Chapter 6
Waves

Abstract Waves, viewed as phenomena extended in both time and space, are introduced in this chapter. The mathematical wave equation is presented together with the concepts of wavelength, period and wave velocity. Also, the mathematical expressions of a wave, in both real and complex notation, are presented, as well as the concepts of transverse and longitudinal waves. The transverse equation of motion of a string, as well as the longitudinal movement of air (or water) molecules when a sound wave passes through the compressible medium, is shown to follow a wave equation.

6.1 Introduction

Everyone has seen circular waves propagating along a water surface (see Fig. 6.1). We are so used to the phenomenon that we barely notice it.

But have you really understood the magic of waves? How come that the wave migrates along the surface of the water without any matter moving at the wave speed? If we throw a ball from point A to point B, the ball moves spatially with all its mass from A to B. But when a wave moves from A to B, there is no corresponding mass that is transported from A to B. What in heaven's name is causing the wave to move along?

Waves are generated when a vibration at one place in space somehow affects the neighbouring area so that it too starts to vibrate, causing in turn another neighbouring area to begin to vibrate, and so on. When we describe this interaction and focus on the explanation of the physics that lies behind wave motion, we study the dynamics of the system. Nevertheless, we start in the same way as in Chap. 2 with "kinematics", that is, with the mathematical description.

A wave can be visualized in three ways:

- We can take a snapshot ("flash image", a high-speed flash photograph) of how the wave looks at a selected time in different parts of space (as a function of position).
- We can record the amplitude as a function of time at *one* place in space as the wave passes this location and plot the result.

A. I. Vistnes, *Physics of Oscillations and Waves*, Undergraduate Texts in Physics,
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Fig. 6.1 Waves that form on water

- We can use a "movie" (animation) that shows how the wave spreads in space as time goes by.

Figure 6.2 shows examples of the first two viewing modes. Imagine standing on a pier and watching waves rolling gently in front of you. You can take a picture of the waves and get something that corresponds to the left part of Fig. 6.2. Take another picture a moment later, and you will see that the wave has moved a little (as indicated in the figure).

Imagine that there is a vertical pole in the water. The water surface then moves up and down the post, and you can record the height as a function of time. This corresponds to the right part of Fig. 6.2. If there are two pegs that stand a little apart, the water surface will not be on top simultaneously on both pins, in general.

For a *harmonic* wave (with a form like that of a sine or cosine function), the first two modes of view will both look like harmonic oscillation: the first as harmonic oscillation as a function of position, the other as harmonic oscillation as a function of time. We know from before that a harmonic oscillation is a solution of a second-order differential equation. If we consider how the wave looks like a function of position (at one point), the result f must be a solution of the differential equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -C_x f \ .$$

If we regard the wave as a function of time as it passes one place in the room, the result must be a solution of the differential equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = -C_t f \ .$$

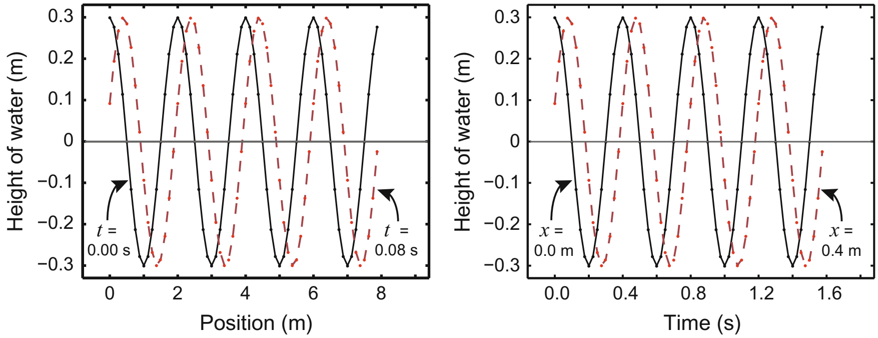

Fig. 6.2 A wave can be considered as a function of position at a certain time, or as a function of time for a particular position. See the text for details

In these equations, x indicates the position and t the time, and C_x and C_t are positive real constants that differ in the two cases. In contrast, the amplitude of the harmonic wave f is the same quantity in each viewing mode. We therefore use the same symbol f in both equations. The amplitude may be, for example, the air pressure of sound waves, or the electric field strength of electromagnetic waves or the height of surface waves at sea.

When we realize that the wave f is the same, regardless of whether we consider the wave as a function of position in space or as a function of time, we can combine the two equations and get:

$$\frac{\mathrm{d}^2 f(x,t)}{\mathrm{d}t^2} = \frac{C_t}{C_x} \frac{\mathrm{d}^2 f(x,t)}{\mathrm{d}x^2} .$$

In the above notation, the dependence of the amplitude on space and time has been explicitly indicated. And when a function depends on more than one independent variable, we use *partial differentiation* and write:

$$\frac{\partial^2 f(x,t)}{\partial t^2} = \frac{C_t}{C_x} \frac{\partial^2 f(x,t)}{\partial x^2} . \tag{6.1}$$

Upon renaming the quotient C_t/C_x as v^2, the above equation takes the form:

$$\frac{\partial^2 f(x,t)}{\partial t^2} = v^2 \frac{\partial^2 f(x,t)}{\partial x^2} . \tag{6.2}$$

This equation is called the "wave equation".

Since the C's were positive real constants, v must be real (and positive).

Remark: We take a short detour to recall what we mean by partial differentiation.

Suppose we have a function $h = h(kx - \omega t)$ and that we wish to find the partial derivative of this function with respect to x. We define a new variable $u = kx - \omega t$ and, using the chain rule, we find:

$$\frac{\partial h}{\partial x} = \frac{\mathrm{d}h(u)}{\mathrm{d}u} \times \frac{\partial u}{\partial x} .$$

It is the second factor on the right-hand side where the implications of partial differentiation strike us first. We have:

$$\frac{\partial u}{\partial x} = \frac{\partial (kx - \omega t)}{\partial x} .$$

Both x and t are variables, but when we calculate the partial derivative with respect to x, we will treat t as a constant! Consequently, we get:

$$\frac{\partial (kx - \omega t)}{\partial x} = k .$$

Similarly, we can go on to deduce the partial derivative with respect to t. In this case, we treat x as constant.

Partial derivatives thus represent the derivative of the function, assuming that all variables are kept constant, except for the one with respect to which the derivative is to be calculated.

We will come across the "wave equation" Eq. (6.2) quite a few times in the book. It may therefore be useful to try to understand it forthwith.

When we discussed oscillations in Chap. 1, we saw that if we know the starting position and the starting speed, e.g. for a shuttle, we can unambiguously calculate how the oscillation will be in the future (so long as the differential equation governing the motion is known).

For waves, it is totally different. Even when we have the exact same wave equation and the very same initial conditions, there are infinitely many different solutions. The reason is that the waves spread out into space, and the shape of the volume the wave is confined in will affect the wave even though the basic differential equation is the same. This is easy to understand if we think of swells rushing towards land. The wave will show enormous local variations, all depending on the landscape, with its rocks and protrusions and recesses. Solving the wave equation therefore requires that we know the initial as well as the boundary conditions. And since there are infinitely many boundary conditions we can imagine, there will also be infinitely many solutions. But once we have specified both initial conditions and complete set of boundary conditions, there is a unique solution.

Since there is such an incredibly wide variety of waves, we often have to resort to simplified solutions to extract at least some typical features. Some such solutions are actually serviceable approximations to real waves in special cases. The most common simplified solution is called *plane wave*, and we will take a closer look at it now.

6.2 Plane Waves

A wave is said to be plane when its amplitude is constant throughout a plane that is normal to the direction of propagation in space. If a wave in three-dimensional space travels in a direction parallel to the x-axis, a planar wave will have an identical amplitude, at any selected instant, throughout an infinite plane perpendicular to the x-axis.

For a plane sound wave that moves in the x-direction, this will in practice mean that, at any time whatsoever, the local air pressure has a maximum everywhere along a plane perpendicular to the x-axis. We call such a plane a "wavefront". For plane waves, the wavefront is plane.

Mathematically, a plane harmonic (monochromatic) wave can be described as:

$$f(x,t) = A\cos(kx - \omega t) \ . \tag{6.3}$$

In this context, k is called *wavenumber* and ω, the angular frequency. If we keep the time constant, for example, at $t = 0$, and start at $x = 0$, we move a wavelength when $kx = 2\pi$. The wavelength λ is therefore precisely this value of x, so that:

$$\lambda = \frac{2\pi}{k} \ .$$

In a similar manner, we can keep the position constant, for example, by setting $x = 0$, and starting at $t = 0$. We find then that, if we want to change the time function by a period, the time must increase by $\omega t = 2\pi$. This time difference is called the *time period* T, and we get:

$$T = \frac{2\pi}{\omega} \ .$$

It may be added that the word "wavenumber" comes from k indicating the number of wavelengths within the chosen unit of length ("how many wave peaks are there in a metre?"), but multiplied by 2π.

We can also apply a similar idea to the angular frequency ω. In that case, we can say that the ω is a "(time) period" which indicates how many periods we have within the chosen unit of time ("how many periods of vibration are there in one second?"), but multiplied by 2π.

The unit for wavenumber measurement is inverse metre, i.e. m^{-1}. Angular frequency unit is actually inverse second, that is, s^{-1}, but in order to reduce the likelihood of confusion with frequency, we often give angular frequencies in *radians per second*.

6.2.1 Speed of Waves

Let us find out how fast the wave travels in the x-direction. Imagine following a peak that corresponds, let us say, to the value 6π for the argument in the cosine function, in which case we will have

$$kx - \omega t = 6\pi \ ,$$

$$x = \frac{\omega}{k} t + \frac{6\pi}{k} \ .$$

We differentiate the expression for position with respect to time, so that we may see how quickly this point moves, and we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} \equiv v = \frac{\omega}{k} \ .$$

The velocity with which the wave travels is thus equal to the ratio between angular frequency and wavenumber. We can rephrase this relation in terms of the wavelength and time period as:

$$v = \frac{2\pi/T}{2\pi/\lambda} = \frac{\lambda}{T} \ .$$

But we know that the frequency is given as the inverse of the period, i.e. $\nu = 1/T$. If we insert this in the last equation, we get a well-known relationship:

$$v = \lambda \nu \ . \tag{6.4}$$

The velocity of a plane wave given in Eq. (6.3) is thus the wavelength multiplied by the frequency (Eq. 6.4). *This is a very important relationship!*

6.2.2 Solution of the Wave Equation?

So far, we have only *asserted* that Eq. (6.3) satisfies the wave equation. We will now verify this, and we get by double differentiation of Eq. (6.3):

$$\frac{\partial^2 f(x,t)}{\partial t^2} = -\omega^2 f(x,t)$$

and

$$\frac{\partial^2 f(x,t)}{\partial x^2} = -k^2 f(x,t) \ .$$

We observe that:

$$\frac{\partial^2 f(x,t)}{\partial t^2} = \frac{\omega^2}{k^2}\frac{\partial^2 f(x,t)}{\partial x^2}$$

or:

$$\frac{\partial^2 f(x,t)}{\partial t^2} = v^2 \frac{\partial^2 f(x,t)}{\partial x^2} \ . \tag{6.5}$$

We see that the plane wave given in Eq. (6.3) satisfies the wave equation, but what about the initial and boundary conditions? Well, here some difficulties arise. If a planar wave should be able to form and remain so, we must initiate a wave that actually has infinite extent and the same amplitude and initial variation in time throughout this infinite plane. There must also be no boundary conditions that affect the wave at any point. If all of these requirements were met, the plane wave would remain plane, but we realize that this is physically unattainable.

However, if we start by considering a wave many, many wavelengths away from the location where it was generated—for example, sunlight as it reaches earth—the so-called wavefront will be quite flat as long as we only consider the light over, for example, a one square metre flat surface normal to the direction of light. If we then follow the light a few metres further, the wave will behave approximately like a plane wave in this limited volume. But if reflected light reaches this volume, we will not have a plane wave anymore!

Remark: The wavefront of light from the sun will in fact not be plane, as indicated above. Due to the angular size of the sun in relation to the wavelengths of visible light, the spatial coherence length is short and the wavefronts irregular. This will be discussed in detail in Chap. 15.

> Plane waves are therefore just an idealization that we can never achieve in practice. The plane-wave description can nevertheless provide a relatively good account over a limited volume when we are far away from bits and bobs that can affect the wave in one way or another.
>
> By "far away" one means that the distance is large relative to the wavelength, from the source of the waves, and from boundaries that distort the wave.

6.2.3 Which Way?

We found above that a plane wave described by the equation

$$f(x, t) = A\cos(kx - \omega t)$$

has a velocity $v = +\omega/k$. That is, the wave propagates in positive x-direction as time passes. With a little practice, we can infer this directly from the argument of the cosine function: if we stay at the same place on a wave (e.g. a peak), the argument must remain unchanged as time increases. And increasing the time t, we can achieve the constancy of the argument only if we compensate by letting x also increase. In other words, the peak of the wave moves towards larger x-values as time increases.

By using similar reasoning, we can easily show that a wave described by:

$$f(x, t) = A\cos(kx + \omega t)$$

propagates towards lower x-values as time increases. Pictorially, for those of us who are accustomed to the x-axis increasing to the right, we can say that the waves described in the first of these ways (with the minus sign) move to the right, and waves described in the other way (with the plus sign) move leftward.

Note that the speed of the wave does not describe speed in the same way as the speed of a ball after it is thrown. The speed of the ball, a physical body, is defined as the time derivative of the position of the ball. For the wave, speed is defined as a more *abstract quantity*, for example, the time derivative of the position in space where the wave has its maximum value. For a sound wave in air, the velocity of the wave is equal to the velocity of, say, a point in space where the local air pressure has a maximum. This can be described as the speed of a "wavefront". We will come back to more complicated relationships later.

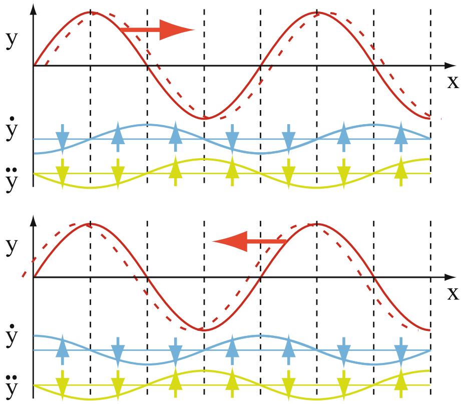

Fig. 6.3 Snapshot of "amplitude" (y *in red*), the time derivative of the amplitude ($\dot{y}$ *in blue*) and the double derivative of the amplitude ($\ddot{y}$ *in green*) in different positions along a wave. The wave as such moves to the right (*top*) and to the left (*bottom*). The dashed red curve shows where the wave is a short time after its current location (*solid curve*)

Figure 6.3 shows a snapshot of "amplitude", the single and double time derivatives of the amplitude at all positions along a wave. The wave as such goes to the right or to the left as the arrows show (top). Note that for a right-going wave, the time derivative of the amplitude will lie a quarter period "in front" of the "amplitude" and the second time derivative a quarter of period "in front" of the first time derivative. For a leftward wave, exactly the same applies, but "in front" of the wave must now mean to the left of the wave.

Let's try to concretize these considerations, but choose a wave on a string (e.g. the one we get just after we swing one end of a long horizontal string up and down a few times). The "amplitude" in this case is very concrete because it simply indicates the position of the string at the spot where the amplitude is measured. The time derivative of the amplitude will then say the vertical velocity to the point along the string we consider, and the double time derivative for this point will then be the vertical acceleration of this point. The wave itself moves in the horizontal direction.

Note that for any arbitrary point along the string, the sign of acceleration at all times is the opposite of the position relative to the equilibrium point. Thus, *the effective force on every element of the wave is always pointing towards the equilibrium state/position.* We hope you realize that this is just as it should be (based on what we learned in Chap. 2).

6.2.4 Other Waveforms

So far, we have considered harmonic waves, i.e. waves with sinusoidal shape. Can waves of another shape satisfy the wave equation?

Let us investigate a wave described by:

$$g(x,t) = G(kx - \omega t) \ .$$

where G can have any shape (but G must be a differentiable function). We introduce a new variable $u = kx - \omega t$, partially differentiate, use the chain rule and get for the left-hand side of Eq. (6.5):

$$\frac{\partial^2 g(x,t)}{\partial t^2} = \frac{\mathrm{d}^2 G(x,t)}{\mathrm{d}u^2} \left(\frac{\partial u}{\partial t}\right)^2$$

$$= \omega^2 \frac{\mathrm{d}^2 G(x,t)}{\mathrm{d}u^2} \ .$$

For the right-hand side, a similar differentiation gives:

$$\frac{\partial^2 g(x,t)}{\partial x^2} = k^2 \frac{\mathrm{d}^2 G(x,t)}{\mathrm{d}u^2} \ .$$

We see that $g(x, t)$ indeed satisfies the wave equation, assuming that k and ω are real constants.

That is, any wave that can be described by a differentiable function and a single argument $(kx - \omega t)$, where k and ω are constant is a solution of the wave equation.

6.2.5 Sum of Waves

What if we have a sum of two different functions, one of which has a slightly different combination of k and ω than the other. The sum function is then given by:

$$g(x, t) = G_1(k_1 x - \omega_1 t) + G_2(k_2 x - \omega_2 t)$$

$$= G_1(u_1) + G_2(u_2) \ .$$

Partial differentiation with respect to time gives:

$$\frac{\partial^2 g}{\partial t^2} = \omega_1^2 \frac{\mathrm{d}^2 G_1(u_1)}{\mathrm{d}u_1^2} + \omega_2^2 \frac{\mathrm{d}^2 G_2(u_2)}{\mathrm{d}u_2^2} \ ,$$

and partial differentiation with respect to position gives:

$$\frac{\partial^2 g}{\partial x^2} = k_1^2 \frac{\mathrm{d}^2 G_1(u_1)}{\mathrm{d}u_1^2} + k_2^2 \frac{\mathrm{d}^2 G_2(u_2)}{\mathrm{d}u_2^2} \ .$$

If these functions are to satisfy the wave equation

$$\frac{\partial^2 g}{\partial t^2} = v^2 \frac{\partial^2 g}{\partial x^2} \ .$$

We must require that the time derivative should equal v^2 times the second spatial derivative. We assume that this demand can be met, and then get the insertion and arrangement of the terms:

$$(\omega_1^2 - v^2 k_1^2) \frac{\mathrm{d}^2 G_1(u_1)}{\mathrm{d}u_1^2}$$

$$= -(\omega_2^2 - v^2 k_2^2) \frac{\mathrm{d}^2 G_2(u_2)}{\mathrm{d}u_2^2} \ .$$

Since G_1 and G_2 can be chosen freely, this equation cannot be satisfied in general unless

$$(\omega_1^2 - v^2 k_1^2) = (\omega_2^2 - v^2 k_2^2) = 0$$

and this implies that

$$v = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2}$$

and one is led to conclude that the two waves must travel with the same velocity!

We have now established that *the sum of two (or more) waves travelling at the same speed will satisfy the wave equation if each of the sub-waves does.*

We have also shown that *if a wave consists of several components that move with different speeds, we will not be able to describe the time development of the wave by using a single wave equation.* Then the waveform will change as the wave moves (an effect we call *dispersion* in Chap. 8).

6.2.6 Complex Form of a Wave

We can use complex description for a wave in the same way we did it for oscillations.

A plane harmonic wave in the x-direction can be descried in terms of complex quantities as:

$$f(x, t) = A e^{i(kx - \omega t + \phi)} \ . \tag{6.6}$$

Similarly, we can describe a plane harmonic wave travelling along an arbitrary direction $\mathbf{k}/|\mathbf{k}|$, where $\mathbf{k}$ is a so-called wave vector, in the following manner:

$$f(\mathbf{r}, t) = A e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)} \ . \tag{6.7}$$

where $\mathbf{k} \cdot \mathbf{r}$ is the dot product between the position vector and the wave vector.

Since f should normally be real, we must either take the real part of the above expressions or seek some other safeguard. An elegant and common way to avoid this problem is to add the complex conjugate ("c.c.") of the expression and divide by 2:

$$f(\mathbf{r}, t) = \frac{1}{2} A e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)} + \text{c.c.} \ . \tag{6.8}$$

This form of representation can be used for both real and complex A.

6.3 Transverse and Longitudinal

There are several types of waves. One classification is based on which direction "the amplitude" has in relation to the direction of propagation of the wave. But since the "amplitude" can be almost anything, and not necessarily something that moves in space, such considerations often turn out to be misleading. It is safer to base the classification on the symmetry properties of the wave, and we shall attempt to do this in what follows.

For sound waves, "the amplitude" is a pressure change. For sound waves in air, this is a pressure change in air, and likewise for sound in other materials. Pressure changes occur locally because air molecules move in the same (or opposite) direction as the direction of propagation of the wave.

> *It is the local rotational symmetry axis of the air pressure that determines the direction of propagation of the wave.* By saying that one means that the local air pressure varies in the same manner irrespective of which direction we take to be the normal to the direction in which the wave travels (thus, we have cylindrical symmetry). Such a wave is called *longitudinal* (lengthwise).

However, air molecules do not move from, say a speaker to my ear, when I listen to music. It is tempting to say that each air molecule fluctuates (statistically) back and forth relative to an equilibrium point. The problem, however, is that there is no equilibrium point because Brownian displacements of the air molecules are usually greater than the displacements caused by the passage of sound. However, the movement due to the sound is "collective" for many air molecules, while individual movements are more chaotic. That way, the sound wave can survive after all. The amplitude of the oscillations due to the sound wave alone is usually much smaller than one millimetre (even smaller for sound in metals).

Transverse waves are the other main type of waves. The best-known example is that of electromagnetic waves. When the physicists at the beginning of the nineteenth century realized that light had to be described by waves (and not as particles as Newton had convinced physicists to believe for over a hundred years), they had trouble explaining polarization. The reason is that they assumed that light waves were longitudinal, as they thought all waves to be. Only when Fresnel suggested that the light waves were transverse, were they able to fathom polarization.

> A transverse wave has an "amplitude" perpendicular to the wave propagation direction (transverse: "turned across"). By that we mean that *the physical parameter we call "the amplitude" does not have local rotational symmetry about the axis indicating the direction of wave motion*. There is no cylindrical symmetry.

For electromagnetic waves, the electric and magnetic field is “the amplitude”. Electrical and magnetic fields are vectors and have a direction in space. That an electromagnetic wave is transverse means that the electric and magnetic field are in a direction perpendicular to the direction along which the wave propagates. Then the rotation symmetry is automatically broken. (It is sufficient with symmetry breaking within a limited volume in space, of the order one half of the wavelength in all directions.)

Note that there is no relocation of anything material across an electromagnetic wave! Many imagine that there is something that moves across an electromagnetic wave, similar to the water level in a surface wave of water. That is wrong. If we depict electric fields as vector arrows at points along the propagation direction, then the arrows will extend and retract. But these arrows are mere aids for thought and have no existence of their own. They only indicate the size and direction of the abstract quantities electric and magnetic fields at the different positions in space. We will discuss common misconceptions when we treat electromagnetic waves in Chap. 9.

Some waves (proclaim to) have a portmanteau character, lying between longitudinal and transverse. Surface waves on water are an example. Here, water molecules move back and forth in the direction of propagation, as well as up and down in a perpendicular direction.

6.4 Derivation of Wave Equation

We have previously given a mathematical expression for a wave and arrived (through quasi-reverse reasoning) at a differential equation with solutions displaying wave behaviour. We will now start with a physical system and derive the pertinent wave equation. We will do this for oscillations on a string and for sound waves in air/liquid. It is considerably more difficult to derive an equation for surface waves in water, and we will just settle for an approximate solution without a derivation. Subsequently, we will also deduce the equation for an electromagnetic wave. Surface waves on water and electromagnetic waves will be discussed in later chapters.

6.4.1 Waves on a String

The starting point is a wave along a string. We consider a small segment of the string, more specifically a segment that is small in relation to the effective wavelength. Figure 6.4 shows the segment along with forces that work on it. The wave is assumed to propagate in the horizontal direction (x-direction), and the equilibrium position of the string when there are no waves on it is also horizontal. The wave is assumed to be purely transverse so that the result is solely in the vertical direction of the figure (y-direction). It should be noted that the amplitude in the vertical direction is *exceedingly*

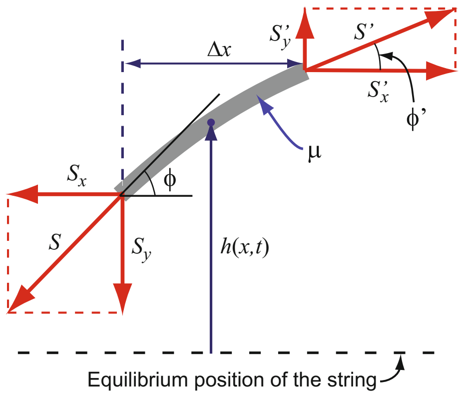

Fig. 6.4 Forces that act on a small segment of a string suffering transverse motion. See the text for details

small in relation to the length of the piece under consideration. We expand the vertical scale in the figure to get some visual help when important relationships are to be entered.

It is assumed that the stiffness of the string is so small that the forces S and S' that work at each end of the string are *tangential* aligned along the string.[1] The mass centre for the segment will still change position $h(x, t)$ relative to a mean position (the equilibrium position of the string when there is no wave). The movement of the segment must be described by Newton's second law.

Newton's second law will be applied separately to the horizontal and vertical directions, and we take the horizontal first. Since the string is assumed to have a purely transverse movement, the centre of mass of the string segment does not move (notably) in the x-direction. Consequently, the sum of forces in the horizontal direction must be equal to zero, in other words:

$$S_x = S\cos\phi = S'\cos\phi' = S'_x \; .$$

This is accomplished automatically (to second order in ϕ) if $S = S'$, since ϕ is a very small angle (remember, according to Taylor's theorem, $\cos\phi \approx 1 - \phi^2 + \cdots$).

Newton's second law, when applied in the y-direction, gives:

$$\sum F_y = ma_y \; . \tag{6.9}$$

The string has a linear mass density (mass per length) equal to μ, and the length of the segment is Δx. The mass of the segment is therefore $m = \mu\Delta x$.

Let $h(x, t)$ denote the position of the midpoint of the segment relative to the equilibrium position when there is no wave on the string. Also, since $S \approx S'$, it follows from Eq. (6.9):

[1] For sufficient small wavelengths, this approximation cannot be used. The limiting wavelength depends on the stiffness of the material of the string.

$$S \sin \phi' - S \sin \phi = \mu \Delta x \left(\frac{\partial^2 h}{\partial t^2} \right)_{\text{midpoint}} . \quad (6.10)$$

The subscript of the last parenthesis indicates that the double derivative of the centre of mass is calculated in the middle of the Δx range, i.e. in the middle of the segment.

Since the ϕ and ϕ' angles are very small, a Taylor expansion provides:

$$\sin \phi \approx \phi \approx \tan \phi$$

and likewise for ϕ'; further, $\sin \phi$ can be replaced by $\tan \phi$ in the above expression. But the tangent indicates the slope, which can also be written as $\partial h / \partial x$. Since there is an increase both at the beginning and the end of the segment, we get:

$$\sin \phi' - \sin \phi \approx \left(\frac{\partial h}{\partial x} \right)_{(x+\Delta x)} - \left(\frac{\partial h}{\partial x} \right)_{x} .$$

This can be rephrased as:

$$\frac{\left(\frac{\partial h}{\partial x} \right)_{(x+\Delta x)} - \left(\frac{\partial h}{\partial x} \right)_{x}}{\Delta x} \Delta x \approx \left(\frac{\partial^2 h}{\partial x^2} \right)_{\text{midpoint}} \Delta x .$$

Make sure that you recognize the second derivative in the above expression!

If this expression is inserted in Eq. (6.10), one obtains:

$$S \left(\frac{\partial^2 h}{\partial x^2} \right)_{\text{midpoint}} \Delta x \approx \mu \Delta x \left(\frac{\partial^2 h}{\partial t^2} \right)_{\text{midpoint}} .$$

Since both derivatives refer to the same point (midpoint), this index can now be dropped. Upon cancelling Δx and carrying out some straightforward manipulations, one is led to the result:

$$\frac{\partial^2 h}{\partial t^2} \approx \frac{S}{\mu} \frac{\partial^2 h}{\partial x^2} .$$

The desired equation follows as soon as we replace the sign for approximate equality with an equality sign:

$$\frac{\partial^2 h}{\partial t^2} = \frac{S}{\mu} \frac{\partial^2 h}{\partial x^2} . \quad (6.11)$$

We have shown that the transverse motion of a strand can be governed by the wave equation. The speed of the wave is easily deduced:

$$v = \sqrt{\frac{S}{\mu}} \,. \tag{6.12}$$

One solution of this equation is:

$$h(x, t) = A \cos(kx - \omega t + \phi)$$

where A is the amplitude, k the wavenumber, ω the angular frequency and ϕ an arbitrary phase angle. In the first instance, all four quantities can be chosen freely, apart from the fact that k and ω must conform to the relation:

$$v = \sqrt{\frac{S}{\mu}} = \frac{\omega}{k} \,.$$

In other words, there are three degrees of freedom in the wave motion, and it is perhaps most common to choose these as amplitude, frequency and phase (phase indicates in practice the choice of zero point for time). The initial conditions determine these, but the boundary conditions too play an enormous role, and they can cause the solution in practice to become a standing wave even if the initial conditions alone indicate something completely different.

Before we leave the wave equation that describes the movement of a string, it may be useful to recall the starting point for our derivation:

- Newton's second law holds.
- The wave is purely transverse.
- The force acting at each end of a segment of the string is tangentially directed (i.e. a purely geometric assumption).
- The angle between the tangent line to the string at any point and the equilibrium line is very small all along the string.
- Only when the angle between the tangent line to the string and the equilibrium line is different at each end of a segment of the string, do we get a net force that performs work on this segment. This corresponds to the fact that there must be a *curvature* on the segment under consideration for it to experience a net force.

Based on these simple assumptions, one is able to infer that a delicate interplay between forces, position and time is responsible for propagating the wave along the string. You are advised to think about what this interaction is in fact. What is actually propelling the wave? What makes the amplitude increase, and what causes it to diminish? It is not only Mona Lisa who conceals something intriguing!

In Chap. 8, we return to the basic requirements for a wave to move. We then base ourselves on numerical methods because these provide extra insight precisely into this context.

6.4.2 Waves in Air/Liquids

Derivation of the wave equation for movement in air/liquids is more complicated than the case considered in the last section. One reason for this is that we now work with a three-dimensional medium. To make the derivation manageable, we limit ourselves to a plane, longitudinal wave, which in effect allows positional changes to be described in terms of only one spatial dimension (plus time). Even so our presentation will be only approximative, but will hopefully reveal the two main mechanisms behind the waves in air and liquids: (1) the mechanical properties of a compressible medium and (2) Newton's second law.

Mechanical properties

In our context, the most important property of air and liquids is that they are relatively *compressible*; that is, it is possible to compress a certain amount of gas or liquid to a smaller volume than it originally had. Air can be compressed relatively more easily than liquids and liquids relatively more easily than solids. (This was why we did not discuss compressibility of the vibrating string in Sect. 6.4.1.] Figure 6.5 illustrates the nomenclature used in the following derivation.

Suppose that a small amount of gas/liquid with volume V expands or compresses to a new volume of $V + dV$ as a response of a change in the pressure from P to $P + dP$. It is assumed that dV and dP may be positive or negative, but their magnitudes are always small relative to V and P, respectively.

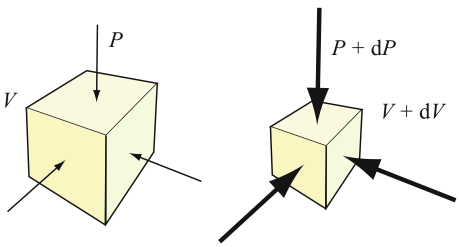

Fig. 6.5 A gaseous volume element can be compressed slightly if the external pressure increases. If dP is positive, dV will be negative

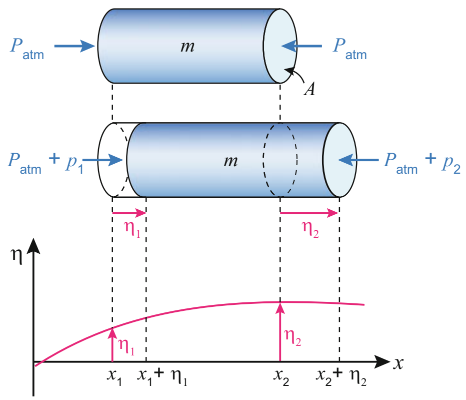

Fig. 6.6 With longitudinal movement of a gas or liquid volume, position, pressure and volume will change, but only in one spatial dimension (here in the x-direction). In the upper part of the figure, the gas volume is at equilibrium, but in the lower part a snapshot of a dynamic situation is given where the same amount of material has moved and changed volume compared to the upper part. Note that p, x and η are functions of both time and space, while the cross section A and the mass of gas or liquid are fixed

The ability of a material to withstand volume changes when the pressure is increased is called the "bulk compressibility module" for the material. It is defined in the following manner:

$$K = -\frac{dP}{dV/V} \; . \tag{6.13}$$

The unit of both the bulk compressibility module and pressure is pascal (abbreviated Pa) where 1 pascal = 1 Pa = 1 N/m^2

Let us now apply this point of view to sound waves. Figure 6.6 is based on a situation where pressure changes and movements of a gas volume occur only in one direction, namely the x-direction. For a given value of x, there are no changes in pressure when we move in the y- or z-direction.

We choose to follow the movement of an arbitrary cylinder-shaped portion of the continuous medium and assume that a negligible number of molecules will be exchanged between this cylinder and the surroundings while we consider the system. The cross section of the cylinder will not change in time, but the cylinder will move in the $\pm\, x$-direction and will change in length (volume) with time.

Figure 6.6 shows our limited volume at equilibrium (no sound) and at an arbitrary time of a dynamic situation. The cross section A in the yz-plane is fixed, and the positions of the bounding surfaces of the cylinder are x_1 and x_2 in the equilibrium and $x_1 + \eta_1$ and $x_2 + \eta_2$ in the dynamic situation.

In our attempt to arrive at a wave equation, we must find relationships between positions, pressures and volumes and use Eq. (6.13) for the model in Fig. 6.6. We choose to write the expression in a slightly different form:

$$dP = -K\frac{dV}{V} \ . \tag{6.14}$$

where K is the "bulk compressibility modulus".

We apply the symbols in Fig. 6.6 in Eq. (6.14) and get:

$$\frac{p_1 + p_2}{2} = -K \ \frac{(x_2 + \eta_2 - x_1 - \eta_1)\,A - (x_2 - x_1)\,A}{(x_2 - x_1)\,A} \ .$$

where the mean value of the pressure changes at the two ends of the cylinder is chosen for the dP term.

We would like to go to the limit where $\Delta x = x_2 - x_1$ goes to zero. This is strictly not permitted given the assumptions about the size of the chosen cylinder of gas or liquid.

We need to make an "acoustic approximation" characterized by the following:

- $\Delta x = x_2 - x_1$ is large relative to the average length of the free paths of air molecules between collisions with other air molecules in their chaotic movement.
- Δx is small relative to the wavelength of the sound waves.
- The displacements η are small compared to Δx, which means that the sound is weak.

The first point ensures that the gas or liquid volume under consideration is reasonably well separated from neighbouring areas. It also ensures that there are a large number of molecules within the selected volume, so that we can disregard individual molecules and treat the contents of the volume as quasi-continuous.

The next point ensures that pressure differences between the end faces are small compared to the pressure variation when the wave passes. The last point just ensures that the displacement of the gas volume is small compared with the length of the gas volume.

Under these approximations, the statement

$$dP = \frac{p_1 + p_2}{2} = p(x, t) \ .$$

is justified and yields:

$$p = -K\frac{\eta_2 - \eta_1}{x_2 - x_1} \ .$$

They also justify that the right side of this expression can be approximated to

$$p = -K \frac{\partial \eta}{\partial x} \,. \tag{6.15}$$

This equation provides a relation between pressure and displacement.

Newton's second law

We will now apply Newton's second law on the system. The mass of the volume element is equal to the mass density ρ_0 multiplied by the equilibrium volume. If the positive x-axis is also defined as the positive direction for F and the acceleration a, one can write:

$$\sum F = ma$$

where we sum over the forces acting on the volume element in the x-direction.

Applied to our cylinder of air or liquid:

$$(P_{\text{Atm}} + p_1)A - (P_{\text{Atm}} + p_2)A = A(x_2 - x_1)\rho_0 \frac{\partial^2 \eta}{\partial t^2}$$

where the acceleration is the double time derivative of the displacement of the gas volume. Rearranging the terms give:

$$p_1 - p_2 = (x_2 - x_1)\rho_0 \frac{\partial^2 \eta}{\partial t^2}$$

$$\frac{p_2 - p_1}{x_2 - x_1} = -\rho_0 \frac{\partial^2 \eta}{\partial t^2} \,.$$

Again, we would happily have gone to the $\Delta x \rightarrow 0$ limit, but with the limitations we have imposed, that is not permissible. However, as long as we adhere to each of the last two points of acoustic approximation, the left-hand side would not change significantly if we made Δx smaller. We get roughly

$$\frac{\partial p}{\partial x} = -\rho_0 \frac{\partial^2 \eta}{\partial t^2} \,. \tag{6.16}$$

Then, substitution of Eq. (6.15) into Eq. (6.16) and rearranging of terms give:

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{K}{\rho_0} \frac{\partial^2 \eta}{\partial x^2} \,. \tag{6.17}$$

We have thus arrived at the wave equation for this system as well. We then realize that if small volumes of air (or liquid) molecules are displaced in an oscillatory manner, as indicated in the derivation, the result of the displacement will spread as a wave. The speed of the wave is given by:

$$v = \sqrt{\frac{K}{\rho_0}} \,. \tag{6.18}$$

In other words, the speed of sound increases if the gas/liquid is hard to compress, but decreases with the mass density of the medium through which sound travels.

The expression for the wave velocity bears close resemblances to the comparable expression for a wave on a string (Eq. 6.12). The wave velocity was then $v = \sqrt{S/\mu}$ where S was the force trying to bring the string back to equilibrium. In our case, K is a measure of the force (pressure) trying to bring the volume back to equilibrium. For the string, μ is the mass per length, while in our case ρ_0 is the mass per volume.

It should be borne in mind that the foregoing derivation is based on a number of approaches. If one uses a more rigorous approach, one arrives not at the simple swing equation but at a nonlinear equation that can only be solved numerically. However, for weak sounds and for normal air pressure, the solution of the latter equation would be quite close to that found by using the simpler wave equation.

We have chosen to ignore another aspect of sound waves in air. When a gas expands or contracts, there is also a change in temperature. We implicitly assumed that there has been no exchange of thermal energy from different volume elements in the gas or the liquid through which the sound wave is transmitted—an adiabatic approach. It can be justified for weak sounds, and that is precisely what we have treated above.

It is quite common to use gas laws instead of the definition of compressibility modulus in the derivation of wave equation for sound waves through a gas. Our choice was dictated by the consideration that this chapter should be comprehensible to those without significant knowledge of statistical physics and thermodynamics. In addition, we think that the concept of compressibility is useful for understanding the underlying mechanism of wave propagation in gases and liquids.

Remark: It is interesting to note that the speed of sound in air is lower (but still not very much lower) than the median of the speed of air molecules between intermolecular collisions on account of their chaotic thermal movement. For nitrogen at room temperature and atmospheric pressure, the maximum in the probability distribution of the molecular speed is about 450 m/s. Those interested in the topic can read more about this under “Maxwell–Boltzmann distribution”, e.g. in Wikipedia.

6.4.3 Concrete Examples

The calculation we made to deduce the wave equation for movements in air and liquids is quite rough. We started out with Newton’s second law, used the validity of what that lies in the definition of the compressibility modulus, plus some other less significant details, and came to the wave equation. Can such an easy description provide a useful estimate of the speed of sound?

Let's try to calculate the sound speed in water. The compressibility modulus for water (at about atmospheric pressure) is given as $K = 2.0 \times 10^9$ Pa. The density of water is $\rho \approx 1.0 \times 10^3$ kg/m^3. If these values are entered into the expression of the sound speed in Eq. (6.18), the result is:

$$v_{\text{water}} \approx 1.43 \times 10^3 \text{ m/s}$$

The literature value for sound velocity in water is 1402 m/s at 0 °C and 1482 m/s at 20 °C. In other words, the conformity is actually good!

Let us then try to calculate the sound speed in air. Then a problem arises because the compressibility modulus is usually not given as a general table value, since the value depends on what pressure we consider. Instead, we start with the gas law:

$$PV^\gamma = \text{constant} \qquad (\gamma = C_p/C_v),$$

where C_p is the specific heat capacity at constant pressure, and C_v is the specific heat capacity at constant volume. It is assumed that the changes in volume and pressure take place so that we do not supply energy to the gas (adiabatic conditions). For sound with normal intensity, this requirement is reasonably well satisfied, but not for very loud sound.

A general differentiation of the gas law gives:

$$dP\, V^\gamma + P\, d(V^\gamma) = 0$$

$$V^\gamma dP + \gamma V^{\gamma-1} dV P = 0\ .$$

Upon combining this with Eq. (6.13), one gets

$$K = -\frac{dP}{\frac{dV}{V}} = \gamma P\ .$$

The ratio of the two specific heats for air is known to be

$$\gamma = \frac{C_p}{C_v} = 1.402\ .$$

Since a pressure of one atmosphere equals 101,325 Pa, it follows that the value of the bulk modulus for air under atmospheric pressure (under adiabatic conditions) is:

$$K = 1.402 \times 101{,}325 \approx 1.42 \times 10^5 \text{ Pa}.$$

Standard tables show that the mass density of air at atmospheric pressure and about 20 °C is $\rho = 1.293\,\text{kg/m}^3$. With all relevant data at hand, we are able to deduce the speed of sound in air:

$$v_{\text{air}} = 331\,\text{m/s}.$$

The value turns out to be 344 m/s.

Not all the data used above refer to 20 °C and one atmosphere pressure. No wonder, then, that the calculated and experimental values are not in complete agreement. Nevertheless, the calculated value is "only" about 4% too low. It indicates that our calculations and the formula found for the speed of sound in gases/liquids are reasonably good.

Remarks: The tables also provide the data for the bulk modulus for metals, and by using the same formula (derived for gases and liquids), we get values that are close to the tabulated values but the discrepancy is larger than that for air and water. For example, we calculate the speed of sound in steel to be 4510 m/s, whereas the actual value is 5941 m/s. For aluminium, the calculation leads to 5260 m/s, but the experimental value is 6420 m/s.

We should also bear in mind that in metals sound is able to propagate as a transverse wave instead of or in addition to a longitudinal wave, for example when the metal piece is shaped as a rod. The speed of a transverse sound wave in a metal depends on the rigidity of the metal, with the result that transverse waves often have lower speeds than longitudinal waves. If we strike a metal rod, we usually get transverse and longitudinal waves at the same time, and the latter usually have a higher frequency (after the standing waves have developed).

6.4.4 Pressure Waves

In the above derivation, we saw that the effective motion of small volumes of gas or liquid can follow a wave equation. It is interesting to see how much displacement is undergone by the small volumes of fluids when a wave passes, but usually it is more interesting to describe the wave in the form of *pressure changes*. Sound waves are usually detected with a microphone, and the microphone is sensitive to small variations in the pressure. The transition can be carried out as follows.

A possible solution of the wave equation Eq. 6.17 is as follows:

$$\eta(x,t) = \eta_0 \cos(kx - \omega t) \; . \tag{6.19}$$

When switching to pressure waves, we use the definition of the compressibility modulus again, more specifically Eq. (6.15) that was derived earlier:

$$p(x,t) = -K\frac{\partial \eta(x,t)}{\partial x} \; .$$

By combining Eqs. (6.19) and (6.15), one gets:

$$p(x, t) = kK\eta_0 \sin(kx - \omega t) \equiv p_0 \sin(kx - \omega t) \ . \tag{6.20}$$

The result shows that wave motion in a compressible medium can be described both as displacements of tiny volumes of the medium or as pressure variations. There is a phase difference between these waves, and a fixed relationship between the amplitudes. If the amplitude of displacement of the tiny volumes (with thicknesses significantly less than the wavelength) is η_0, the amplitude of the pressure wave is $k\ K\ \eta_0$.

6.5 Learning Objectives

After working through this chapter, you should be able to:

- Write down the standard wave equation (for a plane wave).
- Explain amplitude, wavenumber, wavelength, period, frequency, phase, wave velocity and the formula $f\lambda = v$.
- Give a mathematical expression for a harmonic plane wave as well as any arbitrarily shaped wave, which moves in a specified direction. For a harmonic plane wave, you should also be able to provide a mathematical description based on Euler's formula.
- Explain how a wave can be visualized either as a function of time or as a function of position.
- Explain the difference between a longitudinal and a transverse wave, and give at least one example of each.
- Derive the wave equation for a transverse vibration on a string.
- Know the main steps in the derivation of the wave equation for a pressure wave through, for example, air (sound wave).
- Calculate approximately the speed of sound in water using material/mechanical properties for water.

6.6 Exercises

Suggested concepts for student active learning activities: Wave velocity, amplitude, wavelength, plane wave, wave equation, transverse, longitudinal, Taylor expansion, compressible medium, compressibility modulus.

Comprehension/discussion questions

1. Present an example of the equation for oscillatory motion and an example of the wave equation. What types of information should we have in order to find a concrete solution of each of these two types of differential equations?
2. Does the velocity of waves as described in Eq. (6.6) depend on the amplitude? Explain the answer.
3. During thunderstorms, we usually see the lightning before we hear the thunder. Explain this. Some believe that we can determine the distance between us and the lightning by counting the number of seconds between our seeing the lightning and hearing the thunder. Can you find the connection?
4. Suppose that a long string hangs from a high ceiling almost down to the floor. Suppose that the string is given a transverse wave motion at the lower end and that the wave then rises to the ceiling. Will the wave speed be constant on the way up to the ceiling? Explain the answer.
5. If you stretch a rubber band and pluck it, you hear a kind of tone with some pitch. Suppose you stretch more and pluck again (have a go at it yourself!). How is the pitch now compared to the previous one? Explain the result. (hint: the length of a vibrating string is equal to half the wavelength of the fundamental tone.)
6. When we discussed sound waves, we said (with a modifying comment) that each air molecule swings back and forth relative to an equilibrium point. This is in a way totally wrong, but still the picture has a certain justification. Explain.
7. The difference between a longitudinal and a transverse wave is linked in the chapter to symmetry. How?
8. Finally, in Sect. 6.4.1, an overview was given of the essential assumptions made in the derivation of the wave equation for motion along a string. Attempt to set up a corresponding list for the derivation of the wave equation in air/water.
9. Our derivation of the wave equation for a pressure wave in a fluid is rather lengthy and full of details. In spite of this, can you actually point out the physical mechanisms that determines the speed of sound in air or water?
10. Discuss sound waves with regard to energy.
11. For surface waves on water: can you determine, if you know the height of the water surface at *one point* on the surface as a function of time, (a) where the wave comes from, (b) wavelength and (c) whether the height (amplitude) is the result of waves from one or more sources? Use your own experience and the photograph in Fig. 6.1.

Problems

12. Check whether the function $y(x, t) = A \sin(x + vt)$ satisfies the wave equation.
13. What characterizes a plane wave? Mention two examples of waves that are not plane and give an example of an (approximate) plane wave.
14. State a mathematical expression for a plane moving in the negative z-direction.
15. Is this a plane wave: $S = A \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)$? Here $\mathbf{k}$ is the wave vector which points in the direction of wave propagation at the point $\mathbf{r}$, and $\mathbf{r}$ is an arbitrarily

chosen position vector, ω is the angular frequency and t the time. A is a real scalar. Justify your answer.

16. Explain in your own words how we can see from the mathematical expressions that a wave $A\cos(kx - \omega t)$ moves towards larger x-values as time passes, while the wave $B\cos(kx + \omega t)$ moves opposite the way.
17. A standing wave can be expressed as $g(x, t) = A\sin(kx)\sin(\omega t)$. Show by direct substitution that a standing wave is also a solution of the wave equation for $v = \omega/k$ (we will return to standing waves in Chap. 7).
18. What is the wavelength of a 100 Hz sound wave in air and in water?
19. When we take ultrasound images of foetuses, hearts, etc., the image quality depends on the wavelength not being more than about 1 mm. Sound waves in water/tissues have a speed of about 1500 m/s. What frequency must the ultrasound have? Is the word "ultrasound" an apt term?
20. How long is the wavelength of FM broadcast at 88.7 MHz? And what wavelength does your mobile phone have if it operates on 900, 1800 or 2100 MHz?
21. A young human ear can hear frequencies in the range of 20–20,000 Hz. What is the wavelength in air at each of these limits? (The speed of sound in air is about 340 m/s.)
22. A 2 m metal string weighing 3×10^{-3} kg is held under tension roughly like a guitar string. Clamped at one end, it is stretched slightly above a table surface and bent over a smooth round peg at the edge of the table (see Fig. 6.7); the other end of the string is attached to a freely hanging object weighing 3 kg, which provides the tension.
    (a) Calculate the speed of a transverse wave along the horizontal part of the string.
    (b) Would the velocity of the wave change if we change the length of the horizontal part of the string (i.e. how much of the 2-m-long string is located between the clamped point and the round edge)?
    (c) How long should the horizontal part of the string be in order that it may vibrate at 280 Hz if you pluck at it? (Hint: Assume that the string is then half a wavelength long.)
    (d) How heavy should the bob be in order to make the frequency twice than that in the previous task (assuming that the length does not change)?
23. Write a program in Matlab or Python that samples the sound signal reaching the microphone input of a PC when a microphone is connected and plot the signal with the correct timing along the x-axis. You may use program snippet 2 in the end of Chap. 5 for part of the program.
    Sample the sound as you sing a deep "*aaaaaa*". Is the sound wave harmonic? What is its frequency?
24. In Fig. 6.8, there is a wave along a string (a small section) at three neighbouring times. Base your answer on the figure and explain:
    - What is the direction of the net force acting on each of the eight segments of the string at time t (ignore gravity).

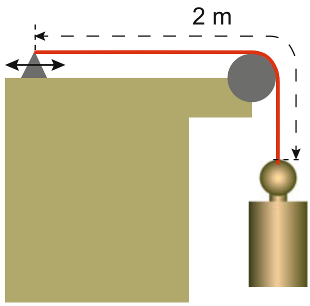

Fig. 6.7 Experimental setup in the following problem

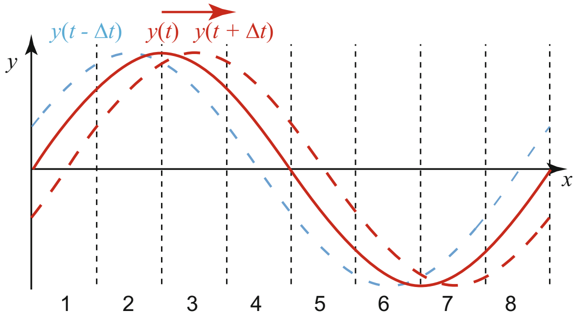

Fig. 6.8 A wave along a piece of a string at three neighbouring instants

- Explain in detail your arguments for finding the force, especially for segments 2, 4, 5, 6 and 7.
- What is the direction of the velocity of each of these segments at time t?
- At first, it may seem that there is a conflict between force and velocity. Explain the apparent conflict.
- The last point is related to the difference between Aristotle's physics and Newton's physics. Do you know the difference?
- How does the energy vary for an element along the string when the wave passes by?
- Elaborate on the expression "the wave brings with it the energy".

25. Make an animation of the wave $A\sin(kx - \omega t)$ in Matlab or Python. Choose yourself values for A, k, ω, and the ranges for x and t. Once you have got this animation working, try to animate the wave $A\sin(kx - \omega t) + A\sin(kx + \omega t)$. Describe the result.
You may use program snippet 3 in the end of Chap. 5 for part of the program (also available at the "Supplementary material" web page for this book is available at http://www.physics.uio.no/pow).
26. Read the comment article "What is a wave?" By John A. Scales and Roel Snieder in Nature vol. 401, 21 October 1999 page 739–740. How do these authors define a wave?