{2}c\varepsilon_0\varepsilon_r E^2$$

where u_{tot} is the energy density in the wave, and c is the speed of light in the medium under consideration. The energy balance sheet comes out to be:

$$\tfrac{1}{2}c_1\varepsilon_0\varepsilon_{r1}E_i^2 = \tfrac{1}{2}c_1\varepsilon_0\varepsilon_{r1}E_r^2 + \tfrac{1}{2}c_2\varepsilon_0\varepsilon_{r2}E_t^2 \ ,$$

$$c_1\varepsilon_{r1}(E_i^2 - E_r^2) = c_2\varepsilon_{r2}E_t^2 \ ,$$

$$c_1\varepsilon_{r1}(E_i + E_r)(E_i - E_r) = c_2\varepsilon_{r2}E_t^2 \ .$$

But, since $E_i + E_r = E_t$:

$$c_1\varepsilon_{r1}(E_i - E_r) = c_2\varepsilon_{r2}E_t \ .$$

Let us examine the constants appearing above. To this end, we recall the expression, given earlier in this chapter, for the speed of light:

$$c_1 = \frac{c_0}{n_1} \approx \frac{c_0}{\sqrt{\varepsilon_{r1}}} \ .$$

Multiplying by ε_{r1} and replacing the $\approx$ sign with equality, we get

$$c_1\varepsilon_{r1} = \frac{c_0}{\sqrt{\varepsilon_{r1}}}\varepsilon_{r1}$$
$$= c_0\sqrt{\varepsilon_{r1}} = c_0 n_1 \ .$$

Substituting this expression (and its counterpart for medium 2) in Eq. (10.2), we obtain

$$n_1(E_i - E_r) = n_2 E_t \ . \tag{10.3}$$

Third step: Combine

We combine now Eqs. (10.2) and (10.3) and eliminate, to begin with, E_t in order to find a relation between E_i and E_r:

$$n_1 E_i - n_1 E_r = n_2 E_i + n_2 E_r$$
$$(n_1 - n_2)E_i = (n_1 + n_2)E_r \ .$$

The ratio between the amplitudes of the reflected and transmitted waves is found to be:

$$\frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2} \ . \tag{10.4}$$

We see that the right-hand side can be positive ($n_1 > n_2$), negative ($n_1 < n_2$) or zero ($n_1 = n_2$).

For $n_2 > n_1$, the ratio is negative, which means that E_r has a sign opposite to that of E_i (i.e. to say, E_r is in the opposite direction to that indicated in Fig. 10.1).

For $n_2 < n_1$, the expression in Eq. (10.4) is positive, which means that E_r has the same sign as E_i (i.e. E_r has the direction shown in Fig. 10.1).

Let us conclude by combining Eqs. (10.2) and (10.3) by eliminating E_r in order to find a relation between E_i and E_t. This gives:

$$n_1 E_i - n_1 E_t + n_1 E_i = n_2 E_t \ .$$

The ratio between the amplitudes of transmitted and incident waves is easily found:

$$\frac{E_t}{E_i} = \frac{2n_1}{n_1 + n_2} \ . \tag{10.5}$$

We see that the electric field of transmitted wave has always the same sign as that of the incident wave.

Equations (10.4) and (10.5) provide the relationship between electric fields on both sides of the interface. When we judge how much of the light is reflected and transmitted, we want to look at the intensities. We have already seen that the intensities are given by expressions of the type:

$$I_i = \frac{1}{2} c_1 \varepsilon_0 \varepsilon_{r,1} E_i^2 \approx \frac{1}{2} c_0 \varepsilon_0 n_1 E_i^2 \ .$$

We are led to the following relation between the intensities:

$$\frac{I_r}{I_i} = \frac{n_1 E_r^2}{n_1 E_i^2} = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 \tag{10.6}$$

and

$$\frac{I_t}{I_i} = \frac{n_2 E_t^2}{n_1 E_i^2} = \left(\frac{2n_1}{n_1 + n_2} \right)^2 \times \frac{n_2}{n_1} \ . \tag{10.7}$$

If we choose to look at what happens at the interface between air and glass (refractive index 1.00 and 1.54, respectively), we get:

Reflected:

$$\frac{I_r}{I_i} = \left(\frac{0.54}{2.54} \right)^2 \approx 0.045 \ .$$

Transmitted:

$$\frac{I_t}{I_i} = \left(\frac{2}{2.54} \right)^2 \times 1.54 \approx 0.955 \ .$$

Thus, we see that about 4.5% of the intensity of light normally incident on an air–glass surface is reflected, while about 95.5% is transmitted. This is the case when the glass surface has not received any special treatment (“anti-reflection coating”).

Finally, it may be noted that the reflection at the surface leads to the creation of some standing waves in the area in front of the interface.

10.3 Obliquely Incident Waves

10.3.1 Snel's Law of Refraction

Willebrord Snel of Royen was born in the Netherlands in 1580. He later changed his name to Willebrord Snellius and died in 1626. His name should be written either as Snel or Snellius, but it is most commonly spelled as Snell. We have chosen the original name Snel.

Snel's law of refraction gives us the relation between the inclination of a light ray before it strikes an interface between two materials and its inclination after the interface.

The law of refraction can be derived in several ways. We will use "Fermat's principle" which is also called *principle of minimum time*. Fermat's principle is expressed in our times by saying that *the optical path length must be stationary*. Speaking a little imprecisely, this means that for the route along which light transports energy ("where light actually goes"), optical path length is the same (in the first approximation) for an array of optical paths that are close to one another. This means that the optical path length must be a maximum, minimum or stationary for small variations in the selected path. When we deduce Snel's law of refraction, we use the minimum as the criterion.

We refer to Fig. 10.3. A beam of light is sent from the point P in a medium with refractive index n_1 to P' in a medium with refractive index n_2. We assume in the figure that $n_2 > n_1$. Since light travels faster in medium 1 than in medium 2, the shortest time to cover the distance between the two points will be achieved by travelling a little longer in medium 1, instead of travelling along the straight line connecting the two points. If we use the symbols in the figure, it follows that the time for travel is:

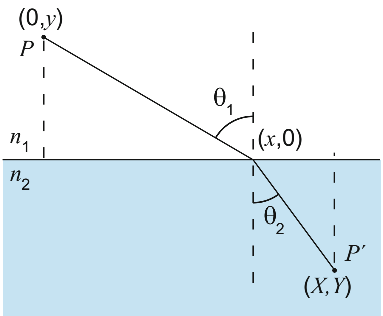


Fig. 10.3 In the derivation of Snel's law of refraction, we use the coordinates given in this figure. The angles θ_1 and θ_2 are called the "incident angle" and the "refraction angle", respectively. See also the text

$$t = \frac{\sqrt{x^2 + y^2}}{c_0/n_1} + \frac{\sqrt{(X - x)^2 + Y^2}}{c_0/n_2}$$
$$= \frac{1}{c_0}\left(n_1\sqrt{x^2 + y^2} + n_2\sqrt{(X - x)^2 + Y^2}\right) .$$

The independent variable here is x and the minimum time can be determined by setting $\mathrm{d}t/\mathrm{d}x = 0$, and this gives:

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{c_0}\left(n_1\frac{\frac{1}{2} \times 2x}{\sqrt{x^2 + y^2}} + n_2\frac{\frac{1}{2}(X - x) \times 2 \times (-1)}{\sqrt{(X - x)^2 + Y^2}}\right) = 0 ,$$
$$n_1 x\sqrt{(X - x)^2 + Y^2} - n_2(X - x)\sqrt{x^2 + y^2} = 0 ,$$

$$\frac{n_1}{n_2} = \frac{(X - x)\sqrt{x^2 + y^2}}{x\sqrt{(X - x)^2 + Y^2}} = \underbrace{\frac{X - x}{\sqrt{(X - x)^2 + Y^2}}}_{\sin\theta_2} \times \underbrace{\frac{\sqrt{x^2 + y^2}}{x}}_{1/\sin\theta_1} .$$

We arrive finally at the refraction law commonly attributed to Snel:

$$\frac{n_1}{n_2} = \frac{\sin\theta_2}{\sin\theta_1}$$

or

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 . \tag{10.8}$$

Fermat's principle has clear links to Huygen's principle and also the thinking behind quantum electrodynamics (CED). The waves follow all possible ways, but in some cases the waves reinforce each other, and in other cases, they will oppose each other. In other words, it is interference that is actually running the show, and the central idea behind this phenomenon is the role played by the relative phases of the different contributions. The "minimum time" criterion achieves the desired result automatically, since minimum time means that many waves, which we can imagine to have been sent from P, will take close to the minimum travel time and all these waves will automatically have the same phase and therefore interfere constructively.

10.3.2 Total Reflection

Total reflection is of course an important effect that anyone who likes to dive underwater knows well. The point is that if light goes from a medium with refractive index n_1 to a medium with index n_2 and $n_1 > n_2$, the "incidence angle" θ_1 will be smaller than the "refraction angle" θ_2 for the transmitted beam. We can first send the beam normally to the interface and then gradually increase the angle of incidence. The refraction angle will then gradually increase and will always be greater than the incidence angle.

Sooner or later, we will have an angle of incidence that leads to a refraction angle of almost 90°. If we increase the angle of incidence further, we will not be able to satisfy Snel's law, because the sine of an angle cannot exceed unity.

The incidence angle (θ_c) for which the angle of refraction is 90°, called the "critical angle", is found by setting $\theta_1 = \theta_c$ and $\theta_2 = 90°$ in Snel's law:

$$n_1 \sin\theta_c = n_2 \sin\theta_2 = n_2 \sin 90° = n_2 \ .$$

The critical angle of incidence can be expressed as:

$$\sin\theta_c = \frac{n_2}{n_1} \ . \tag{10.9}$$

If the angle of incidence is increased beyond the critical angle, there will no longer be a transmitted beam. Everything will be reflected from the interface back into the original medium, leading to a phenomenon called *total reflection*.

If we are underwater and look up at the surface, the critical angle will be given by:

$$\sin\theta_c = \frac{1.00}{1.33}$$
$$\theta_c = 48.8° \ .$$

If we try to look at the surface along a greater angle than this (relative to the vertical), the water surface will merely act as a mirror.

Total reflection is used to a large extent in today's society. Signal cables for the Internet and telephony and almost all information transfer now largely take place via optical fibres. For optical fibres having a diameter that is many times the wavelength (so-called multimode fibres), it is permissible to say that total reflection is at work here.

An optical fibre consists of a thin core of super-clean glass. Outside this core is a layer of glass whose refractive index is very close to, but slightly less than that of the core, the difference being about 1%. The consequence is that the critical angle becomes very close to 90°. This means that only the light that moves very nearly parallel to the fibre axis is reflected at the interface between the inner core and the next layer of glass outside. It is important that the waves are as parallel as possible to the axis so that pulses transmitted into the fibre should retain their shape before being relayed.

In many optical fibres, the diameter of the inner glass core is only a few times the wavelength. Such fibres are called single-mode fibres, and most are used in telecommunications and similar applications. For single-mode fibres, it is really misleading to explain the waveform in the fibre with total reflection. Instead, we must use Maxwell's equations directly with the given geometry. The wave image inside the fibre can no longer be considered a plane wave as we find it in vacuum far from the source and from disturbing boundary conditions. The boundary conditions imply a completely different solution. We will come back to this when we deal with waveguides in Chap. 16.

Single-mode fibres are challenging to work with because the cross section of the fibre is very small and the light entering the fibre must have a direction very close to the fibre direction. It is therefore difficult to get light *into* the fibre without too much loss. Standardization of coupling devices, however, makes it easy for telecommunication equipment, but it is quite a challenge to connect light into a fibre from a beam in air in a laboratory.

It is much easier to get light into multimode fibres because they have larger cross sections and the direction of the incoming light is not as critical. Multimode fibres, however, are not suitable for long-distance communications since pulses "fade out" after travelling relatively short distances.

10.3.3 More Thorough Analysis of Reflection

We will now look more closely at reflection and transmission when a (more or less) plane electromagnetic wave strikes obliquely an interface between two media. We make the same assumptions as mentioned in the beginning of the chapter that the interface is plane, "infinitely large and infinitely thin".

> A major challenge in the derivation that will follow consists of keeping track of geometry. Waves are inclined towards the interface, and the outcome depends on whether the electric field that meets the interface is parallel to the interface or inclined obliquely with respect to it. You may want to spend some time to understand the decomposition of the electric field vector E in Fig. 10.4 before reading further.

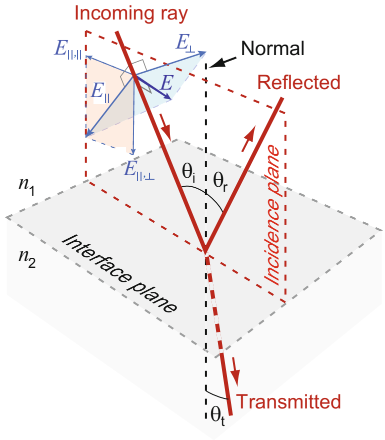


Fig. 10.4 Geometrical details for discussing the propagation of an electromagnetic ray inclined obliquely towards a plane interface between two media. The electrical field vector of the ray is resolved into a component normal to the incidence plane and a component parallel to this plane (lying in the plane of incidence). The latter component is further resolved into a component that is parallel to the interface and one that is normal to the interface. See the text for details

We draw a "ray" travelling obliquely towards the interface. We draw a normal to the interface at the point where the ray meets the interface. The plane containing the incident ray and the normal will be called the *plane of incidence*. The angle between the incident beam and the normal will be denoted by θ_i. See Fig. 10.4.

The reflected beam will lie in the input plane and have the same angle with the incident ray as the incident beam, i.e. $\theta_i = \theta_r$. The transmitted beam will also be in the same plane as the other rays, but it makes an angle θ_t with the normal (extended into medium 2).

We shall not go into any detailed proof that the three rays are in the same plane, but Maxwell's equations are symmetrical with regard to time. It is believed that if one solution of Maxwell's equations is an incident beam that divides into a reflected and a transmitted beams, then another solution is that where the reflected and transmitted waves can be considered as two incident rays coming *against* the interface and combining into a single output ray (similar to the original incident ray, but with the opposite direction of motion).

Since we can reverse, at least hypothetically, the time course for what is happening, it means that the solution must have a certain degree of symmetry. One consequence is that the three rays must lie in the incidence plane.

We *start* by assuming that all three rays lie in the incidence plane and $\theta_i = \theta_r$ in Fig. 10.4, and then use Maxwell's equations to determine how much of the incoming energy that is reflected and transmitted at the interface.

However, the wave has an arbitrary polarization. This means that the electric field $\vec{E}$, which is perpendicular to the incident beam, may have any angle relative to the incidence plane. The result for the component of electric field which lies in the incidence plane $E_{\parallel}$ is slightly different from that for the component perpendicular to the incidence plane $E_{\perp}$.

First step: $E_{\perp}$

We start by treating the component of electric field perpendicular to the incidence plane. This component will at the same time be parallel to the interface, which was also the case for the wave incident at the interface (discussed in the previous section). Faraday's law used as in Fig. 10.2 gives as before:

$$E_{i,\perp} + E_{r,\perp} = E_{t,\perp}$$

where i, r and t again represents incoming, reflected and transmitted. $\perp$ indicates the component that is perpendicular to the incident plane, which in turn is parallel to the interface. However, we do not pursue this component in detail.

It is more interesting to look at the component that lies in the incidence plane, but the treatment here is a little more complicated. The component *lying in the incidence plane* can be resolved into a component that is *normal to the interface* and one that is *parallel to the interface*.

In Fig. 10.4, we have tried to indicate that the electric field of the incoming wave has components both normal and parallel to the *incidence plane*, and that the latter component, $E_{\parallel}$, can in turn be resolved into a component $E_{\parallel,\parallel}$ parallel to and a component $E_{\parallel,\perp}$ perpendicular to the *interface/boundary surface* (Note: For simplicity reasons, we drop the vector notation for all components of the electric fields.).

In Fig. 10.5, *only* the component of the electric field parallel to the incident plane is drawn. Decomposition of this component is, respectively, $E_{\parallel,\parallel}$ and $E_{\parallel,\perp}$. The first

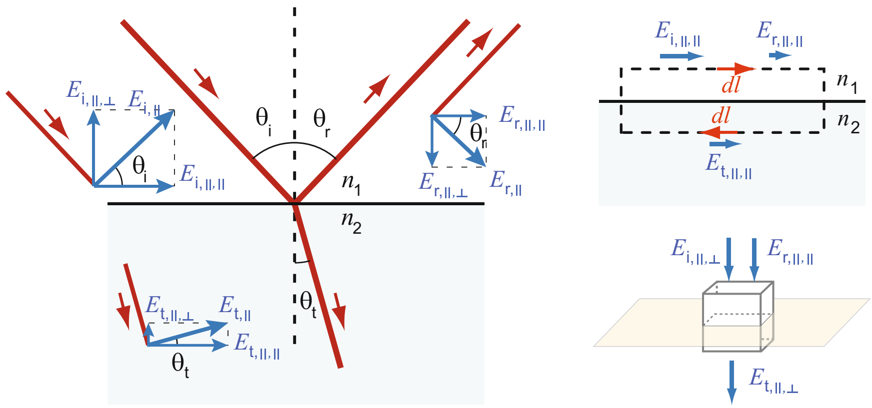


Fig. 10.5 Components of the electric field in the incoming plane for incoming, reflected and transmitted rays. The field decomposition on the left is drawn separately for the incoming, reflected and transmitted rays so as to avoid clutter. The diagrams on the right specify the positive directions of the components in the mathematical treatment. See also the text for details

part of the subscript indicates component relative to the incidence plane, and the second part indicates the component with respect to the interface.

From Fig. 10.4, we see that $E_{\|,\|}$ is perpendicular to $E_{\perp}$ (the component of electric field normal to the incidence plane), although both are parallel to the interface. Also note that $E_{\|,\perp}$ is perpendicular to the interface and thus parallel to the normal (defining the plane of incidence).

Second step: $E_{\|,\|}$

We can apply Faraday's law to the $E_{\|,\|}$ components of incident, reflected and transmitted waves, and we find, just as for waves incident normally on the interface:

$$E_{i,\|,\|} + E_{r,\|,\|} = E_{t,\|,\|} \ .$$

The positive direction is defined in the right part of the figure. It follows then that:

$$E_{i,\|} \cos\theta_i + E_{r,\|} \cos\theta_r = E_{t,\|} \cos\theta_t \ .$$

Since $\theta_i = \theta_r$, we are finally led to state:

$$E_{i,\|} + E_{r,\|} = \frac{\cos\theta_t}{\cos\theta_i} E_{t,\|} \ . \tag{10.10}$$

Third step: Gauss' law

We need yet another equation to eliminate one of the three quantities in order to find a relation between the other two. For the case where the ray was normal to the interface, we used an energy balance sheet to get an equation. In the present case of oblique incidence, it will not be so easy, since we have to take into account many components at the same time. Instead, we choose to use Gauss's law for electric fields on a small cube with surfaces parallel to the interface and the incidence plane. The cube has sides with area A and normal to the $d\vec{A}$, and we write:

$$\oint \vec{D} \cdot \mathrm{d}\vec{A} = Q_{\text{free,enclosed}} \ .$$

The advantage of this choice is that all components of the electric field that are parallel to the interface will give zero net contribution to the integral. They enter and leave the side surfaces in the same medium, and these field components are approximately constant along the surface as long as we allow the cube to have a side length small compared with the wavelength. On the other hand, we get contributions from the component that is normal to the end faces of the cube that are parallel to the interface; see Fig. 10.5. By specifying how we define positive field directions in the right part of the same figure, follow:

$$D_{i,\parallel,\perp} + D_{r,\parallel,\perp} = D_{t,\parallel,\perp} \ ,$$

$$\varepsilon_0 \varepsilon_{r1} E_{i,\parallel,\perp} + \varepsilon_0 \varepsilon_{r1} E_{r,\parallel,\perp} = \varepsilon_0 \varepsilon_{r2} E_{t,\parallel,\perp} \ .$$

We use now the relation $n \approx \sqrt{\varepsilon_r}$, and get:

$$n_1^2 E_{i,\parallel,\perp} + n_1^2 E_{r,\parallel,\perp} = n_2^2 E_{t,\parallel,\perp} \ .$$

Using the definition of positive directions for the vectors in the right part of Fig. 10.5, it follows that:

$$-n_1^2 E_{i,\parallel} \sin\theta_i + n_1^2 E_{r,\parallel} \sin\theta_r = -n_2^2 E_{t,\parallel} \sin\theta_t \ .$$

We invoked Snel's law of refraction (derived above):

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

and moreover $\theta_i = \theta_r$. We then eliminate θ_t and get:

$$-n_1^2 E_{i,\parallel} \sin\theta_i + n_1^2 E_{r,\parallel} \sin\theta_i = -n_2 E_{t,\parallel} n_1 \sin\theta_i \ .$$

Dividing throughout by $n_1^2 \sin\theta_i$, we get:

$$E_{i,\parallel} - E_{r,\parallel} = \frac{n_2}{n_1} E_{t,\parallel} \ . \tag{10.11}$$

Fourth step: Combining

We now have two equations that connect $E_\parallel$ for incoming, reflected and transmitted waves. One equation can be used for eliminating one of the three quantities and obtaining the relationship between the two others. For example, if we subtract Eq. (10.10) from Eq. (10.11), we get:

$$2E_{r,\parallel} = \left(\frac{\cos\theta_t}{\cos\theta_i} - \frac{n_2}{n_1} \right) E_{t,\parallel} \ . \tag{10.12}$$

Details, Brewster angle

Equation (10.12) is in fact interesting in itself, because it shows that the contents of the parenthesis can be made to vanish. When this happens, no part of the incident wave will be reflected if E lies in the incidence plane (because then $E_\perp = 0$). The incident angle θ_i where this happens is called *the Brewster angle*. Let us explore this special case in some detail. The condition is that:

$$\frac{\cos\theta_t}{\cos\theta_i} = \frac{n_2}{n_1} \ .$$

Using Snel's law once again, we get:

$$\frac{\cos\theta_t}{\cos\theta_i} = \frac{\sin\theta_i}{\sin\theta_t}$$

$$\sin\theta_i \cos\theta_i = \sin\theta_t \cos\theta_t \ .$$

We know that $\sin(2x) = 2\sin x \cos x$, thus

$$\sin(2\theta_i) = \sin(2\theta_t) \ .$$

We also know that $\sin x = \sin(\pi - x)$, which implies

$$\sin(2\theta_i) = \sin(\pi - 2\theta_t) \ .$$

This relation will be satisfied if

$$2\theta_i = \pi - 2\theta_t \qquad \text{or} \qquad \theta_i = \pi/2 - \theta_t \ .$$

With $\theta_i = \theta_r$, we are finally led to the result:

If

$$\theta_r + \theta_t = \pi/2 \tag{10.13}$$

there will be no reflected light with polarization parallel to the incidence plane. Then, the angle between the reflected and transmitted rays equals $\pi/2$ as indicated in Fig. 10.6.

Since the angles of incidence and reflection are equal, it is easy to show that the angle where we have no reflected light with polarization in the incidence plane is characterized by the angle between reflected and transmitted rays being 90°.

We wish to find an expression for the angle ($\theta_i \equiv \theta_B$) for which this holds, and start with:

$$\frac{n_2}{n_1} = \frac{\cos\theta_t}{\cos\theta_i}$$

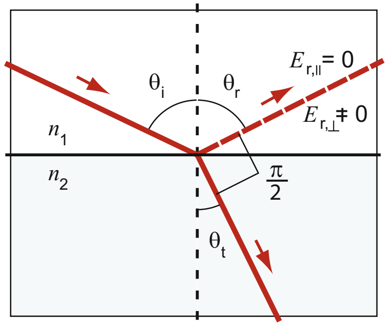


Fig. 10.6 When the angle between the reflected and transmitted rays is 90°, there is no electric field parallel to the incident plane in the reflected ray

and combine this with $\cos\theta_t = \cos(\pi/2 - \theta_i) = \sin\theta_i$ to get:

$$\tan\theta_i = \frac{n_2}{n_1} \equiv \tan\theta_B \ . \tag{10.14}$$

The angle θ_B is called *Brewster's angle*. At the interface between air and glass with refractive index 1.54, we find:

$$\tan\theta_B = \frac{1.54}{1.00}$$
$$\theta_B \approx 57^\circ \ .$$

Since $\theta_r + \theta_t = \pi/2$, we can easily determine θ_t. The result is about 33°.

It may be worth noting that there will also be no reflection (for light with electric vector parallel to the incidence plane) if the light goes from glass to air. For this case, we have:

$$\tan\theta_B = \frac{1.00}{1.54}$$
$$\theta_B \approx 33^\circ \ .$$

In other words, the Brewster effect can occur when light enters a new medium, regardless of whether the refractive index becomes higher or lower! By comparison, total reflection (which we will return to if a little) occurs only when the light hits a medium with a lower refractive index.

Fig. 10.7 Unpolarized light reflected at an air–glass interface can be fully polarized when the angle of incidence is equal to the Brewster angle. These photographs show this. The picture on the left is taken without a polarization filter. The picture on the right is taken with a polarization filter oriented so as to let only light polarized parallel to the incidence plane. All reflection is removed at the Brewster angle, and we look directly at the curtains behind the glass window practically without any reflection. This means that, at the Brewster angle, all the reflected light is fully polarized in a direction perpendicular to the incidence plane (parallel to the air–glass interface). Note that reflections on the painted surface are affected similarly to reflections from the glass. NB: Many modern windows are now surface treated in different ways. Then, we do not get any direct interface between air and glass, and the Brewster effect as described disappears totally or in part

10.3.4 Brewster Angle Phenomenon in Practice

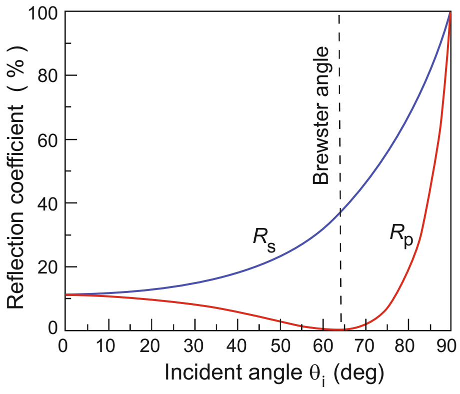
It is actually relatively easy to observe that light reflected from a surface at some angles is fully polarized.

The essential point is that ordinary unpolarized light can be decomposed into light with polarization parallel to the incidence plane and perpendicular to it. For the component parallel to the incidence plane, we can achieve zero reflection if the light comes in at the Brewster angle. In that case, the reflected light will be completely polarized normal to the incidence plane. We can observe this by using a polarization filter that only lets through light polarized in a certain direction. Figure 10.7 shows an example of this effect.

10.3.5 Fresnel's Equations

In order to arrive at relations involving reflection and transmission, we used Maxwell's equations, but these laws were derived long before Maxwell systematized electromagnetic phenomena in his equations. Fresnel derived equations which describe reflection and transmission already in the first half of the nineteenth century. You can read more about this e.g. in Wikipedia under the keyword "Fresnel equations". Here we will present only two formulas and a graph. In Eqs. (10.15) and (10.16), and in Fig. 10.8, the reflection coefficient is given for light fully polarized perpendicular to the incidence plane (R_s) and fully polarized parallel to the incidence plane (R_p) [The suffixes s and p are from German: *Senkrecht* (vertical) and *parallel*, respectively.]. The reflection coefficient refers to intensities, so in our language use, for example,

Fig. 10.8 Reflection and transmission coefficients of electromagnetic waves directed obliquely at an interface between two media with refractive index $n_1 = 1.0$ and $n_2 = 2.0$. The subscript s indicates that the electric field component of the wave is normal to the incidence plane, and the index p that the component is parallel to the incidence plane



$$R_s = \left(\frac{E_{r,\perp}}{E_{i,\perp}}\right)^2 .$$

The complete expressions can be written as follows:

$$R_s = \left(\frac{n_1 \cos\theta_i - n_2\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_i\right)^2}}{n_1 \cos\theta_i + n_2\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_i\right)^2}}\right)^2 , \tag{10.15}$$

and

$$R_p = \left(\frac{n_2 \cos\theta_i - n_1\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_i\right)^2}}{n_2 \cos\theta_i + n_1\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_i\right)^2}}\right)^2 . \tag{10.16}$$

The transmission can be found by using the relations $T_s = 1 - R_s$ and $T_p = 1 - R_p$.

If the light falling on the surface is totally unpolarized (with all polarizations equally present), the total reflection is given by $R = (R_s + R_p)/2$.

Figure 10.8 gives the reflection as a percentage for different angles of incidence. The figure applies to $n_1 = 1.0$ and $n_2 = 2.0$. For a wave that approaches the interface normally, the reflection is about 11% and of course independent of the polarization direction. The Brewster angle for these refractive indices is about 63°, and for this angle, the reflection is about 36% for waves polarized normally to the incidence plane.

Note further that the reflection coefficient goes to 1.0 (100%) when the angle of incidence goes to 90°. This applies to both components of the electric field.

10.4 Polarization

We have already mentioned polarization a great deal in this chapter, meaning the direction of the electric field vector when an electromagnetic wave travels through space.

However, polarization is not always in a particular *plane*. The electric field of an electromagnetic wave may change direction in a systematic manner as the wave moves. If we draw an electric field vector at closely spaced points along the line of propagation, the tip of all the field vectors may describe, for example, a helix with one turn per wavelength. In that case, the wave is said to be circularly polarized.

Figure 10.9 shows four different forms for polarization, where elliptical polarization is intermediate between linear polarization (polarization in a plane) and circular polarization.

It might seem that linear polarization is very different from circular, but the fact is that it is quite simple to switch from one to the other. Start by considering a plane

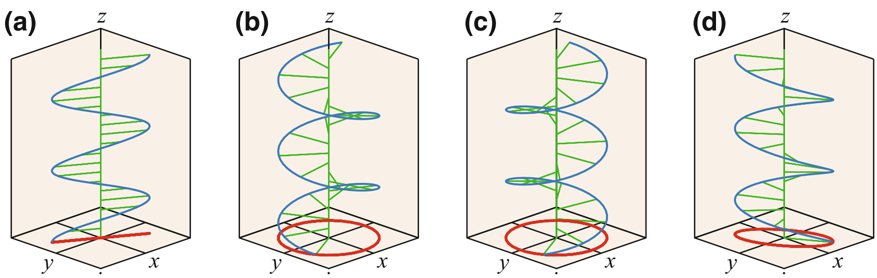


Fig. 10.9 Four different polarizations of a plane electromagnetic wave travelling in the z-direction. The green bars perpendicular to the z-axis indicate the size and the direction of the electric field at some z-values, all at the same time. The blue curves mark the tip of the electric field vector drawn from every point along the z-axis. The red curve shows the projection of the blue curve onto the xy-plane. **a** Plane polarized wave with the polarization plane −60° relative to the xz-plane. **b** Left-handed circular polarized wave. **c** Right-handed circular polarized wave. **d** Elliptically polarized wave, in this case 45% circular, 55% linear, the plane for the linear polarization is −30° relative to the xz-plane

linearly polarized electromagnetic wave moving in the z-direction. The polarization lies in a plane between the xz-plane and the yz-plane (similar orientation as in part a of Fig. 10.9). We can say that $E_x(t)$ and $E_y(t)$ vary "in step" or "in phase".

Mathematically, we can describe the wave on the left of Fig. 10.9 in the following way:

$$\vec{E} = E_x \cos(kz - \omega t)\,\vec{i} + E_y \cos(kz - \omega t)\,\vec{j}$$

where $E_x < E_y$.

If we delay the x-component by a quarter period compared to y-component (e.g. by using a quarter wave plate), and the amplitudes are equally large, polarization is circular (similar to c in Fig. 10.9), and the polarization follows a spiral as on a normal screw. We say that we have a right-handed circular polarization because the polarization direction follows our fingers on the right hand if we grasp the axis that indicates the direction of propagation, with the thumb pointing in this direction.

However, if we advance x-component by a quarter of a period compared to y-component, the polarization is left-handed circular (as for b in Fig. 10.9).

Mathematically, we can describe a left-handed circularly polarized wave (as **b** in Fig. 10.9) as follows:

$$\vec{E} = E_x \cos(kz - \omega t)\,\vec{i} + E_y \sin(kz - \omega t)\,\vec{j}$$

where $E_x = E_y$. The electric field in the x-direction is, as we see, shifted a quarter period (or a quarter wavelength) relative to the electric field in the y-direction.

The polarization of a plane electromagnetic wave can be specified either in terms of two plane polarized waves with orthogonal polarizations as basis vectors, or with a right-handed and a left-handed circularly polarized wave as basis vectors.

Be sure that you understand what is meant by a "plane, electromagnetic wave with (e.g. right-handed) circular polarization".

10.4.1 Birefringence

In the previous section, we claimed that it is easy to change from linear polarization to circular or vice versa. All that is needed is to change the phase of the time variation of one component of the electric field with respect to the other. But how do we achieve such a phase change in practice? Change in phase corresponds to a time delay, and a delay can be achieved if the wave moves more slowly when the electric field vector has one direction in space compared to when the field vector has a direction perpendicular to the first.

There exist materials in which waves polarized in one direction have a different velocity than waves polarized in a direction perpendicular to the first. This means that the refractive index is different for the two polarizations. Such materials are called birefringent (meaning doubly refracting).

A glass cannot be birefringent because it is matter in a disordered state, where bonds between atoms have all possible directions in space. To get a birefringent material, there must be a systematic difference between one direction and another, and this difference must be constant within macroscopic parts of the material (preferably an entire piece of the material). A birefringent material is therefore most often a crystal. Calcite crystals are a well-known example of birefringent material and will be described in some detail in the next sub-chapter.

It is interesting to note that birefringence was first described by Danish scientist Rasmus Bartholin in 1669.

It is possible to make a thin slice of a calcite crystal that has just the thickness required to delay the waveform by a quarter period in one component of electric field vector as compared to the perpendicular component perpendicular. Such a disc is called a “quarter wave plate”. A quarter wave plate will ensure that linearly polarized light is transformed into circularly polarized or vice versa. A quarter wave plate will only work optimally for a relatively narrow wavelength range. When such a plate is bought, the wavelength for which it is to be used must be specified.

Two different refractive indices in one and the same material give rise to a peculiar phenomenon. The upper part of Fig. 10.10 shows how a straight line looks when we see it through a calcite crystal oriented in a special way. The orientation is such that we see *two* lines instead of one. It is easy to understand the term “birefringent material” when we see such a splitting of an image.

We can imagine that the light from the line (surrounding area) has all possible linear polarization directions. Light with a particular polarization travels at a different speed compared to light polarized along a perpendicular direction. That is, the refractive indices for light with these two polarizations are different, which is why we see two lines through the crystal.

The last two pictures in the figure show how the line looks when we interpose a polarization filter between the crystal and our eyes. For a specific orientation of the filter, we allow the passage of light with only one polarization direction. By rotating the filter in one direction, we only see one line through the crystal. If we rotate the filter 90°, we only see the other line through the crystal. This is a good indication that the two refractive indices are linked to the polarization of the light through the crystal.

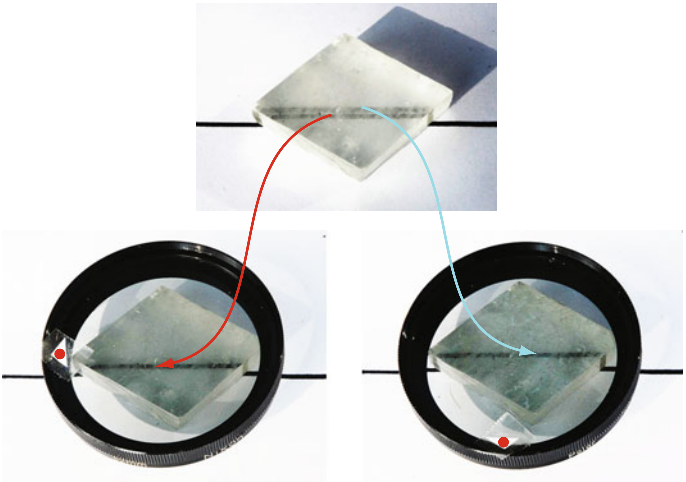


Fig. 10.10 Upper part of the figure shows a straight line viewed through a birefringent substance (oriented in a well-chosen manner). We see *two* lines! These are due to the fact that light with different polarization has different refractive indexes through the crystal. This can be demonstrated by holding a linear polarization filter in front of the crystal. If we orient the polarization filter in one way, we only see one of the two lines, but if we rotate the polarization filter by 90°, we see only the other line. A mark is made on the filter to show the rotation made between the two lower pictures

Remark: So far, we have set the relationship between electric field strength $\vec{E}$ and electric flux density (or the displacement vector) $\vec{D}$ as follows:

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$$

where ε_0 is the permittivity in empty space, and ε_r is the relative permittivity (also called the dielectric constant). Both of these quantities have been simple scalars, and therefore, the vectors $\vec{D}$ and $\vec{E}$ have been parallel.

In terms of the components, the equation can be written as:

$$D_i = \varepsilon_0 \varepsilon_r E_i \tag{10.17}$$

where $i = x, y,$ or z.

In birefringent materials, this simple description no longer holds. Electric field directed in one direction could provide the polarization of a material (e.g. calcite) also in a different direction. To incorporate this behaviour into mathematical formalism, the scalar ε_r must be replaced with a tensor with elements $\varepsilon_{r,i,j}$ where i and j correspond to x, y and z. Then, Eq. (10.17) is replaced by:

$$D_j = \varepsilon_0 \varepsilon_{r,i,j} E_i \ . \tag{10.18}$$

This is just one example of how a simple description needs refinement when a physical system displays properties that lie beyond the realms of the most elementary.

We mention these details to remind you that one of the tasks of physics is to provide mathematical modelling of the processes we observe. When the processes in nature are complicated, correspondingly complicated mathematical formalism is needed.

10.4.2 The Interaction of Light with a Calcite Crystal

All light originates in some process involving matter. When it is created, light acquires a polarization determined by the geometric constraints that are a part of the process whereby light is created. When light passes through vacuum, its polarization does not change, but as soon as it interacts with matter again, polarization can change. There are many different mechanisms that affect the polarization of light. This means that by studying change of polarization that accompanies the passage of light through matter, we can gain more knowledge of the material. A collective name for all such studies is "polariometry".

To get an idea of the mechanism responsible for the change in the state of polarization, let us discuss what happens when light is sent through a piece of mineral calcite. The chemical formula of calcite is $CaCO_3$, and we will consider calcite crystals. These are "birefringent"; that is, when we consider an object through a clear calcite crystal, the object looks double. The unit cell in a calcite crystal is relatively complicated.[1] Figure 10.11 is a perspective sketch of four $CaCO_3$ as some of the molecules are located within the unit cell. All the $CaCO_3$ molecules in the crystal are oriented so that the carbonate groups (CO_3^{2-}) are approximately in a plane perpendicular to a preferred direction called the optic axis. The orientation of the carbonate groups is such that there is a significant degree of rotational symmetry around the optic axis.

In Fig. 10.11b, we have indicated what happens when light passes the crystal with a polarization parallel to the carbonate plane. When the electric field is aligned as shown, the electron clouds around each atomic core will undergo a slight displacement relative to the core. Each atom then acquires a polarization (redistribution of electrical charge). Energy is stolen from the electromagnetic field of the light and temporarily stored in the polarization of the crystal. When the electric field then goes to zero and increases again in the opposite direction, it will induce polarization of the crystal again, but now with the opposite displacements of the electron clouds relative to the atomic nuclei.

However, we do not build more and more polarization as time passes. The stored energy in the polarization of the material will in some way act as "antennas" and generate electromagnetic waves. These waves have the same frequency as those which created the polarization originally. It is this polarization of the material and re-emitting of electromagnetic waves from the small induced dipoles in the material which causes light to move at a slower speed in the crystal compared with vacuum. As soon as the wave goes out of the crystal, there is no matter to polarize (when we ignore air) and the light velocity becomes the same as in vacuum.

Now comes something exciting! If we send light into the calcite crystal so that the electric field in the light wave has a direction *perpendicular* to the carbonate planes as in Fig. 10.11c, we will, as before, have displacement of the electron clouds relative to the atomic nuclei. But now the electron cloud is shifted across the carbonate planes.

[1] See for example Wikipedia for "calcite".

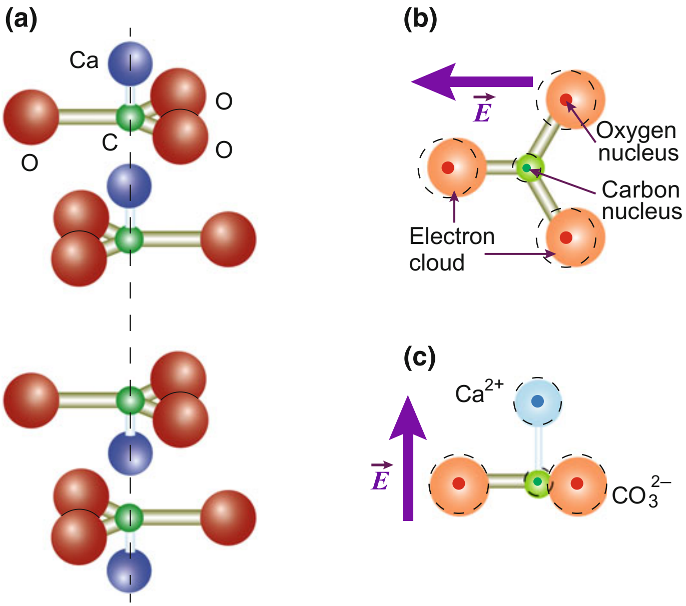


Fig. 10.11 Calcite is built up by the atomic groups $CaCO_3$. Part **a** gives a perspective drawing that indicates how these groups are oriented relative to each other. There is a large degree of symmetry around the direction marked with the dashed line, the so-called optic axis. In **b** and **c**, a snapshot of how an external electric field from passing light will polarize the atoms. Dashed circles indicate the location of the electron clouds when there is no external electric field. In **b**, the polarization is across the optic axis, whereas in **c** the polarization is along the optic axis

Due to the special symmetry of the crystal, light polarized in the direction of the symmetry axis will have a smaller charge polarization than when the polarization of light is perpendicular to the symmetry axis.

The result is that when light is transmitted through the crystal, the beam will be split into two, an "ordinary ray" with polarization normal to the optic axis and an "extraordinary ray" with a polarization perpendicular to the former. Snel's law does not apply. At about 590 nm, the refractive index for the ordinary ray is $n_o = 1.658$ and for the extraordinary ray $n_e = 1.486$.

It is quite natural that effects similar to that seen in calcite are observed only in crystalline materials, or at least materials with different properties in one direction compared to another (anisotropic media). However, we can have similar effects also for an initially isotropic material if it has been exposed to stress in a certain direction on account of which it is no longer isotropic. An isotropic plastic material can be made slightly anisotropic by, for example, bending or stretching it. By the way, some types of plastics are often slightly anisotropic if they are made by moulding where the molecules have been given a certain uniformity locally as the plastic was pressed into the mould from a particular point of feeding.

A Challenge
In this chapter, we have used the word "polarization" for two widely different conditions. We used the word when we mentioned different electric permittivities (which are connected with the difference between the electric field $\vec{E}$ and the electric field strength $\vec{D}$). This reflects how much we can deform the electron clouds relative to the nuclei of the atoms and generate a polarization (asymmetry) in charge distributions. We also used the word when we distinguished between e.g. linear and circular polarization. Make sure you fully understand the difference between these two different (but still related) terms with the same name. Otherwise, you should discuss with fellow students and/or teaching-assistant/lecturer.

10.4.3 Polarization Filters

Linear polarization filters
When we discussed the Brewster angle, we saw an example of a linear polarization filter. Roughly speaking, we can say that such a filter (if it is thick enough) peels off one component of the electric field vector in the electromagnetic waves (visible light). If the light is totally unpolarized initially, the intensity will be halved after the light has passes through a linear polarization filter.

What does the term "unpolarized light" mean? It is actually a little difficult to explain. We have seen that Maxwell's equations can give us plane and polarized (or circularly polarized) waves. Is that not true for all electromagnetic waves? Well, it is true, but light is usually generated from a very large number of sources that are independent of each other. When we turn on the light in a room, for example, light is generated from every fraction of a millimetre of filament in an old-fashioned light bulb and the light emitted from each part is independent of the other parts. All the waves pass through the room and the contributions will, to a large degree, overlap in time and space. The result is that if we follow the polarization at a tiny point in the room, polarization will still change during a fraction of a millisecond. There is also quite a different time development at a small point in the room and at another point just a few millimetres away from the first.

We shall describe such more or less chaotic waves in Chap. 15 when we refer to coherence. Unlike chaotic light (unpolarized light), for example, we have laser light and it is exciting to see how waves are added to each other, but that will come later.

Let us assume *for now* that we have a horizontal light ray with unpolarized light. Using a linear polarizing filter, we can make sure all the transmitted light has electric field that is aligned horizontally.

If we insert another such filter, and orient it just like the previous one, any light transmitted by filter 1 will be transmitted also by filter 2.

If filter 2 is rotated 90° so that it can only let through vertically polarized light, there will be no such light coming through filter 1. Then, *no* light will emerge from filter 2 (left part of Fig. 10.12).

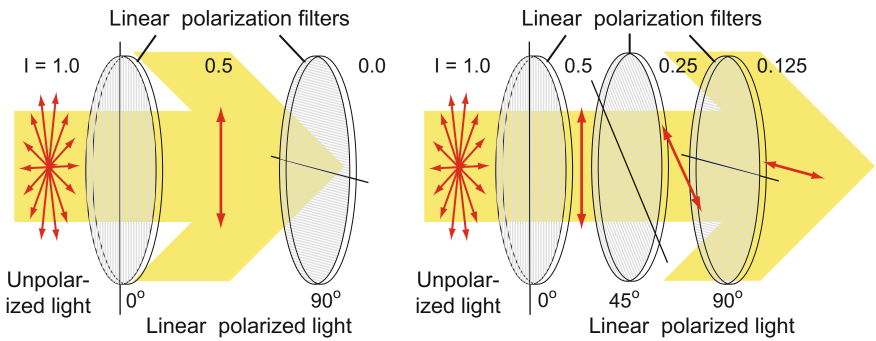


Fig. 10.12 Two sequential linear polarization filters aligned 90° apart, transmit no light (*left part*). If a third filter is placed between the first two, with the polarization direction different from the other two, some light will in fact go through the filters (*right part*)

However, if we, for example, rotate filter 2 by 45° relative to filter 1, light with horizontal polarization after filter 1 will actually have a component also in the direction of filter 2. Light that now passes filter 2 acquires polarization 45° relative to the polarization it had after filter 1. We *change* polarization, but the amplitude of the electric field is now less than what it was before filter 2 (Only the E field component in the direction of filter 2 is transmitted).

The intensity of the light passing through filter 2 is given by Malus's law:

$$I = I_0 \cos^2(\theta_2 - \theta_1) \; . \tag{10.19}$$

Here, I_0 is the intensity of the light after it has passed filter 1. The argument for the cosine function is the difference in the angle of rotation between filters 1 and 2. Malus's law only applies to linear polarization filters!

Now let us start with two polarizers with polarization axes perpendicular to each other, and place a third polarization filter between the first two. If we choose an orientation other than 90° relative to the first, we get light through all three filters (right part of Fig. 10.12). This is because the middle filter has changed the polarization of the light before it hits the last filter.

It is important to note that a polarization filter of this type actually plays an active role as it *changes* the polarization of light passing through it.

Remarks:
We will now present a picture that might serve as a useful analogue to what is happening in a linear polarization filter. Imagine that the filter consists of pendulums that can swing in only one plane. If we attempt to push the pendulums in a direction along which they can actually swing, the

pendulums will swing. A swinging pendulum can propagate its motion to a neighbouring pendulum of the same type, and so a wave can propagate through the material.

However, if we try to push the pendulums in a direction in which they *cannot* swing, there will be no oscillations. No wave can then propagate through the medium. If we push obliquely, the pendulums will swing, but only in the direction along which they can actually swing. This means that the swing direction in the wave will change when the wave propagates through the medium, but we get a reduction in the wave because only the component of our push that is along the swinging plane will be utilized in the oscillations.

10.4.3.1 Circular Polarization Filters in Photography *

A circular polarization filter is basically a filter that only lets through circularly polarized light. There are two variants of such filters, one type that allows right-handed circularly polarized light to pass through, and another type that passes through the left-handed circularly polarized light. A purely circular polarization filter has the same effect even if it is rotated around the optic axis.

Today, however, there is a completely different type of filter called the circular polarization filter. We are thinking of polarization filters used in photography. In many photographic devices, autofocus is based on circularly polarized light. If we want a polarization filter in front of the lens, the filter must be made such that circularly polarized light reaches a detector inside the device.

Such a circular polarization filter is assembled in a very special way. When light enters the filter, it first meets an ordinary linear polarization filter. Just behind this filter is a so-called quarter wave plate with a special orientation. As a result, the light is first converted into pure linearly polarized light, and subsequently converted into almost completely circularly polarized light. The light that enters the camera is therefore circularly polarized and the autofocus works.

We will look closely at the details in this context.

A quartz wave plate is made of a birefringent material, for example calcite. We have already seen that in a birefringent substance the phase velocity of light polarized in a certain direction differs from the phase velocity of light polarized perpendicular to the aforementioned orientation.

In polarization filter used in photography, the orientation of the birefringent substance is chosen so that the electric vector after the linear polarization filter forms 45° with each of the two special directions in the birefringent substance. We decompose the electrical vector as shown in Fig. 10.13. The E_o component will go through the substance with a certain phase velocity (i.e. a certain wavelength), while the E_e component passes through the substance at a different phase velocity (and wavelength).

By choosing a certain thickness of the birefringent substance, we can arrange E_o to have just one-quarter wavelength difference from what E_e has when it leaves the filter. In that case, we achieve exactly what we want, namely that linearly polarized light is transformed into circularly polarized light.

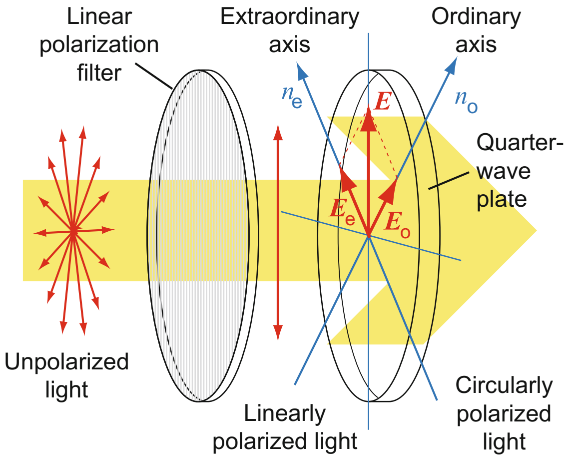


Fig. 10.13 Schematic drawing of a so-called circular polarization filter used in photography. The light passes through an ordinary linear polarization filter and then through a quarter wave plate. The two parts are close to each other. The orientation of the plate is chosen so that mean wavelengths are converted from linear to circular polarized light

Looking further at this argument, we discover that we cannot get a perfect transformation from linear to circular light for all wavelengths in the visible range at the same time. In practice, therefore, the thickness of the birefringent substance will be chosen such that the centre wavelengths (around spectral green) get optimal conversion while other wavelengths get a less perfect transformation. It does not matter because autofocus must have some light that is circularly polarized and does not need perfect circular polarization for all wavelengths.

Remarks:
A linearly polarized wave can be considered as a sum of a right-handed and a left-handed circularly polarized wave, and a circular polarized wave can be considered as a sum of two linearly polarized waves with polarization perpendicular to each other (and phase shifted). This means that we can combine circular polarization filters and linear polarization filters in different ways.

However, filter combinations in which the photographic circular polarization filters are included give a lot of surprises just because these filters are composed of two elements.

If we place two photographic circular polarizing filters with the inner surfaces facing each other, the light will pass through both filters with approximately the same intensity as after the first filter. Intensity is almost independent of the angle of rotation of one filter relative to the other. This is due to the fact that the light after passing the first filter is approximately circularly polarized and thus has a circularly symmetrical E-field distribution (when considering intensity).

If, on the other hand, we place two such filters with the outer surfaces against each other, two linear polarization filters will follow in succession in the light path. The pair of filters will then behave like two ordinary linear polarization filters, and the intensity of the emerging light is given by Malus's law (Eq. 10.19).

The special construction of the filters means that in photography, we achieve the same effect as with linear polarization filters, from the photographic point of view. Polarizing filters are used to remove reflection (as shown in Fig. 10.7) and to remove the effect of haze in the atmosphere (since light from the haze is partially linearly polarized). With the help of polarizing filters, we can achieve a big contrast between blue sky and white clouds, which adds extra life to the pictures (see Fig. 10.22).

10.4.4 Polariometry

We have previously seen that a polarization filter sandwiched between two crossed polarization filters causes light to escape through the combination of three filters. This gives us an excellent starting point for studying certain material properties. Any material that changes the polarization of light will ensure, when interposed between crossed polarizers, that some light passes through the arrangement. For example, many plastic objects will have differences in optical properties in different parts of the article depending on how the plastic material flowed into a mould prior to and during curing. Anisotropy in different parts of the material causes the polarization direction of light to rotate slightly or the conversion of some plane polarized light to circularly polarized light. The effect is often dependent on the wavelength. As a result, we can get beautiful coloured images through the crossed polarization filters if we send white light through the arrangement.

Figure 10.14 shows the image of a plastic box for small video cassettes in the crossed configuration. I could have used white light (e.g. from the sun or from a filament lamp), two crossed linear polarization filters and the plastic cassette holder between the filters. However, since the filters available to me were not as big as the cassette holder, I chose instead to use a computer screen as the source of plane polarized light. Many data monitors, cellular phone monitors and some other displays based on liquid crystal technology, give rise to plane polarized light. I placed the plastic holder directly against the computer screen and used a polarization filter just in front of the camera lens.

As shown in Fig. 10.14, anisotropies in the plastic are revealed well by polariometry. Variants of this method are used for many different materials and in many different contexts in industry and research. You can purchase specialized equipment for this type of analysis.

It may be useful to remember that light from, for example, mobile phones is usually linearly polarized. If you wear polarized glasses, you may experience a black screen and think something is wrong with your mobile phone, while the picture is completely normal when you do not wear the glasses!

10.4.5 Polarization in Astronomy

In recent years, several studies have been conducted on the polarization of the light from the sun and from distant light sources in the universe. Admittedly, it is not exactly what astronomers are primarily occupied with. Usually, the challenge is to gather enough light to get good pictures or spectroscopic data. If we insert a polarization filter, we lose half the intensity of light. And if we want information about the polarization of the light, we would like to have at least two photographs taken with light polarized along perpendicular directions. This means that a study based on a straightforward procedure will take at least four times as long as one image taken without paying any regard to polarization.

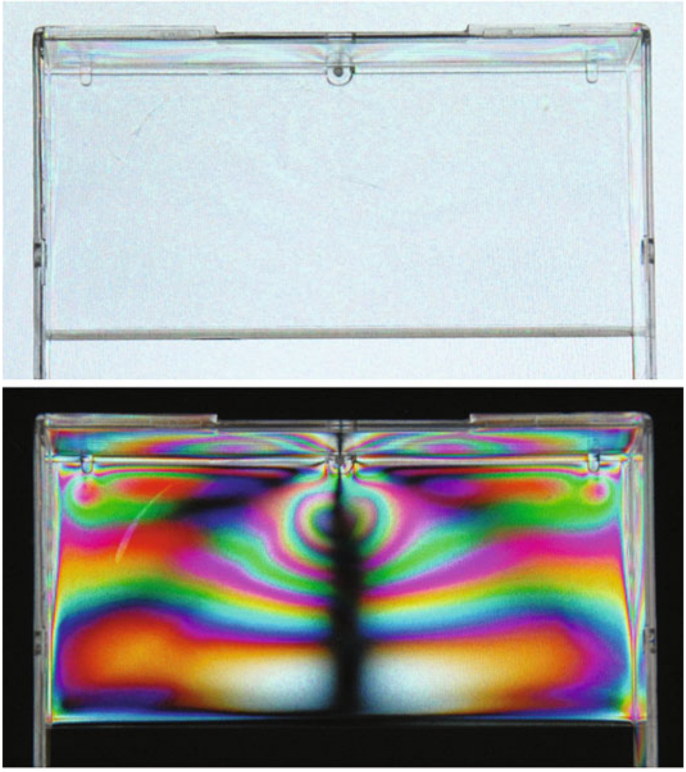


Fig. 10.14 Photography of a plastic box for small video cassettes in polarized light (*top*), and when sandwiched between two crossed linear polarizers (*bottom*). Light can be transmitted only if the polarization of the light transmitted by the first polarizer changes as it passes the plastic

The reason why polarization is still interesting in astronomy is much the same as for the polariometry of different materials. For example, let us consider light from the sun. The light may be emitted as unpolarized light in processes known as "blackbody radiation" (radiation from a hot body). However, the light will interact with plasma and atoms along its way to the earth. If the electrons in a plasma are influenced by a strong "quasi-static magnetic field", the movement of the electrons will not take place equally easily in all directions (remember the cross-product in the expression of the Lorentz force).

When for example the light from a part of the sun passes the electrons in a plasma, the electromagnetic field of the electromagnetic wave (the light) will set the electrons in the plasma into motion. Without any quasi-static magnetic field, the electrons will oscillate in the same direction as the electric field in the light, and the polarization will not change. But if there is a powerful quasi-static magnetic field present, the electron oscillation could get a different direction than the electric field of the light. The result is that a polarization is imparted to the light that is conveyed to us. The

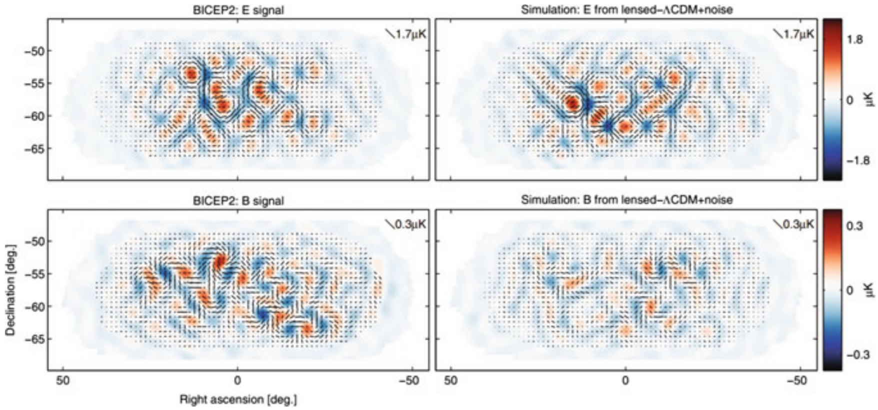


Fig. 10.15 Polarization in the electromagnetic waves originating from the Big Bang has recently been mapped. Figure taken from [1] under a CC BY-SA 3.0 license

direction of polarization will tell us something about the magnitude and direction of the quasi-static magnetic field in that part of the sun where the light came from.

A number of other factors can also affect the polarization of light from astronomical objects, and there is much that needs to be mastered! In the coming years, polariometry will undoubtedly give us information about astronomical processes that until recently was unavailable by any other means. Please read the article "Polarization in astronomy" in Wikipedia. There are also many postings and YouTube videos available on the Web that mention the so-called POLARBEAR Consortium project that uses the polarization of electromagnetic waves in space exploration (Fig. 10.15).

10.5 Evanescent Waves

In Chap. 9, we distinguished between near field and far field and pointed out that many known relationships between the electric and magnetic field of electromagnetic waves apply only in the far field. We mentioned that the near field extends no further than a few calculated wavelengths beyond the sources or structures that cause the near field.

This recognition has grown in the last decade and has had a big impact in e.g. optics. When we derived the expression for total reflection above, we relied entirely on Snel's law and on manipulating a few mathematical expression.

However, if we use Maxwell's equations in a more thorough analysis of total reflection, we will realize that *an electric field on the inside of the glass at*

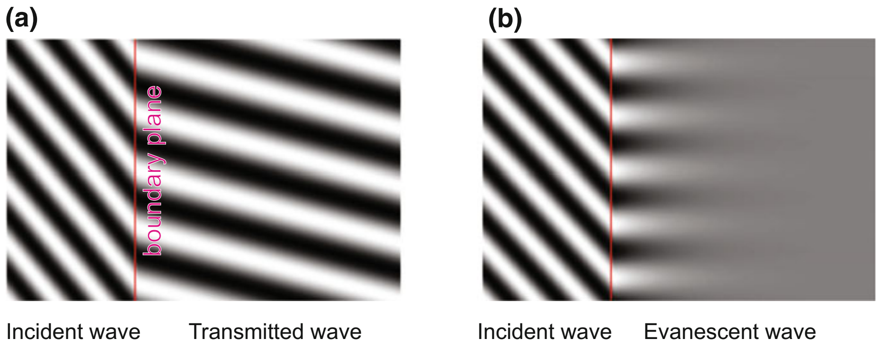


Fig. 10.16 Evanescent waves at total reflection (*right*). The boundary line between two media with different refractive indices is marked with a red line. Only incoming and transmitted/evanescent waves are displayed; that is, the reflected wave is not shown. Aluminum, CC BY-SA 3.0 GNU Free Documentation License 1.2. Modified from original [2]

total reflection cannot end abruptly at the interface between glass and air. The electric field must decrease gradually.

A detailed solution of Maxwell's equations for the region near the interface shows some kind of standing wave where the amplitude (and intensity) decreases exponentially when we move away from the interface (see Fig. 10.16). This standing wave is called an evanescent wave, and it fades over a distance of the order of a wavelength.

Evanescent waves are found in many situations, not only when total reflection takes place. A very important example is the interface between metal and air or metal and another dielectric. In the metal, however, electrons will move along the interface in a special way. We call this phenomenon "plasmons" ("surface plasmon polariton waves"). Plasmons (collective electron motion) are a powerful contributor to how the electromagnetic field will change in the region near the interface between the two materials, and consequently also the evanescent waves outside the metal.

Evanescent waves are now very popular in physics research, not least because we have also had a significant development in nanotechnology recently. Today we can make structures much smaller than the wavelength of light. The result is, among other things, that smart ways have been found to improve resolution, for example, in microscopy. In Chap. 13, we will describe diffraction, and according to the classical analysis of diffraction, we could never achieve a better resolution than the so-called diffraction-limited resolution. Today, however, for special geometries, we can surpass this limit.

The evanescent waves are primarily significant in distances less than about $\lambda/3$ away from the interface. There is room for much creativity over the coming years in the field of evanescent wave research and utilization of these!

10.6 Stereoscopy

People have a well-developed “depth perception”. The two pictures captured by our eyes are slightly different because the two eyes are 6–7 cm apart. This means that the viewing angle for nearby objects is different for each eye, but almost identical for remote objects.

Ever since the infancy of photography, people have experimented with taking pairs of photographs that correspond to the pictures formed on our retinas, the so-called stereoscopic pair. We could look at the pictures individually through special stereoscopes (see Fig. 10.17). This technique was also used in commercial “ViewMaster” binoculars that were highly popular in the 1970s. The technique is still employed in modern Virtual Reality technology.

Three-dimensional images can also be created by placing a stereoscopic picture pair on top of each other, where the image to the left eye is transmitted through a blue-green colour filter while the image to the right eye is through a reddish colour filter. The resulting image looks somewhat strange, with reddish and blue-green objects side by side (see Fig. 10.18). When such an image is viewed through so-called anaglyph glasses which are red on the left side and blue-green on the right, the blue-green parts of the image will be visible through the red filter (little colour drops and the object looks dark). The red parts of the image will pass through the red filter as well as white light and will only look white and become “invisible”.

The use of a crude form of colour coding serves its purpose in anaglyph glasses, but colour reproduction is not satisfactory for many purposes.

That is where polarization filters come in, and polarization of light is just perfect for this purpose. We have two eyes and need to “code” two pictures so that one picture

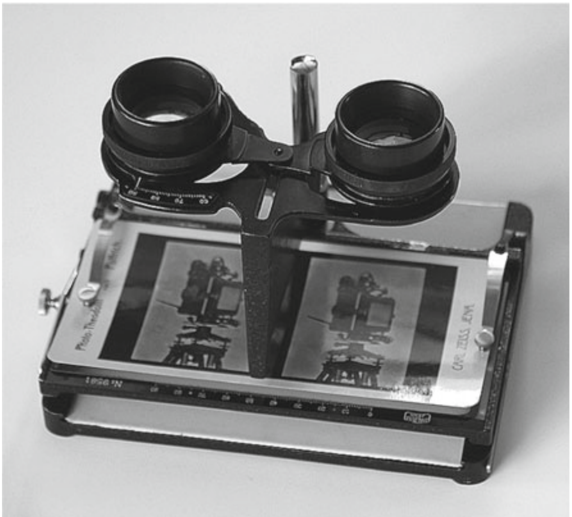


Fig. 10.17 Stereoscopic picture pairs with accompanying lenses were developed already over 100 years ago. The picture shows a 1908 stereo viewer used for looking at a pair of stereoscopic photographs

Fig. 10.18 Stereoscopic images can be made by adding two colour-coded images on top of each other. The images are traversed through coloured glasses ("anaglyph glasses") to ensure that a stereoscopic pair of photographs is perceived only by the eye each image is made for

reaches one eye and another picture the other eye. If we allow the light from one image to be horizontally polarized and the light from the other image is vertically polarized (we assume that we now look horizontally against the pictures), by using a horizontal axis polarization filter on one glass and vertical axis on the second glass, we will achieve exactly what we want.

The use of linearly polarized light works well as long as we see a movie and the head is held straight up (so that the interocular axis is horizontal). But if we cock our head at 45°, each of the eyes will get as much light from each of the two pictures. In that case, we will see double images on both eyes.

However, if we use circularly polarized light, we will forego this disadvantage. The two projectors needed for stereoscopy must provide right-handed and left-handed circularly polarized light. The glasses must have the corresponding polarization filters.

Several hundred movies have been recorded with stereoscopic techniques so far (see list on Wikipedia under "List of 3D movies"). Many stereoscopic televisions have been on the market. They are based on spectacles that makes sure that every alternate picture is presented to only one eye. Commercial photographic appliances and camcorders intended for the consumer market already exist. Only the future will show how large a share will stereoscopic images/movies get.

For the sake of curiosity, we conclude with a stereoscopic image that can be viewed without any aids (see next page). It is formed by many point pairs that are positioned so that one point in each pair fits one eye and the other point in the pair fits the other eye. A variety of books based on this principle have been made in many variants and with different effects (Fig. 10.19).

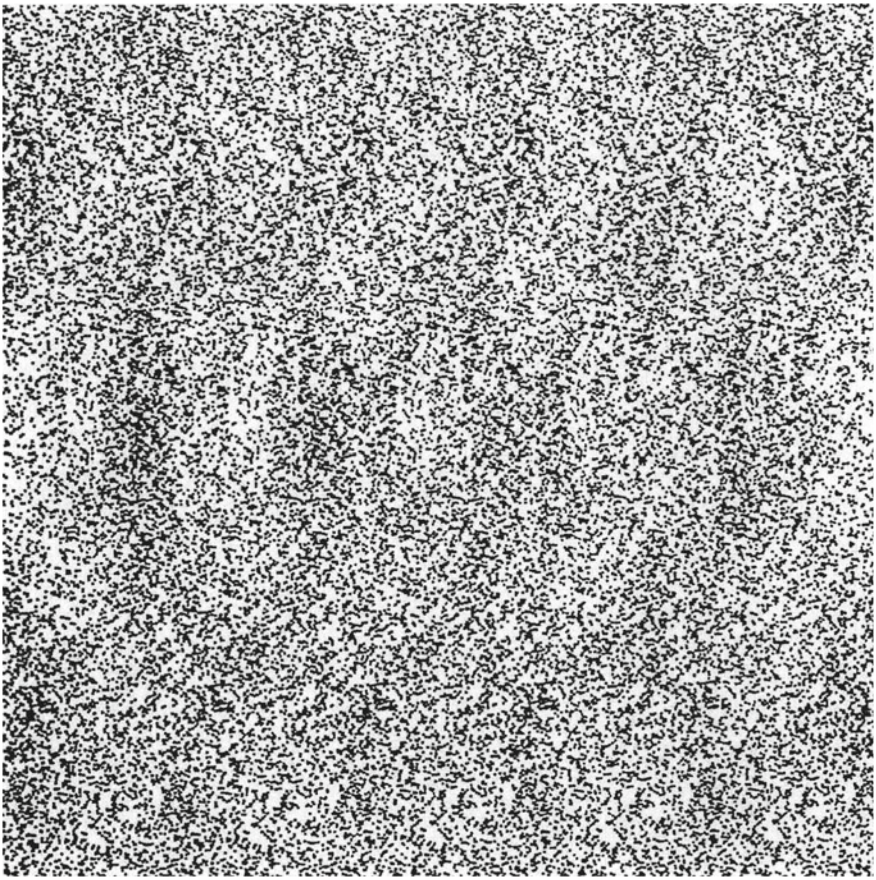


Fig. 10.19 Stereoscopic image made using dots. The illustration was made by Niklas En [3]. Reproduced with permissions. Bring the book (or screen) all the way up to the nose and let the book (screen) recede slowly, very slowly from the face. Do not try to focus on the dots in the image, but let the actual stereoscopic overall image come into focus (perhaps focusing on "infinite" at the start). This stereoscopic image will, when you finally notice it, appear to be almost as far back from the paper (screen) as your eyes are in front of the paper

10.7 Learning Objectives

After working through this chapter, you should be able to:

- Using Maxwell's equations to deduce on your own, the relationships between incident, reflected and transmitted waves when a planar electromagnetic wave strike normally a flat interface between two different dielectric materials.

- Explain "Fermat's principle" (also called the principle that optical distance must be stationary). Apply this principle to derive Snel's law of refraction and the law according to which "the angle of incidence equals the angle of emergence" when light is reflected from a flat surface.
- Explain the phenomenon "total reflection" and be able to give an example of the use of total reflection in modern technology.
- Explain the calculation of reflection and transmission when a planar electromagnetic wave is incident obliquely at a flat interface between two media (especially keeping track of the two components of the electric field in the calculations).
- Explain the phenomenon associated with the Brewster angle, and set up a mathematical expression for this angle.
- Define the reflection coefficient and transmission coefficient.
- Explain the difference between a linearly and a circularly polarized plane electromagnetic wave, and state mathematical expressions for the two examples.
- Explain what characterizes a birefringent material and explain how we can use such a material to transform a linearly polarized wave into a circularly polarized wave.
- Explain what happens when light is sent through several subsequent polarization filters, and be able to state Malus's law.

10.8 Exercises

Suggested concepts for student active learning activities: Homogeneous, thin interface, Fermat's principle, total reflection, optical fibre, incidence angle, refraction angle, Brewster phenomena, birefringence, quarter wave plate, polariometry, rotational symmetry, plane/circular/elliptical polarization, right-handed circular, linear polarizing filter, crossed polarizers, circular polarizing filter in photography, evanescent waves, stereoscopy, anaglyph glasses.

Comprehension/discussion questions

1. Can water waves and/or sound waves in the air be reflected and transmitted (as we have seen for transverse waves)?
2. When we see a reflection in a window, we often see two images slightly displaced in relation to each other. What causes this? Have you even seen more than two pictures every now and then?
3. You direct a laser beam to a glass plate. Can you achieve total reflection? Explain.

4. How can you decide if your sunglasses are of the polaroid type or not?
5. How can you determine the polarization axis of a single linear polarization filter?
6. Name two significant differences between total reflection and the Brewster angle phenomenon.
7. The speed of sound waves in air increases with temperature, and the air temperature can vary appreciably with height. During the day, the ground often becomes hotter than air, so that the temperature of the air near the ground is higher than slightly further up. At night, the ground is cooled (by radiation) and we can end up with the temperature in the air being lowest near the ground and rising slightly (before it gets cooler still further up). Can you use Fermat's principle to explain that we often hear sounds from distant sources better at night than in the day?
8. Why does the sea look bright and shiny when we look at a sunset in the ocean?
9. Is it possible to create a *plane* electromagnetic wave which is simultaneously *circularly* polarized. As usual: Justify the answer!
10. When referring to Fig. 10.4, it was said that Maxwell's equations are symmetrical with regard to time. If one solution is as given in Fig. 10.4, another solution will be the one where all the rays go in the opposite direction. Would it be possible to demonstrate this in practice? (Look especially at the light that comes from the bottom towards a interface. Would there not be a reflected beam down in this case?) Do you have any examples of experimental situations similar to this case?

Problems

11. A light source has a wavelength of 650 nm in vacuum. What is the velocity of light in a liquid with refractive index 1.47? What is the wavelength in the fluid?
12. Light passes through a glass that is completely immersed in water. The angle of incidence of a light beam that strikes the glass–water interface is 48.7°. This corresponds to the critical angle where the transition from some transmission to pure total reflection occurs. Determine the refractive index of the glass. The refractive index of water at 20 °C at 582 nm is 1.333.
13. Assume that one (multimode) optical fibre has a difference in the refractive index of 1% between the glass in the inner core where the light is and the surrounding layer of glass. Determine the maximum angle (relative to the fibre axis) the light may have and yet get total reflection. Determine the minimum and maximum time a short pulse will use to travel 1.0 km along the fibre (due to different angles of the light beam in the fibre). What will be the largest bit rate (pulses per second) that can be transmitted along the fibre (if we just consider this difference in efficient path length only)?
14. When a parallel unpolarized light beam hits a glass surface with an angle of 54.5°, the reflected beam is fully polarized. What is the value of the refractive index for the glass? What angle does the transmitted beam have?
15. A horizontal unpolarized ray of light passes through a linear polarization filter with polarization axis turned 25.0° from the vertical. The light ray continues through a new, identical polarization filter whose axis is turned 62.0° from the

vertical. What is the intensity of the light after it has gone through both filters compared to the intensity before the first filter?

16. A horizontal unpolarized ray of light passes through a linear polarization filter with polarization axis turned $+15.0^\circ$ from the vertical. The light ray continues through a new, identical polarization filter whose axis is rotated -70.0° from the vertical.
(a) What is the intensity of the light after it has gone through both filters compared to the intensity before the first filter?
(b) A third polarization filter, identical with the other two, is then put in, but now with the axis turned -32.0° from the vertical. The third filter is placed *between* the other two. What is the intensity of the light now going through all three filters?
(c) Would the result be different if the third filter was located *after* the other two instead of between them?

17. Show that if we send a thin beam of light through a flat glass plate of uniform thickness, the beam passing through the glass will have the same direction as the incoming beam, but with a parallel displacement. Show that the parallel displacement d is given by:

$$d = t \sin(\theta_a - \theta_b)/\cos(\theta_b)$$

where t is the thickness of the glass plate, θ_a is the angle of incidence, and θ_b is the angle between the normal and the refracted ray in the glass.

18. Show mathematically that the angle of incidence is equal to "angle of reflection" (angle between the reflected beam and the normal to the reflecting surface at the point of incidence) using Fermat's principle.

19. A birefringent material has a refractive index of n_1 for light with a certain linear polarization direction and n_2 for light with polarization perpendicular to the first. If this material is to be used as a quarter wave plate for light wavelengths of 590 nm, light with one polarization must travel a quarter wavelength more within the plate than light with perpendicular polarization. Show that the plate must have a (minimum) thickness given by:

$$d = \lambda_0/[4(n_1 - n_2)]$$

where λ_0 is the wavelength in vacuum (air). Find the minimum thickness of a quarter wave plate made of calcite ($n_o = 1.658$ and $n_e = 1.486$, where the suffix o in n_o stands for "ordinary" and can correspond to our n_2, while the suffix e stands for "extraordinary" and can correspond to our n_1). What is the next thickness that will provide quartz wave plate function? What is the function of a quarter wave plate?

20. Determine how much a beam of light is deviated if it passes through an equilateral triangular glass prism in such a way that the light beam inside the prism is parallel to a side surface. The glass has a refractive index of n.

Fig. 10.20 A "polarization filter" for radio waves

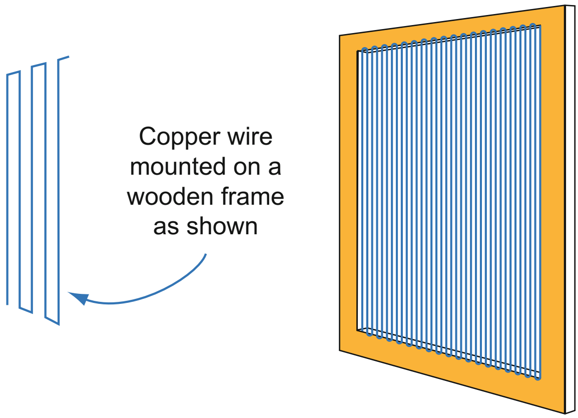
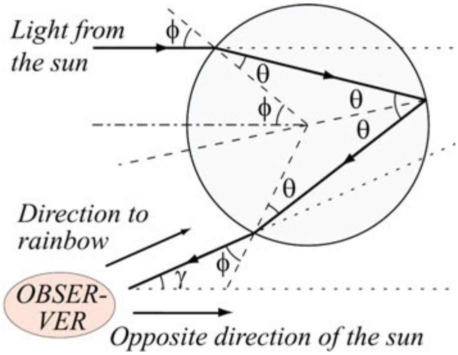


Fig. 10.21 Light that gives us the rainbow goes through the water droplets as shown in this figure at an angle $\phi \approx 60°$



21. Show that Eq. (10.6) can be derived from Eqs. (10.15) and (10.16) in the event that all are valid at the same time.
22. We can create a polarization filter for radio waves using a copper wire stretched over a frame as shown in Fig. 10.20. Explain the mode of action and explain which direction of polarization the radio waves must have to be stopped by the filter and which direction has minimal influence on the waves. It may be that the filter would be even more effective if the filter was made *slightly* differently. Do you have any good ideas in this way?
23. The light paths in raindrops when we see a rainbow are described in the article " The rainbow as a student project involving numerical calculations" written by David S. Amundsen, Camilla N. Kirkemo, Andreas Nakkerud, Jørgen Trømborg and Arnt Inge Vistnes (*Am. J. Phys.* 77 (2009) 795–798).
Figure 10.21 shows the path of the ray. As stated in the figure, the angle $\phi \approx 60°$ for the light rays gives us the rainbow (for details, read the original article).
(a) Calculate the angle θ, given the refractive index of water is approximately 1.333.
(b) Is there total reflection for the light that hits the rear boundary of the water

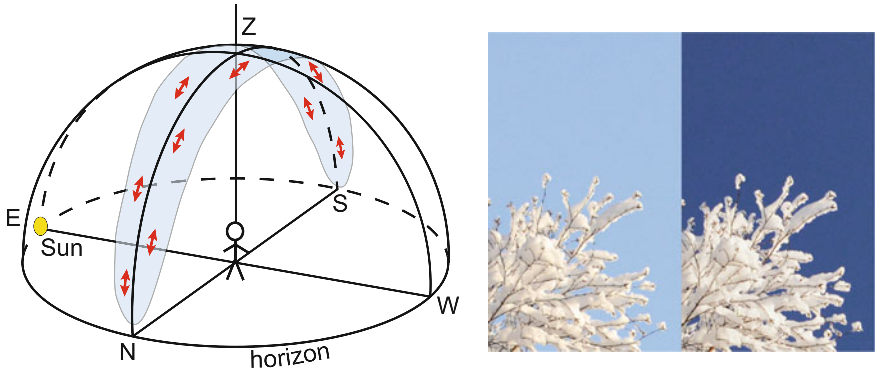


Fig. 10.22 Scattered light from the sun is polarized in a band over the sky 90° away from the direction of the sun. This can be used to remove a good part of the scattered light when we take a photograph. The right part shows images without and with polarization filter

drop for this value of θ?

(c) Calculate the Brewster angle both for the light that goes from air to water and at the interface water to air.
(d) Would you think that the light from a rainbow is highly polarized or not?
(e) How could we experiment experimentally if the light from the rainbow is highly polarized or not? (See Fig. 1.1.)

24. Light from the sun is scattered in the atmosphere, and it is this scattering that makes the sky blue during the day and red in the evening sky before the sun sets. The scattering of light is called Rayleigh scattering, and when the atmosphere is very clean and dry, the scattered light is polarized in a special way. Looking towards the sky in a direction 90° from the direction to the sun, the light will be significantly linearly polarized. The left part of Fig. 10.22 shows the direction of polarization for the region where polarization is most pronounced for a situation where the sun is on the horizon.
This polarization uses photographers occasionally to get a deep blue sky. In the right part of Fig. 10.22, it is shown how the sky looks without polarization filter.
(a) How should the filter be set for maximum effect?
The right part of the Fig. 10.23 shows the photograph of a shining water surface. Reflected sky light is strong for the water surface far away, but there is a pronounced dark area near us (marked with a dashed oval), where almost no light is reflected. We look straight down to the bottom of the water. The image is taken *without* polarization filter (just as we see it by our eyes). The phenomenon can be observed when the sun is near the horizon, and we look in a direction about 90° away from the sun (marked with dashed line in the image).
(b) Can you explain in detail the physical effects that conspire to create a "black hole" in the reflected sky light?
(c) The phenomenon is easiest to see when we stand at least a few metres above

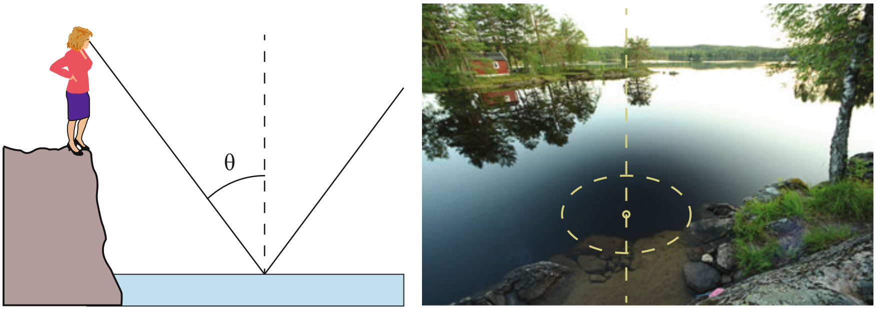


Fig. 10.23 Wide angle image of a shining water surface. There is hardly any light from the dark area (*marked with a dashed oval with a centre*). The sun is on the horizon in a direction 90° to the right from the direction marked with a vertical dashed line. The sky was clear everywhere, but brighter to the right (closer to the sun on the horizon, outside the field of view.)

the surface of the water, as shown in the left part of Fig. 10.23. Can you determine the θ angle in the figure that corresponds to the direction of the darkest part of reflected sky light?

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