

Chapter 10

Electromotive Forces

Electromotive forces induced in windings of electrical machines are analyzed and discussed in this chapter. Analysis includes transformer electromotive forces and dynamic electromotive forces. The rms values, the waveforms, and harmonics are derived for concentrated and distributed windings. For real windings that have conductors distributed in a limited number of slots, the electromotive forces are calculated by introducing, explaining, and using chord factors and belt factors. Discussion includes design methods that suppress low-order harmonics in electromotive forces. This chapter concludes with the analysis of distributed windings with sinusoidal change of conductor density. Calculation of flux linkage and electromotive force in such windings shows that they achieve suppression of all harmonic distortions and operate as ideal spatial filters.

Variation of the flux in the machine windings results in induction of electromotive forces, proportional to the first time derivative of the flux. The voltage balance in each winding is given by equation $u = Ri + d\Psi/dt$, where u denotes the voltage across the winding terminals, i is the electrical current in the winding, R is the winding resistance, while the flux derivative represents the induced electromotive force. In the introductory courses of electrical engineering, the electromotive force is calculated as the first derivative of the flux with a leading negative sign, $e = -d\Psi/dt$. This convention indicates that the induced electromotive force and consequential change in electrical current oppose to the flux changes. Namely, in a short circuited winding ($u = 0$), the flux changes produce the electromotive force which, in turn, gives a rise to electrical current which opposes to the flux changes. Adopting another convention has its own advantages as well. By defining the electromotive force $e = +d\Psi/dt$, the current in the winding is determined as the ratio of the voltage difference ($u - e$) and the resistance R . The electromotive force defined as $e = +d\Psi/dt$ is *opposed* to the voltage. Therefore, it is also called *counter electromotive force*. In this book, the latter convention has been adopted with e denoting $+d\Psi/dt$ and with the voltage balance in each windings being $u = Ri + e$.

10.1 Transformer and Dynamic Electromotive Forces

Electromotive forces are generated due to changes of the winding flux. The flux can change due to changes in electrical currents of the windings or due to motion of the rotor with respect to the stator. In cases where the flux changes take place due to motion, the consequential electromotive forces are called *dynamic electromotive forces*. In cases where the stator windings carry constant currents, dynamic electromotive force appears in the rotor winding which rotates with respect to stator. Constant stator currents create a stationary magnetic field which does not move with respect to the stator. One part of this flux encircles the rotor windings as well. The rotor flux caused by the stator current depends on the relative position between the stator and rotor. When the rotor moves, such rotor flux changes, and this leads to creation of a dynamic electromotive force.

Electromotive force can also arise in cases with no rotor movement. If electrical current in stator conductors is variable, the flux created by the stator winding is variable as well. In part, the rotor flux is a consequence of stator currents. The amount of the rotor flux caused by stator currents is determined by the mutual inductance between the stator and rotor windings. Even with the rotor that does not move, the rotor flux varies due to variable electrical currents in the stator winding. As a consequence, an electromotive force is induced in the rotor winding. It is called *transformer electromotive force*. In a power transformer, alternating currents in the primary winding produce a variable flux which also encircles the secondary winding. Variable flux leads to the *transformer* electromotive force in the secondary winding, providing the means for passing the electrical power from the primary to the secondary side.

Electromotive force $e = Ldi/dt$, which appears in a stand-alone winding due to variation of the electrical current i , is proportional to the coefficient L , the self-inductance of the winding. This electromotive force is called the *electromotive force of self-induction*.

10.2 Electromotive Force in One Turn

For the purpose of modeling electrical machines, it is necessary to calculate the electromotive force induced in concentrated and distributed windings. Figure 10.1 shows an electrical machine with permanent magnets in the rotor magnetic circuit. The magnets are shaped and arranged in the way to create sinusoidal distribution of the magnetic induction in the air gap,

$$B = B_m \cos(\theta - \theta_m). \quad (10.1)$$

Due to rotation of the rotor, the maximum induction $B_m = \mu_0 H_m$ is reached at position $\theta = \theta_m$, where θ_m represents relative position of the rotor with respect to

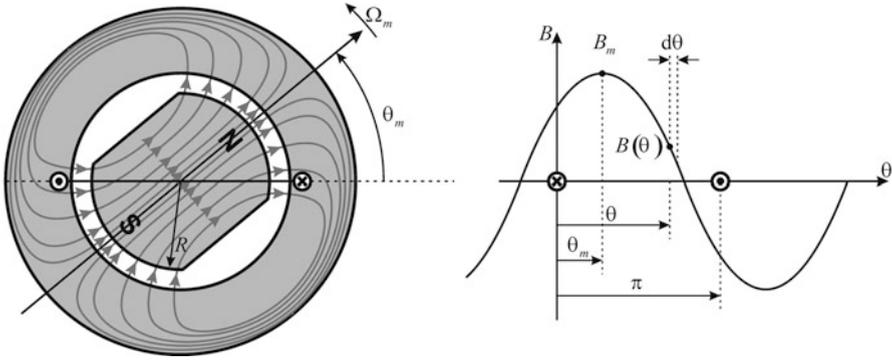


Fig. 10.1 Rotor field is created by action of permanent magnets built in the magnetic circuit of the rotor

the stator. In the case when rotor revolves at a constant speed Ω_m , position of the rotor changes as $\theta_m = \Omega_m t$, and magnetic induction at the observed position θ is equal to

$$B(\theta, t) = B_m \cos(\theta - \Omega_m t).$$

The expression represents a wave of sinusoidally distributed magnetic field which moves along with the rotor. It is of interest to determine the electromotive force induced in one turn of the stator. This turn is shown in Fig. 10.1, and it is made of two diametrical conductors. Conductor denoted by \otimes is at position $\theta_{\otimes} = 0$, while conductor denoted by \odot is at position $\theta_{\odot} = \pi$. The electromotive force in the stator turn can be determined in two ways:

1. By determining the first derivative of the flux encircling the turn
2. By calculating and summing the electromotive forces of individual conductors

10.2.1 Calculating the First Derivative of the Flux in One Turn

First derivative of the flux in one turn determines the counter electromotive force which is induced in the turn. The flux encircling the turn exists due to action of the permanent magnets mounted on the rotor, which create magnetic field in the air gap. Spatial distribution of the magnetic induction in the air gap, along the rotor circumference, is described by expression (10.1). The flux in the considered turn is equal to the surface integral of the magnetic induction over any surface leaning on conductors \otimes and \odot . In order to facilitate the calculation, one should select the surface which passes through the air gap; the expression $B(\theta)$ is readily available. Such a surface is a semicylinder starting from conductor \otimes at position $\theta = 0$,

passing through the air gap in the upper part of the figure and ending up by leaning on conductor \odot at position $\theta = \pi$. The selected semicylinder has length L and diameter R , where L is length and R is diameter of the rotor. Area of the selected surface is

$$S = L \cdot R \cdot \pi. \quad (10.2)$$

The flux in the turn is calculated as the surface integral of the magnetic induction, given by expression (10.1), over the selected surface, the semi cylinder. Since the magnetic field in the air gap is radial, the vectors of the field \mathbf{H} and the magnetic induction \mathbf{B} are perpendicular to the surface. The scalar product of vector \mathbf{B} and the surface element dS , the latter aided with the unit normal on the semicylinder, becomes the product between the amplitudes of \mathbf{B} and dS . Thus, the expression for the flux obtains the form

$$\Phi = \int_S \vec{B} \cdot d\vec{S} = \int_S B \cdot dS.$$

Elementary surface of the semicylinder is $dS = L \cdot R \cdot d\theta$, and expression for the flux in one turn of the stator due to action of permanent magnets assumes the form

$$\begin{aligned} \Phi &= \int_0^\pi d\Phi = \int_0^\pi B \cdot dS = \int_0^\pi B \cdot LR d\theta \\ &= (2LRB_m) \sin \theta_m = \Phi_m \sin \theta_m. \end{aligned} \quad (10.3)$$

Flux in the turn is dependent on relative position between the stator and rotor. Maximum value of the flux in one turn is reached when the rotor comes to position $\theta = \pi/2$. In this position, the flux is equal to Φ_m , where

$$\Phi_m = 2 \cdot R \cdot L \cdot B_m. \quad (10.4)$$

In other positions, the flux in one turn has smaller values, $\Phi(\theta_m) = \Phi_m \sin \theta_m$. When the rotor revolves at a constant speed Ω_m , the rotor position changes as $\theta_m = \Omega_m t$, and the flux in one turn is

$$\Phi = \Phi_m \sin(\Omega_m t),$$

while the (counter) electromotive force in the turn is

$$e_1 = + \frac{d\Phi}{dt} = \Omega_m \Phi_m \cos(\Omega_m t). \quad (10.5)$$

10.2.2 Summing Electromotive Forces of Individual Conductors

It is possible to determine the electromotive force in one turn by summing the electromotive forces of individual conductors that make one turn. In the example presented in Fig. 10.1, the field of permanent magnets revolves with respect to the stator conductors. The peripheral speed of relative motion is $v = R\Omega_m$, where R is the rotor radius and Ω_m is the angular speed of the rotor. In the conductor denoted by \otimes , the electromotive force is

$$e_{\otimes} = L \cdot R \cdot \Omega_m \cdot B_m \cdot \cos(\Omega_m t).$$

At position $\theta = \pi$, where the diametrical conductor \odot is placed, the magnetic induction has the same amplitude and the opposite direction. For this reason, the electromotive force e_{\odot} induced in the conductor denoted by \odot is of the opposite direction, $e_{\odot} = -e_{\otimes}$. The electromotive force in one turn is obtained by summing the electromotive forces in individual conductors. One turn is formed by connecting the two conductors in series. The terminals of the turn are made available at the front side of the cylinder. The other ends of the conductors \otimes and \odot are connected at the rear side of the cylinder. Therefore, when circulating along the contour made by the two conductors, the electromotive forces of conductors e_{\otimes} and e_{\odot} are summed according to $e = e_{\otimes} - e_{\odot}$. Finally, the electromotive force in one turn is

$$e_1 = 2 \cdot e_{\otimes} = 2 \cdot L \cdot R \cdot \Omega_m \cdot B_m \cdot \cos(\Omega_m t). \quad (10.6)$$

According to expression (10.4), the maximum value of the flux in one turn is equal to $\Phi_m = 2LRB_m$, and the previous expression can be written as

$$e_1 = \Omega_m \Phi_m \cos(\Omega_m t), \quad (10.7)$$

which is in accordance with (10.5).

In the considered example, the induced electromotive forces are harmonic functions of time; thus, it is possible to represent them by phasors. Summing the electromotive forces e_{\otimes} and e_{\odot} can be represented by phasors, as shown in Fig. 10.3.

10.2.3 Voltage Balance in One Turn

The voltage balance within one turn is given by equation

$$\begin{aligned} u &= Ri + e_1 \approx \frac{d\Phi}{dt} = 2\Omega_m LRB_m \cos \theta_m \\ &= \Omega_m \Phi_m \cos \theta_m = E_m \cos \theta_m, \end{aligned} \quad (10.8)$$

where e_1 represents (counter) electromotive force. If the resistance R of the turn is sufficiently low, the voltage drop Ri can be neglected, and the electromotive force e_1 is equal to the voltage across the terminals of the turn. Since the electromotive force is a sinusoidal function of time, its rms (*root mean square*) value is given by (10.9), where f is the frequency in Hz, that is, the number of rotor revolutions per second. Therefore, the electromotive force is proportional to the maximum value of the flux and to the frequency,

$$E_{1rms}^{turn} = \frac{E_m}{\sqrt{2}} = \frac{(2\pi f)\Phi_m}{\sqrt{2}} = 4,44f \Phi_m. \quad (10.9)$$

10.2.4 Electromotive Force Waveform

Preceding analysis dealt with an electrical machine with permanent magnets on the rotor which create magnetic field with sinusoidal distribution $B(\theta)$ along the air gap circumference. With constant rotor speed, sinusoidal electromotive force is induced in stator turn shown in Fig. 10.1.

If the speed of rotation varies, the electromotive force induced in the turn may deviate from sinusoidal change. In cases when distribution of magnetic field is sinusoidal, but the speed of rotation changes in time, $\Omega_m(t)$, the electromotive force in the turn is

$$\begin{aligned} e_1 &= + \frac{d}{dt} (\Phi_m \sin(\Omega_m t)) \\ &= \Omega_m \Phi_m \cos(\Omega_m t) + t \cdot \Phi_m \cos(\Omega_m t) \cdot \frac{d\Omega_m}{dt}. \end{aligned}$$

This expression represents a harmonic function of time if the speed of rotation is constant, that is, in cases where $d\Omega_m/dt = 0$.

In cases where the permanent magnets create a non-sinusoidal periodic distribution of the magnetic field in the air gap, the induced electromotive force assumes a non-sinusoidal function of time. Let $B(\theta - \theta_m)$ be a periodic function specifying the change of the magnetic induction along the air gap circumference. At position of the conductor \otimes , magnetic induction is equal to $B_{\otimes} = B(0 - \theta_m) = B(-\theta_m)$. If the rotor revolves at a constant speed, the electromotive force of the turn calculated according to expression (10.6) is equal to

$$e_1 = 2 \cdot e_{\otimes} = 2LR\Omega_m \cdot B_{\otimes} = 2LR\Omega_m \cdot B(-\Omega_m t). \quad (10.10)$$

The obtained expression shows that the electromotive force waveform is determined by the function $B(\theta - \theta_m)$, expressing the spatial distribution of the magnetic induction originating from the rotor permanent magnets. Therefore, the form

of the function $B(\theta - \theta_m)$ determines the waveform $e_1(t)$ of the electromotive force in one turn. In cases with non-sinusoidal distribution $B(\theta - \theta_m)$ of the magnetic field in the air gap, the electromotive force induced in the turn (Fig. 10.1) will be non-sinusoidal as well.

Question (10.1): Assume that synchronous generator supplies electrical loads and comprises concentrated stator winding with all the conductors located in two slots on the inner surface of the stator. These slots are diametrically positioned grooves in the magnetic circuit facing the air gap. Permanent magnets of the rotor create the induction $B(\theta) = B_m \operatorname{sgn}[\cos(\theta - \theta_m)]$ in the air gap. Determine and sketch the form of the voltage supplied to the electrical load.

Answer (10.1): In accordance with (10.8), the voltage across terminals of the stator winding is equal to

$$u \approx 2NLR\Omega_m B(\Omega_m t) = (2NLR\Omega_m B_m) \operatorname{sgn}[\cos(\Omega_m t)].$$

10.2.5 Root Mean Square (rms) Value of Electromotive Forces

The AC voltages and currents in electrical engineering are characterized by their *rms* value (*root mean square*). Sinusoidal voltages and currents are mostly described in terms of their *rms* value instead of their peak values. *Root mean square (rms)* value of an AC voltage corresponds to DC voltage that would result in the same power when applied to resistive loads. Namely, when an AC voltage with the *rms* value of U is applied to the resistance R , the power dissipated in resistive load will be $P = U^2/R$. The same power is obtained when a DC voltage U is applied across the same resistance. Therefore, the *rms* value of AC voltages is also called *equivalent DC voltage*. For sinusoidal voltages, their *rms* value is obtained by dividing their peak value by the square root of two. The *rms* value can be defined as well for periodic non-sinusoidal voltages. For a voltage that changes periodically with a period T , the *rms* value is calculated according to

$$U_{rms} = \sqrt{\frac{1}{T} \int_0^T u^2 dt}.$$

The *rms* value can be also defined for AC currents, using the previous expression. For sinusoidal currents, their *rms* value is $\sqrt{2}$ times lower than their peak value. Expressions (10.11) and (10.12) give the *rms* value of the electromotive force in one turn and *rms* value of the electromotive force in one conductor,

$$E_{1rms}^{turn} = \frac{E_m}{\sqrt{2}} = \frac{(2\pi f)\Phi_m}{\sqrt{2}} = 4,44f \Phi_m \quad (10.11)$$

$$E_{rms}^{con} = 2,22f \Phi_m. \quad (10.12)$$

The angular frequency $\omega = 2\pi f$ of the electromotive force is determined by the speed of rotation Ω_m . The rms value of an induced electromotive force can be expressed in terms of the frequency f and the flux as $E_{rms} = 4.44 f \Phi_m$. For a concentrated winding consisting of N turns, the rms value of the electromotive force is given by equation

$$E_{rms}^{wind} = 4,44Nf \Phi_m. \quad (10.13)$$

10.3 Electromotive Force in a Winding

Electrical machines usually have a number of windings. Most induction and synchronous machines, also called AC machines, have their stator designed for the connection to a three-phase system of alternating voltages and currents. Therefore, most AC machines have three windings on the stator, also called phase windings. Some authors use the term *stator winding* to describe the winding system comprising three-phase windings.

Each winding has one or more turns. Individual turns are connected in series. The ends of this series connection are usually made available at machine terminals. In this section, the electromotive force induced in a winding is calculated for concentrated and distributed windings. This electromotive force determines the voltage across the machine terminals.

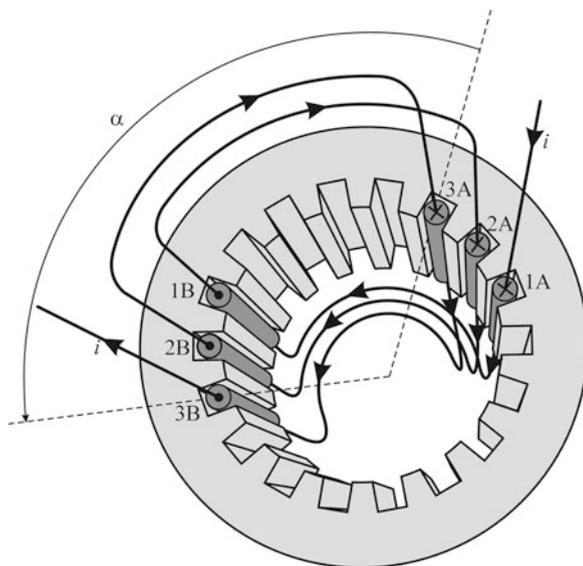
10.3.1 Concentrated Winding

Conductors making a winding can be concentrated in two diametrically positioned grooves, constituting a concentrated winding. Since all the turns of a concentrated winding reside in the same position, they all have the same flux and the same electromotive force. The rms value of the electromotive force in one turn, made of two diametrically positioned conductors, is given by expression (10.12). For a concentrated winding with N turns, the rms value of the induced electromotive force is given by expression (10.14).

10.3.2 Distributed Winding

Windings are usually made by placing conductors in a number of equally spaced slots along the machine circumference. Individual turns are spatially shifted and have different electromotive forces. For this reason, the electromotive force

Fig. 10.2 Distribution of conductors of a winding having fractional-pitch turns and belt distribution in $m = 3$ slots



induced in a distributed winding is not equal to the product of the number of turns N and the electromotive force induced in one turn. The process of calculating the electromotive force of a distributed winding is explained in this section.

The turns making a winding are often made of conductor pairs that do not have diametrical displacement. Namely, the two conductors making one turn may not have angular displacement of π . Displacement of the two conductors is also called *pitch* of the turn. With pitch lower than π , one obtains *fractional-pitch turn* or *fractional-pitch coil*.

In distributed windings, conductors of a winding may be placed in several adjacent slots. A group of conductor is called a *winding belt*. In Fig. 10.2, a sample winding is depicted with three series-connected turns, 1A–1B, 2A–2B, and 3A–3B. Conductors 1A and 1B belong to one turn, and they are placed in the slots at an angular distance of α . With $\alpha = \pi$, the conductors are diametrically positioned, making a *full-pitch coil*. In the case when $\alpha < \pi$, conductors 1A and 1B reside on a chord. In this case, turn 1A–1B is *fractional-pitch turn*.

The electromotive force induced in a fractional-pitch turn is smaller than in the case of a full-pitch turn. The reduction is determined by a coefficient called *chord factor*.

Conductors 1A, 2A, and 3A are placed in three adjacent slots which make a *winding belt*. In the same way, conductors 1B, 2B, and 3B are placed in other three adjacent slots. The electromotive forces induced in the turns 1A–1B, 2A–2B, and 3A–3B are not equal due to spatial displacement of corresponding turns. Therefore, electromotive forces in individual turns do not reach their maximum value at the same instant. The spatial shift between magnetic axes of individual turns results in a phase shift between corresponding electromotive forces. For this reason, the peak and

rms value of the winding electromotive force is smaller than the product obtained by multiplying the number of turns (3) and the peak/rms value of the electromotive force in one turn. This reduction is determined by the coefficient called the *belt distribution factor* or *belt factor*. Calculation of the electromotive force of a distributed winding requires both the chord factor and the belt factor to be known.

10.3.3 Chord Factor

The expressions for electromotive force obtained so far are applicable to turns made of diametrically positioned conductors, namely, to full-pitch turns. There is a need explained later on requiring the two conductors making one turn to be placed at an angular distance smaller than π . The reason for applying fractional pitch may be the desire to shorten the *end turns*, the wires at machine ends that connect the two conductors in series hence completing the turn. In most cases, the fractional-pitch turns are used to reduce or eliminate the undesired higher harmonics that may appear in the electromotive force.

As an example, one can start from the machine shown in Fig. 10.2 and assume that the spatial distribution of the magnetic inductance $B(\theta)$ in the air gap is not sinusoidal. For the purpose of discussion, the function $B(\theta)$ is assumed to be $B_{m1}\cos(\theta - \theta_m) + B_{m5}\cos 5(\theta - \theta_m)$. Based upon that, the electromotive forces e_{\otimes} and e_{\odot} induced in the conductors constituting one turn comprise the component at the basic angular frequency $\omega = \Omega_m$, but also the fifth harmonic at frequency of $5\Omega_m$. In cases where conductors \otimes and \odot are positioned at angular distance π , the electromotive forces induced in them are of the same shape. The fundamental harmonic component of the electromotive force depends on B_{m1} . At the same time, higher harmonics of the magnetic induction determine the higher harmonics of the electromotive force. By connecting diametrically positioned conductors in series, the electromotive force at the fundamental frequency is doubled, but so is the unwanted electromotive force of the fifth harmonic.

If the conductors are placed so that the angular distance between them is $\alpha = 4\pi/5$, there is a phase shift between the electromotive forces e_{\otimes} and e_{\odot} which depends on the angular distance α . In the course of the rotor motion, the instant of passing of the north magnetic pole of the rotor against the stator conductor \odot is delayed by $\Delta t = \alpha/\Omega_m$ with respect to the instant of passing of the same pole against the conductor \otimes . The electromotive force of conductor \odot has angular frequency Ω_m , and it is phase shifted by α with respect to the electromotive force of conductor \otimes . In Fig. 10.1, the angular distance between the two conductors making the turn is $\alpha = \pi$. In Fig. 10.2, there are turns with $\alpha < \pi$. With $\alpha = \pi$, electromotive forces in conductors \otimes and \odot have the opposite sign. They are connected in series by the end turn which provides the current path between the ends of conductors at the same machine end. With such connection, the electromotive forces with opposite sign actually add up. For that reason, the electromotive force induced in a single turn in Fig. 10.1 is two times larger than electromotive forces of individual conductors.

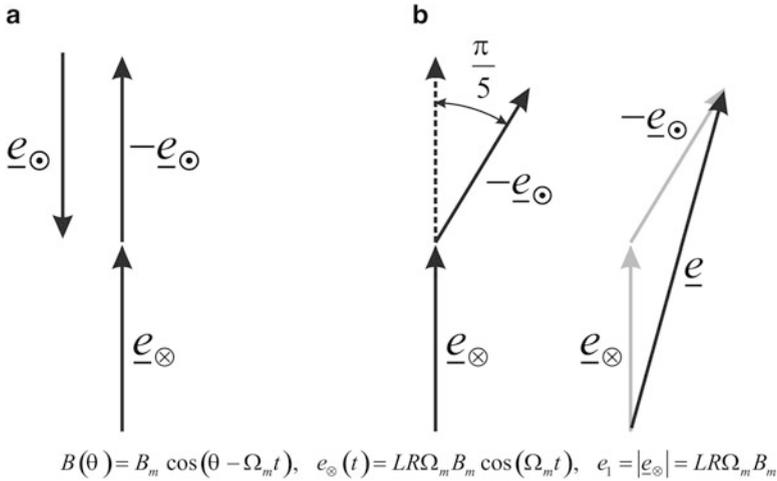


Fig. 10.3 Electromotive forces of conductors of a turn. (a) Full-pitched turn. (b) Fractional-pitch turn

Electromotive forces in individual conductors can be represented by phasors \underline{e}_{\otimes} and \underline{e}_{\odot} . With peripheral rotor speed of $v = R\Omega_m$, and with $B(\theta) = B_m \cos(\theta - \theta_m)$, both phasors have the same amplitude $|\underline{e}_{\otimes}| = |\underline{e}_{\odot}| = e_1 = LvB_m$, where L denotes the axial length of the machine. The phasor of the electromotive force in the turn, shown in Fig. 10.3a, is obtained by summing the phasors \underline{e}_{\otimes} and \underline{e}_{\odot} , $\underline{e} = \underline{e}_{\otimes} - \underline{e}_{\odot}$. With angular distance between conductors of α , $\underline{e}_{\otimes} = e_1$, $\underline{e}_{\odot} = e_1 e^{-j\alpha}$ and $(-\underline{e}_{\odot}) = e_1 e^{-j\alpha + j\pi}$. By placing the conductors at angular distance $4\pi/5$, phasors of electromotive forces \underline{e}_{\otimes} and \underline{e}_{\odot} are not be collinear, and this is illustrated in part (b) of Fig. 10.3. With respect to e_{\otimes} , electromotive force $-e_{\odot}$ is phase shifted by $\pi/5$. Therefore, the amplitude of the resulting electromotive force in one turn $e_{\otimes} - e_{\odot}$ is slightly smaller than the sum of amplitudes LvB_m of electromotive forces in individual conductors.

Phasor diagrams can be constructed for the fundamental harmonic but also for each of the higher harmonics. With $B(\theta) = B_{m1} \cos(\theta - \theta_m) + B_{m5} \cos 5(\theta - \theta_m)$, electromotive forces have the fundamental component of frequency Ω_m and amplitude determined by B_{m1} and the fifth harmonic of frequency $5\Omega_m$ and amplitude determined by B_{m5} . For the fifth harmonic of electromotive forces, the phase difference is multiplied by 5, as well as the frequency. Since $5 \times \pi/5 = \pi$, the fifth harmonic of the electromotive force e_{\otimes} is in opposition with the fifth harmonic of the electromotive force $(-e_{\odot})$. Thus, they will be mutually canceled. Despite the presence of the fifth harmonic in the spatial distribution of $B(\theta)$, the electromotive force induced in the considered turn will not contain the fifth harmonic. One of the consequences of shortening the turn pitch is reduction of the first, fundamental harmonic of the electromotive force. The electromotive force in one turn formed by the fractional-pitch conductors can be determined from the phasor diagrams of

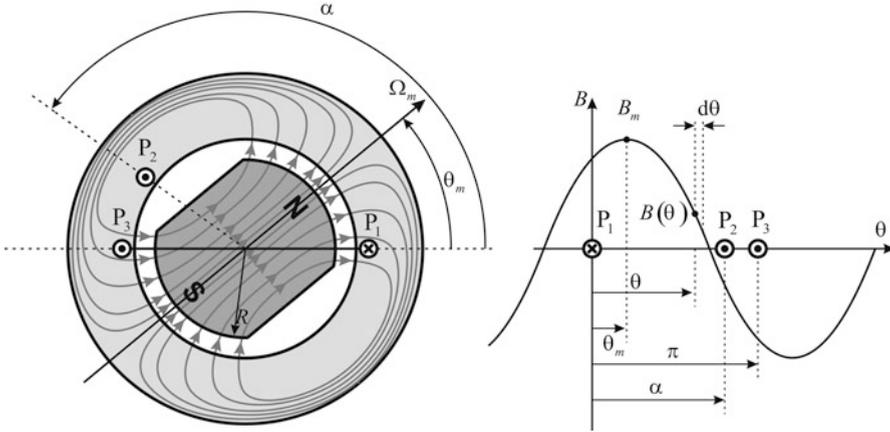


Fig. 10.4 Electromotive forces in a fractional-pitch turn

Fig. 10.3 but also by calculating electromotive force of turn P1–P2 in Fig. 10.4, which is the next step in the analysis.

Question (10.2): How do we place the conductors making one turn in order to eliminate the seventh harmonic of the induced electromotive force? It is assumed that the spatial distribution $B(\theta)$ comprises the seventh harmonic.

Answer (10.2): By placing conductors at the angular distance of $6\pi/7$ and by making their series connections in the way that leads to adding of electromotive forces at the fundamental frequency, phasors of electromotive forces at the fundamental frequency are shifted by $\pi/7$, while phasors of the seventh harmonic are shifted by π . Therefore, by summing the electromotive forces of the two conductors, the seventh harmonic is eliminated.

* * *

Electromotive force induced in the turn made of conductors P1 and P2, shown in Fig. 10.4, placed at angular distance $\alpha < \pi$ is smaller than the electromotive force induced in the turn P1–P3 which is obtained by connecting diametrically positioned conductors. Reduction of the electromotive force due to this *fractional-pitch* setting of conductors is determined by the coefficient k_T called *chord factor*. The coefficient k_T can be determined by investigating the electromotive force induced in the turn P1–P2, and its value is always $k_T \leq 1$.

Figure 10.4 shows cross section of an electrical machine with the stator turn P1–P2 having fractional-pitched conductors. It is assumed that the magnetic field in the air gap has sinusoidal distribution and that it rotates along with the rotor. Variation of the magnetic induction is depicted as well. The presented spatial distribution of the magnetic induction can be accomplished by insertion of permanent magnets into the rotor magnetic circuit.

The electromotive force of turn P1–P2 can be determined by calculating derivative of the flux encircled by the turn. The flux of the turn P1–P2 is determined by expression

$$\Phi = \int_0^{\alpha} B \cdot dS = \int_0^{\alpha} B \cdot LRd\theta = LRB_m \int_0^{\alpha} \cos(\theta - \theta_m) \cdot d\theta, \quad (10.14)$$

where L is axial length of the machine, R is radius of the rotor, while B_m is the maximum value of sinusoidally distributed magnetic induction which comes as a consequence of permanent magnets on the rotor. By calculating the integral, one obtains

$$\begin{aligned} \Phi &= LRB_m \cdot \sin(\theta - \theta_m) \Big|_0^{\alpha} = LRB_m [\sin(\alpha - \theta_m) - \sin(-\theta_m)] \\ &= \left[2LRB_m \sin \frac{\alpha}{2} \right] \cos \left(\frac{\alpha}{2} - \theta_m \right) = \Phi_m^{\alpha} \cos \left(\frac{\alpha}{2} - \theta_m \right). \end{aligned} \quad (10.15)$$

Φ_m^{α} denotes the maximum value of the flux in the turn P1–P2. This value is reached when the rotor is in position $\theta_m = \alpha/2$. Since the maximum value of the flux in the turn P1–P3 is $\Phi_m^{\pi} = 2LRB_m$, it is shown that the peak value for the fractional-pitch turn is reduced by factor of $\sin(\alpha/2)$. In other words, the peak value of the flux in the fractional-pitch turn (Φ_m^{α}) is smaller than the peak value of the flux of the full-pitch turn (Φ_m^{π}), and it is equal to $\Phi_m^{\alpha} = \Phi_m^{\pi} \sin(\alpha/2)$. The electromotive force induced in turn P1–P2 is given by expression

$$e_1(t) = \frac{d\Phi}{dt} = \Omega_m \Phi_m^{\alpha} \sin \left(\frac{\alpha}{2} - \theta_m \right) = E_m \sin \left(\frac{\alpha}{2} - \omega_m t \right). \quad (10.16)$$

The peak value of the electromotive force is equal to

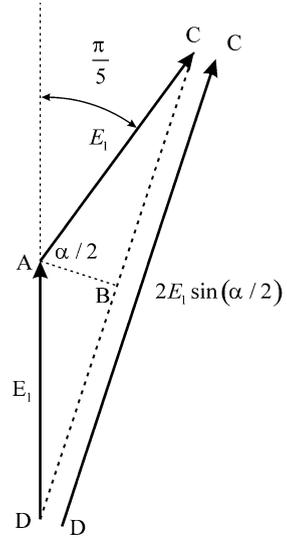
$$E_m = 2\pi f \Phi_m^{\pi} \cdot \sin \frac{\alpha}{2}. \quad (10.17)$$

Coefficient $k_T = \sin(\alpha/2)$ determines reduction of the turn electromotive force due to fractional pitch. The rms value of the electromotive force of the turn is given by expression

$$E_{1rms}^{turn} = \frac{E_m}{\sqrt{2}} = 4,44f \Phi_m^{\pi} \cdot \sin \frac{\alpha}{2}. \quad (10.18)$$

The electromotive forces represented by means of corresponding phasors are depicted in Fig. 10.5. Using the phasors, the procedure of calculating the chord factor can be simplified. Hypotenuse AC of the right-angled triangle ABC represents

Fig. 10.5 Electromotive force of a fractional-pitch turn. The amplitude of the electromotive force induced in one conductor is denoted by E_1 . The amplitude of the electromotive force induced in one turn is determined by the length of the phasor DC



the rms value of the electromotive force E_1 induced in one of the conductors. Opposite to angle $\alpha/2$ is the side BC of the length $E_1 \cdot \sin(\alpha/2)$. The rms value of the electromotive force induced in the turn is represented by phasor DC. It has twice the length of the side BC, and it amounts $2E_1 \sin(\alpha/2) = 2E_1 k_T$. Therefore, due to fractional pitch, the electromotive force in one turn is smaller than $2E_1$. Chord factor $k_T = \sin(\alpha/2)$ is in accordance with the result (10.18).

Within the subsequent sections, winding design methods are discussed that result in reduction of harmonic distortions of the electromotive force and even elimination of certain harmonics of lower order. These methods include placing conductors of one turn along chord instead diameter and arranging winding turns in belts. To facilitate calculation of electromotive forces induced in such windings, analysis results in coefficients called *belt factor* and *chord factor*.

In Sect. 10.5, the electromotive force induced in ideal distributed winding is analyzed, proving that the induced electromotive force of this winding comprises only the fundamental harmonic, notwithstanding the non-sinusoidal distribution of the magnetic induction $B(\theta)$ and non-sinusoidal waveforms of electromotive forces induced in individual conductors. Basically, an appropriate series connection of all the winding conductors can be envisaged so as to cancel all the harmonics in induced electromotive force except for the fundamental. The subsequent analysis will consider the winding with sinusoidal distribution of conductors along the machine circumference. With an ideal sinusoidal distribution of conductors, all the distortions in are canceled, resulting in a sinusoidal electromotive force of the winding. The winding then acts as a *spatial filter*, removing distortions in the spatial distribution $B(\theta)$ and giving a sinusoidal electromotive force.

10.3.4 Belt Factor

Windings of practical machines are made by placing conductors in slots, axially cut grooves in the magnetic circuit. The stator and rotor have large numbers of slots. Figure 8.1 shows a cross section of the magnetic circuit of an electrical machine exposing the usual shapes of the slots and their number.

One turn consists of the two diametrical conductors or the two conductors placed at the ends of the chord. On the other hand, a three-phase stator winding comprises three phases, three separate windings having their magnetic axes displaced by $2\pi/3$. Therefore, the minimum number of slots contained by the stator magnetic circuit of a three-phase machine is $2 \times 3 = 6$. This number of slots is used in cases where each of the phase windings is concentrated, that is, where all conductors in a winding are placed in two diametrical slots. The number of slots is usually higher than 6. One of the reasons for using higher number of slots is the problem associated with placing one half of all the conductors of one winding into only one slot. Such slot would have an extremely large cross section that is not practical for the machine construction. Thus, the conductors are usually distributed in a number of neighboring slots. In this way, a *winding belt* is formed. A belt comprises two, three, or more adjacent slots. Each slot of the belt can accommodate one or more winding conductors. In order to simplify further considerations, one may assume that each slot of the belt contains only one conductor. In cases where each slot has M conductors, all the subsequent conclusions hold, except for the electromotive forces amplitude which has to be multiplied by M .

An example of a winding belt having conductors placed in three adjacent slots is presented in Fig. 10.6 The figure shows the cross section of such winding and phasor diagram showing relation between the electromotive forces induced in the individual windings. The turns 1A–1B, 2A–2B, and 3A–3B have full pitch, namely, these turns are made of diametrically positioned conductors. Phasors E_1 , E_2 , and E_3 represent the electromotive forces of turns 1A–1B, 2A–2B, and 3A–3B, respectively. Angle γ in Fig. 10.6 denotes the distance between the two adjacent slots. The spatial shift between the turns results in the phase shift γ between electromotive forces induced in corresponding turns.

Placing of conductors in winding belts allows elimination or reduction of higher harmonics in the induced electromotive force. Due to the spatial displacement between conductors, the electromotive forces in individual conductors are phase shifted, as explained in the preceding section discussing the chord factor. In cases where a winding belt ranges over m neighboring slots, the electromotive force of the winding can be determined according to equation

$$\vec{E}^{phase} = \vec{E}_1 + \vec{E}_2 + \dots + \vec{E}_m, \quad (10.19)$$

where letters $E_1 \div E_m$ denote the electromotive forces induced in individual turns. Due to the phase shift, the sum of individual electromotive forces gives a resultant

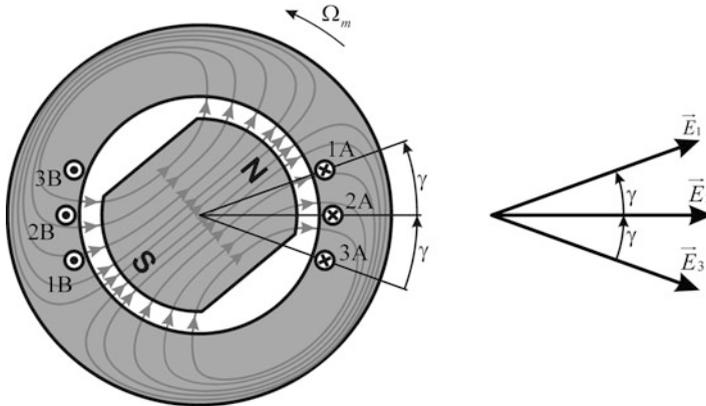


Fig. 10.6 Three series-connected turns have their conductors placed in belts. Each of belts has three adjacent slots (*left*). Phasor diagram showing the electromotive forces induced in the turns 1, 2, and 3 (*right*)

electromotive force with a rms value which is smaller than the sum of the rms values of individual electromotive forces,

$$|\vec{E}_1 + \vec{E}_2 + \dots + \vec{E}_m| < |\vec{E}_1| + |\vec{E}_2| + \dots + |\vec{E}_m|.$$

Adjacent slots are seen from the center of the rotor at an angle of γ , which determines at the same time the phase shift of the electromotive forces in individual turns, said turns comprising conductors placed in the adjacent slots. By using (10.19) and phasor diagram given in Fig. 10.7, it can be shown that distribution of conductors within the winding belts leads to elimination of certain higher harmonics in the induced electromotive force.

10.3.5 Harmonics Suppression of Winding Belt

According to (10.10), variation of the conductor electromotive force is determined by spatial distribution of the magnetic induction in the air gap. If the spatial distribution of magnetic induction contains a higher spatial harmonic of the order n , the time change of the electromotive force induced in a single conductor contains a higher harmonic of the order n . In a full-pitch turn, conductors are placed in diametrically positioned slots, and the electromotive force is twice the electromotive force of a single conductor. Therefore, the harmonic of the order n is present as well by the electromotive force of the turn.

The electromotive force of the winding shown in Fig. 10.6 is equal to the sum of electromotive forces in spatially shifted turns 1A–1B, 2A–2B, and 3A–3B. It is possible to cancel higher harmonics by the proper selection of the angle γ . Diagram

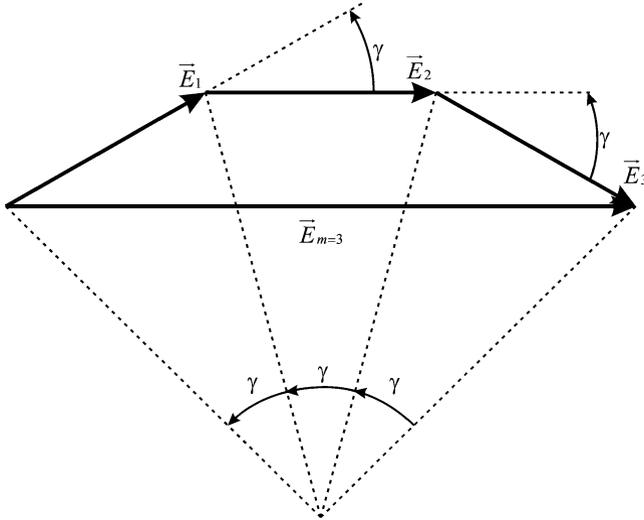


Fig. 10.7 Phasor diagram of electromotive forces in individual turns for the winding belt comprising $m = 3$ adjacent slots

in Fig. 10.6b shows phasors E_1 , E_2 , and E_3 which represent the fundamental harmonic of the electromotive forces in turns 1A–1B, 2A–2B, and 3A–3B. The phasors are shifted by angle γ which represents the spatial shift of the two adjacent turns. Higher harmonics of the electromotive forces can also be represented by phasors. Harmonics of the order n are of the same amplitude in all considered turns, but their initial phases differ due to their space shift. The angular frequency is increased n times for the n -th harmonic. In adjacent conductors, the spatial shift γ produces the phase delay for the n -th harmonic of $n\gamma$. In the phase diagram representing n -th harmonics of the electromotive forces of the turns, phasors E_{1n} , E_{2n} , и E_{3n} are shifted by angle $n\gamma$.

If angle γ in Fig. 10.6a is equal to $2\pi/(3 \cdot n)$, phasors on the n -th harmonic E_{1n} , E_{2n} , и E_{3n} are shifted by $2\pi/3$. Since amplitudes of these phasors are all equal, their phase diagram is represented by an equilateral triangle where the beginning of the first and end of the last phasor coincide. That is, the sum of the phasors of the n -th harmonic in all the three turns is equal to zero. Therefore, in cases where $\gamma = 2\pi/(3 \cdot n)$, harmonics of the electromotive force of the order n are eliminated.

In general, two winding belts making a winding may cover m consecutive slots each. The winding then consists of m turns connected in series. If the angle γ equals $2\pi/(m \cdot n)$, the phase shift of the fundamental component of the electromotive force in the adjacent turns is equal to $2\pi/(m \cdot n)$, while the phase shift of the n -th harmonic is $2\pi/m$. The electromotive force is induced in the winding with n -th harmonic obtained by adding m phasors, each representing the n -th harmonic in a single turn. These m phasors are of the same amplitude, with their initial phases shifted

by $2\pi/m$. The phasor representing the n -th harmonic of the electromotive force of the winding comprising m series-connected turns is

$$\begin{aligned}\underline{E}_n^{wind} &= \underline{E}_{1n} + \underline{E}_{2n} + \underline{E}_{3n} + \dots + \underline{E}_{mn} \\ &= \underline{E}_{1n} \sum_{k=0}^{m-1} e^{-j \cdot k \frac{2\pi}{m}} = 0.\end{aligned}$$

The last expression can be represented graphically. By adding m phasors mutually shifted by $2\pi/m$, the end of the last phasor coincides with the beginning of the first one. The phasor diagram is represented by a polygon of m sides of equal length. The polygon is inscribed within the circle with phasors of the n -th harmonic in individual turns being the chords of the circle.

Therefrom, one can conclude that the resultant electromotive force of the harmonic of the n -th order is equal to zero provided that the angle γ equals $2\pi/(m \cdot n)$. In other words, the n -th harmonic can be eliminated provided that the winding belt spans over m slots is placed at an angular distance of $2\pi/n$.

With the slots distributed in a belt, the fundamental component of the resultant electromotive force is slightly smaller than what would be the electromotive force in a concentrated winding. By using the diagram in Fig. 10.7, it can be noticed that the amplitude of the sum of phasors $E_1 + E_2 + E_3$ is smaller than the sum of individual amplitudes $|E_1| + |E_2| + |E_3|$. This difference appears since the phasors being added are not collinear. Factor k_p is equal to the ratio

$$k_p = \frac{|\vec{E}_1 + \vec{E}_2 + \vec{E}_3|}{|\vec{E}_1| + |\vec{E}_2| + |\vec{E}_3|}$$

and it is called *belt factor*.

In cases where the width of the winding belt is $m = 3$, the fundamental component of the resultant electromotive force is obtained by adding three phasors, as shown in Fig. 10.7, resulting in equation

$$\vec{E}_{m=3}^{wind} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3. \quad (10.20)$$

Phasors E_1 , E_2 , and E_3 are lying on a circle of radius R . For each of the isosceles triangles of Fig. 10.7, it is known that the base of the triangle is equal to twice the product of the triangle side and the sine of one half of the opposite angle. Therefore, the expression $R \sin(\gamma/2) = E_1/2$ links the radius of the circle, the electromotive force E_1 , and the angle γ . At the same time, the amplitude of the resultant phasor is

$$\begin{aligned}E_{m=3} &= 2R \cdot \sin \frac{3\gamma}{2} = 2 \left(\frac{E_1}{2 \cdot \sin \frac{\gamma}{2}} \right) \sin \frac{3\gamma}{2} \\ &= E_1 \frac{\sin \frac{3\gamma}{2}}{\sin \frac{\gamma}{2}} = \left(\frac{\sin \frac{3\gamma}{2}}{3 \cdot \sin \frac{\gamma}{2}} \right) \cdot 3E_1 = k_p \cdot 3E_1,\end{aligned}$$

where k_p stands for the belt factor of the winding with the belt width of $m = 3$ slots. For belt width $m \neq 3$, the amplitude of the resultant phasor representing the fundamental harmonic component is $E_m = 2R \sin(m\gamma/2)$, where radius R and electromotive force E_1 in one turn are related by $R \sin(\gamma/2) = E_1/2$. With E_1 representing the rms value of the fundamental component of the electromotive force induced in a single turn, the rms value of the fundamental component in the winding comprising m series-connected turns is

$$E_m = E_1 \frac{\sin \frac{m\gamma}{2}}{\sin \frac{\gamma}{2}} = \left(\frac{\sin \frac{m\gamma}{2}}{m \cdot \sin \frac{\gamma}{2}} \right) \cdot mE_1 = k_p \cdot mE_1.$$

The belt factor k_p is determined by the expression

$$k_p = \frac{|\vec{E}_{m=3}|}{m|\vec{E}_1|} = \frac{\sin \frac{m\gamma}{2}}{m \cdot \sin \frac{\gamma}{2}}, \quad (10.21)$$

and the rms value of the winding electromotive force can be determined from

$$E_{rms} = E^{phase} = 4,44k_p f N \Phi_m^\pi, \quad (10.22)$$

where $\Phi_m^\pi = 2LRB_m$ is the maximum value of the flux in one full-pitched turn, N is the number of series-connected turns, while $f = \Omega_m/(2\pi)$ is the frequency.

Question (10.3): Due to non-sinusoidal distribution of magnetic induction $B(\theta)$, the electromotive forces induced in conductors comprise higher harmonics. A winding of 3 turns consists of 6 conductors. The conductors are placed at angular distance of $\gamma = 24^\circ$. The resultant electromotive force measured at the winding terminals does not have some of the higher harmonics that are present in electromotive forces of single conductor. What is the order of these harmonics?

Answer (10.3): The resultant electromotive force of the harmonic of the order n is equal to zero in cases where $\gamma = 2\pi/(m \cdot n)$ or $\gamma = q \cdot 2\pi/(m \cdot n)$, where q is an integer. Since $m = 3$ and $\gamma = 2\pi/15$, all harmonics of the order $n = 5q$ are eliminated. These are all the harmonics with the order n being an integer multiple of five.

10.4 Electromotive Force of Compound Winding

Windings are usually made by series connecting the fractional-pitch turns, namely, the turns made of conductors that are not diametrically placed, but reside at the ends of a chord. In addition, conductors of several turns are distributed in adjacent slots that make up one winding belt. For this reason, calculation of the fundamental (first) harmonic component in the electromotive force of the winding should include the belt and pitch factors. These factors are given by expressions

$$k_T = \sin \frac{\alpha}{2}, \quad (10.23)$$

$$k_P = \frac{\sin \frac{m\gamma}{2}}{m \cdot \sin \frac{\gamma}{2}},$$

while the rms value of the electromotive force of a compound winding can be determined from

$$E_{rms} = 4,44 \cdot k_P k_T N \cdot f \cdot (2L \cdot R \cdot B_m), \quad (10.24)$$

where $N = N_C/2$ is the number of turns, B_m is the maximum value of sinusoidally distributed magnetic induction in the air gap, while L and R stand for the machine length and the rotor radius. Quantity f is the frequency of the induced electromotive forces. In the example given in Fig. 10.6, the frequency f is equal to the number of rotor revolutions per second.

The preceding expression can be written in the form

$$E_{rms} = 4,44 k_P k_T N f \Phi_m^\pi,$$

where $\Phi_m^\pi = 2LRB_m$ represents the maximum value of the flux which would have existed in a single full-pitched turn. In distributed windings with fractional-pitch turns, the quantity Φ_m^π is hypothetical, and expression (10.24) is more suitable.

10.5 Harmonics

The waveform of dynamic electromotive force is determined by distribution of the magnetic field in the air gap. In cases where the magnetic field H and induction B vary sinusoidally along the machine circumference, the electromotive force is a sinusoidal function of time, and it does not contain distortions and higher harmonics. The fundamental or basic frequency component is also called *first harmonic*. The term *higher harmonics* refers to any other harmonic of the order $n > 1$. The presence of higher harmonics distorts the waveform and makes it non-sinusoidal. The field in the air gap appears as a consequence of the magnetomotive forces created by electrical currents in conductors but also due to the presence of permanent magnets on the rotor. In both cases, one of the goals which is set up in the course of machine design is to achieve near-sinusoidal distribution of the magnetic field in the air gap, so as to obtain sinusoidal electromotive forces. This goal cannot be accomplished in full for a number of reasons. For one, conductors making the windings do not have sinusoidal distribution as they have to be placed in slots. Magnetic circuit of electrical machines usually has several tens of slots. Thus, there is a relatively small number of slots available for placing conductors. Electrical current in such conductors produces magnetomotive force and magnetic field in the air gap. Deviation from harmonic distribution of conductors leads to appearance

of higher spatial harmonics of the magnetic induction $B(\theta)$ in the air gap. The presence of higher harmonics makes the spatial distribution of $B(\theta)$ a non-sinusoidal function of the angle θ . Similarly, permanent magnets built in the rotor cannot produce an ideal, sinusoidal distribution of the magnetic field, but they create magnetic field comprising higher spatial harmonics.

With the magnetic field $B(\theta)$ in the air gap and with the rotor rotating at the speed of Ω_m , electromotive forces are induced in stator conductors, proportional to the magnetic inductance and the speed. In the conductor at position $\theta = 0$, the induced electromotive force is $E = R\Omega_m LB(0 - \theta_m)$, and it can be written as $E = kB(0 - \Omega_m t)$. In cases when the spatial distribution $B(\theta)$ contains the fifth harmonic, magnetic induction at position $\theta = 0$ is

$$B_5 \cos 5(\theta - \theta_m) = B_5 \cos 5(0 - \theta_m) = B_5 \cos(5\Omega_m t);$$

thus, the fifth harmonic of the electromotive force induced in the conductor placed at $\theta = 0$ is

$$E_5 = R\Omega_m LB_5 \cos(5\Omega_m t).$$

Therefore, higher spatial harmonics of the function $B(\theta)$ result in higher harmonics of the induced electromotive force.

Higher harmonics of the electromotive force create electromagnetic disturbances and contribute to pulsations in electromagnetic torque. They increase the maximum and rms values of the electrical current with respect to the case with sinusoidal electromotive forces. As a consequence, power of losses in electrical machine increases. For this reason, the windings of electrical machines are designed and built with the aim of minimizing the influence of higher harmonics of the magnetic field to the induced electromotive forces. Most often, it is not possible to obtain an ideal, sinusoidal distribution of the magnetic field. For this reason, higher harmonics are reduced by the proper design of the windings.

A winding consists of a number of conductors connected in series. In each conductor, induced electromotive force depends on the rotor speed and the magnetic inductance in the air gap. Whether sinusoidal or not, these electromotive forces are periodic, AC waveforms with their frequency determined by the rotor speed. The initial phase of electromotive force induced in a conductor depends upon angular position of the slot where the conductor is placed. The electromotive force of the winding is the sum of phase-shifted electromotive forces of individual conductors. The conductors may be connected in the way that higher harmonics of the electromotive force are in counter phase; thus, they will mutually cancel in the process of summing. The phase shift is further dependent upon the order n of harmonic, and this makes the process of harmonic elimination more involved. Namely, in cases where a winding is made in the way that one higher harmonic is canceled, it is possible that in the same process, the other higher harmonic is summed up and augmented.

Practical methods of designing the windings specify the way of placing individual conductors in various slots. In most cases, all the series-connected conductors along with their end turns are made of one single, uninterrupted wire. In such cases, the winding design provides a scheme or a map which indicates position and sequence of slots where the wire is to be inserted. The winding design relies on the fact that the most harm comes from the low-order harmonics. Therefore, most winding design schemes are focused on suppressing low-order harmonics.

The waveform $B(\theta)$ of the spatial distribution of magnetic induction is usually symmetrical with respect to the maximum B_m , and it does not contain even harmonics. Therefore, the induced electromotive forces could contain only odd harmonics. Moreover, the odd harmonics of the order $3n$ are not relevant either. This claim is briefly explained below.

For star-connected three-phase windings, the sum of electrical currents is equal to zero ($i_a + i_b + i_c = 0$). The fundamental components of electrical currents are phase shifted by $2\pi/3$, which drives their sum to zero. Considering harmonics of the order $3n$ (*triplian*), their mutual phase shift is $3n \times 2\pi/3 = n \times 2\pi$; hence, they have the same phase. Therefore, higher triplian harmonics in electrical currents cannot exist, as their sum would not be zero. Without the path for electrical currents, the higher triplian harmonics in electromotive forces are not relevant as they do not produce electrical currents. In a star connection, the electromotive force of the phase winding may comprise a triplian harmonic, but it cannot produce any electrical current. The phase of triplian harmonics of phase electromotive forces is equal due to $3n \times 2\pi/3 = n \times 2\pi$. Therefore, the line voltage, being the difference between the two phase voltages, will be free from triplian harmonics.

For the above reasons, practical approaches to winding design are primarily focused on suppressing the fifth and seventh harmonic. Where possible, the next harmonics to be targeted are the eleventh and thirteenth.

10.5.1 Electromotive Force in Distributed Winding

Practical windings of electrical machines are formed by series connecting of the conductors placed in slots. Usually, there are several tens of slots, meaning that the conductors could be placed at one of several tens of discrete positions. Practical windings are designed to have spatial distribution of their conductors as close to sinusoidal as possible. Winding design techniques include the use of fractional pitch of conductors making one turn (fractional-pitch turns) in order to eliminate or reduce some of the higher harmonics. In addition, conductors of the winding are distributed in winding belts, comprising a number of adjacent slots, and this approach contributes as well to elimination or reduction of higher harmonics in the winding electromotive force. For this reason, further considerations are made in order to analyze the resultant electromotive force in windings with near-to-sinusoidal distribution of their conductors along the machine circumference.

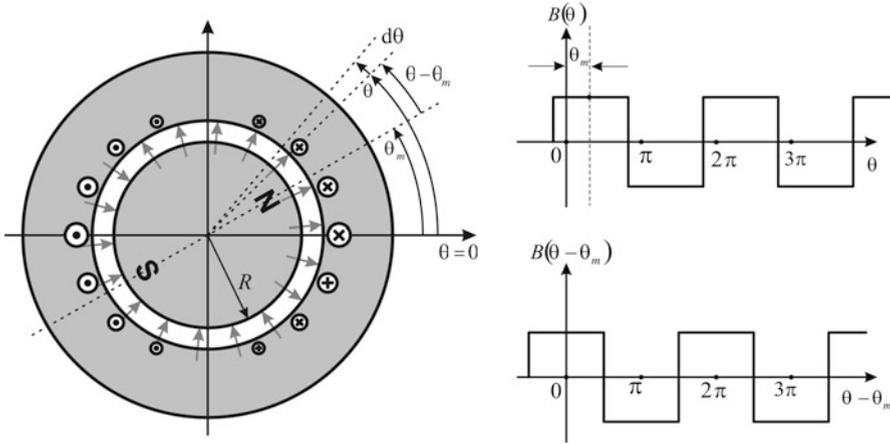


Fig. 10.8 Cross section of an electrical machine comprising one stator winding with sinusoidal distribution of conductors and permanent magnets in the rotor with non-sinusoidal spatial distribution of the magnetic inductance

It will be shown hereafter that an ideal winding with sinusoidal distribution of conductors acts as a *spatial filter* which eliminates completely all the higher harmonics of the resulting electromotive force. In cases where the magnetic induction $B(\theta)$ in the air gap has a non-sinusoidal distribution along the machine circumference, electromotive forces in individual harmonics are non-sinusoidal as well, and they comprise higher-order harmonics. Notwithstanding distortions of $B(\theta)$, the winding with sinusoidal distribution of conductors has a sinusoidal-induced electromotive force, free from higher harmonics and distortions. This statement will be proved at the end of this chapter. Hence, sinusoidal distribution of conductors is an ideal worth striving for. Yet, in practice, it cannot be accomplished since each machine has a relatively small number of slots. Therefore, electromotive forces in practical machines deviate from an ideal sinusoidal form due to the fact that higher harmonics are not completely eliminated.

In the following considerations, the induced electromotive force is calculated for windings where distribution of conductors along the machine circumference is assumed to be perfectly sinusoidal. Analysis is focused on electrical machine shown in Fig. 10.8.

It has permanent magnets on the rotor. The magnets produce the magnetic inductance in the air gap with spatial distribution $B(\theta)$. The function $B(\theta)$ is non-sinusoidal, and it has higher harmonics. The machine under scope has a stator winding with conductors distributed along the inner surface of the stator magnetic circuit. It is assumed that each of these conductors can be placed at an arbitrary location and that the distribution of the conductors along the circumference is sinusoidal. It is going to be proved that in this case, the winding has an induced electromotive force that is sinusoidal, even though the field has a non-sinusoidal distribution in the air gap. The winding plays the role of a spatial filter which

eliminates all harmonics induced in the electromotive force except for the first, fundamental harmonic. Fundamental harmonic of the function $B(\theta)$ is the first member of series (10.25), the member with $i = 1$. It represents the first harmonic of the spatial distribution of the magnetic field. The fundamental harmonic of the winding electromotive force is determined from the first harmonic of the distribution $B(\theta)$ and from the angular frequency $\omega_S = \Omega_m$. Exact expressions are calculated in this section.

In general, magnetic field in the air gap can be the consequence of the rotor currents or permanent magnets built in the rotor magnetic circuit. In both cases, the spatial orientation of the rotor magnetic field depends on the position of the rotor. When the rotor revolves at an angular speed of Ω_m , the rotor magnetic field revolves along with the rotor and has the same speed. As a consequence, electromotive forces are induced in stator conductors. On the basis of Fig. 10.8, distribution of magnetic induction $B(\theta)$ in the air gap is not sinusoidal. It contains higher harmonics, specified by the series (10.25). Calculation of the resultant electromotive force induced in the winding will be carried out with the aim of proving that it contains only the fundamental harmonic.

Magnetic induction created by permanent magnets built in the rotor magnetic circuit can be described by function $B(\theta - \theta_m) = B_m \text{sgn}[\cos(\theta - \theta_m)]$. Over the interval $-\pi/2 < (\theta - \theta_m) < +\pi/2$, the magnetic induction is equal to $+B_m$, while for $+\pi/2 < (\theta - \theta_m) < +3\pi/2$, the induction is $-B_m$. By expanding this function to Fourier series, one obtains

$$B(\theta - \theta_m) = \sum_{i=1}^{\infty} \frac{4}{\pi} \frac{B_m}{2i-1} (-1)^{i+1} \cos[(2i-1)(\theta - \theta_m)]. \quad (10.25)$$

The function contains all odd harmonics, while even harmonics are equal to zero. The amplitude of specific harmonics decrease with their order, $A \sim 1/(2n-1)$. The absence of even harmonics could have been predicted from the fact that the function $B(\theta - \theta_m)$ is symmetrical, $B(x) = B(-x)$. The amplitude of the first harmonic is equal to $4B_m/\pi$.

10.5.1.1 Flux in One Turn

Calculation of the electromotive force induced in the stator winding requires the flux in the winding to be determined first. Then, the electromotive force can be found from the first derivative of the winding flux. Since winding consists of a number of series-connected turns, each one in a different position, the total flux is obtained by adding (integrating) fluxes in individual turns. First of all, it is necessary to determine the flux $\Phi(\theta)$ in one turn. It is assumed that the turn is made of conductors A and B. It is also assumed that the conductor A, denoted by \otimes , resides at position θ , while the conductor B of the same turn, denoted by \odot , resides at

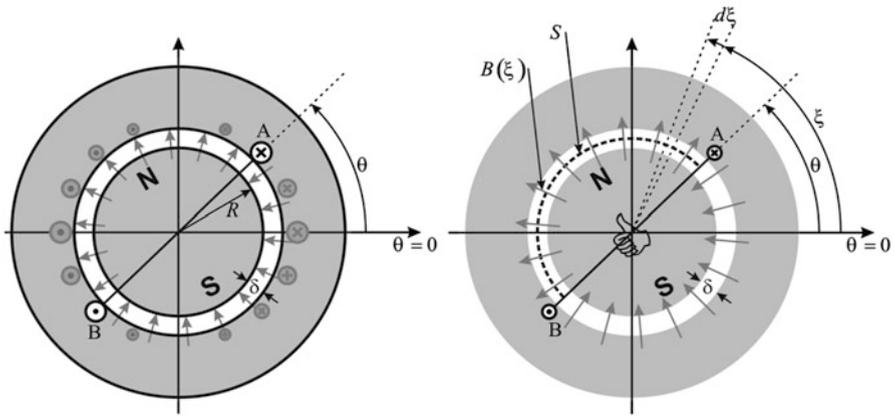


Fig. 10.9 Calculation of the flux in turn A–B (left). Selection of the surface S (right)

position $\theta + \pi$. The considered turn has full pitch. Conductors A and B are shown in Fig. 10.9.

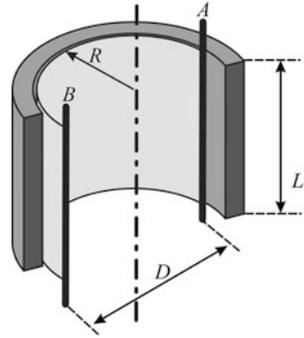
Flux $\Phi(\theta)$ is equal to the surface integral of the vector of magnetic induction \mathbf{B} over the surface S which is leaning on the conductors A and B of the considered turn (Fig. 10.10). The integral is calculated over the surface which lies in the air gap starting from conductor A up to conductor B. The integration surface passing through the air gap is selected because there is an analytical expression for the change of magnetic induction B in terms of the angle θ . The reference direction is determined by the right-hand rule. The flux is determined by integrating the quantity $B(\xi)dS = B(\xi)LRd\xi$ between the limits θ and $\theta + \pi$, as indicated by expression

$$\begin{aligned} \Phi(\theta) &= \int_{\theta}^{\theta+\pi} B(\xi) L \cdot R d\xi \\ &= \int_{\theta}^{\theta+\pi} \left\{ \sum_{i=1}^{+\infty} \frac{4}{\pi} \frac{B_m}{2i-1} (-1)^{i+1} \cos[(2i-1) \cdot (\xi - \theta_m)] \right\} L \cdot R d\xi. \end{aligned} \quad (10.26)$$

The integration is carried out over the surface S passing through the air gap, the elements of which are $dS = LRd\xi$, where L is the axial length of the machine and $R = D/2$ is radius of the rotor cylinder. Expression $B(\xi)$ represents radial component of the magnetic induction at an arbitrary position ξ within the interval $[\theta .. \theta + \pi)$. The result of the integration is the flux determined by equation

$$\Phi(\theta) = LRB_m \frac{8}{\pi} \sum_{i=1}^{\infty} \frac{(-1)^i}{(2i-1)^2} \sin[(2i-1) (\theta - \theta_m)]. \quad (10.27)$$

Fig. 10.10 Semicylinder S extends along the air gap starting from conductor A and ending at conductor B



10.5.1.2 Sinusoidal Distribution of Conductors

Resultant flux ψ_S in the winding is calculated by summing all the fluxes in individual turns. The distributed winding shown in Fig. 10.9 has the line density of conductors along the machine circumference expressed by $N'(\theta)$,

$$N'(\theta) = N_m \cos \theta.$$

The highest density of conductors is at positions $\theta = 0$ and $\theta = \pi$. Within a very small element of the machine circumference $Rd\theta$, there are dN conductors,

$$dN = N'(\theta) \cdot R \cdot d\theta.$$

All the conductors with reference direction \otimes are placed within the interval $[-\pi/2 .. \pi/2)$. Each conductor having reference direction \otimes is connected in series with diametrically placed conductor of direction \odot . These two conductors are connected at machine ends, and they make one turn. The flux $\Phi(\theta)$ within one turn is determined by expression (10.27). Considered turn has the conductor \otimes in position θ and the conductor \odot positioned at $\theta + \pi$.

Resultant flux Ψ_S of the winding is determined by integrating

$$d\Psi_S = N_m \Phi(\theta) R \cos(\theta) \cdot d\theta$$

over the interval $[-\pi/2 .. \pi/2)$,

$$\Psi_S = \int_{-\pi/2}^{\pi/2} N_m \Phi(\theta) R \cos(\theta) \cdot d\theta. \tag{10.28}$$

By introducing $\Phi(\theta)$ from expression (10.27), one obtains that the flux of the winding is

$$\Psi_S = N_m L R^2 B_m \frac{8}{\pi} \sum_{i=1}^{\infty} \frac{(-1)^i}{(2i-1)^2} \int_{-\pi/2}^{+\pi/2} \sin[(2i-1)(\theta - \theta_m)] \cos \theta \cdot d\theta \quad (10.29)$$

If the integrand of the integral (10.29) is written in the form

$$\begin{aligned} f(\theta, i) &= \sin[(2i-1)(\theta - \theta_m)] \cos(\theta) \\ &= 1/2[\sin(2i\theta - (2i-1)\theta_m) + \sin((2i-2)\theta - (2i-1)\theta_m)], \end{aligned}$$

while the value of $(N_m L R^2 B_m 8/\pi)$ is denoted by k , the result (10.29) can be represented as the sum in which each element is an integral of the function $f(\theta, i)$ within limits $-\pi/2$ and $\pi/2$,

$$\Psi_S = k \sum_{i=1}^{\infty} \frac{(-1)^i}{(2i-1)^2} \int_{-\pi/2}^{+\pi/2} f(\theta, i) \cdot d\theta.$$

For $i \geq 2$, each of the elements of the function $f(\theta, i)$ is a sine function with an integer number of its periods on the interval $[-\pi/2 .. \pi/2)$. Therefore, the integral of $f(\theta, i)$ over the interval has a non zero value only for $i = 1$. Any other of the sum but the first is equal to zero. For the first member of the sum, for $i = 1$, the integrand becomes

$$f(\theta, 1) = [\sin(2\theta - \theta_m) + \sin(-\theta_m)]/2.$$

Introducing this expression in (10.29), for $i = 1$, one obtains

$$\Psi_S^1 = N_m L R^2 B_m \frac{8}{\pi} \frac{(-1)^1}{(2-1)^2} \int_{-\pi/2}^{+\pi/2} \sin(\theta - \theta_m) \cos(\theta) \cdot d\theta,$$

which gives the resultant flux in the winding. It has been demonstrated that only the first (fundamental) harmonic of the non-sinusoidal distribution $B(\theta)$ of the magnetic induction produces the flux in the winding which has sinusoidal distribution of its conductors. Since

$$\begin{aligned} \int_{-\pi/2}^{+\pi/2} \sin(\theta - \theta_m) \cos(\theta) \cdot d\theta &= \frac{1}{2} \int_{-\pi/2}^{+\pi/2} [\sin(2\theta - \theta_m) + \sin(-\theta_m)] \cdot d\theta \\ &= -\frac{1}{4} \cos(2\theta - \theta_m) \Big|_{-\pi/2}^{+\pi/2} + \frac{1}{2} \sin(-\theta_m) \cdot \theta \Big|_{-\pi/2}^{+\pi/2} = -\frac{\pi}{2} \sin \theta_m, \end{aligned}$$

the value of the flux Ψ_S^1 is equal to

$$\Psi_S^1 = N_m L R^2 B_m \frac{8(-1)}{\pi} \left(-\frac{\pi}{2} \sin \theta_m \right) = 4N_m L R^2 B_m \sin \theta_m. \quad (10.30)$$

The remaining elements of the sum in expression (10.29) have $i > 1$, and they are all equal to zero. For $i = 2$, the integrand is

$$f(\theta, 2) = [\sin(4\theta - 3\theta_m) + \sin(2\theta - 3\theta_m)]/2.$$

The integral of this function on the interval $[-\pi/2 .. \pi/2]$ is equal to zero. The same hold for any $i > 1$. Therefore, it can be concluded that the induced electromotive force does not have any higher harmonics.

10.5.1.3 Flux of the Winding with Arbitrary Distribution of Conductors

The previous calculation has been carried out in order to demonstrate that a winding with sinusoidal distribution of conductors acts as a spatial filter and removes all the higher harmonics from the electromotive force waveform. It is also of interest to derive the expression for the winding flux in a more general case, where distribution of conductors is described by an arbitrary function $N'(\theta)$.

The first step in calculating the winding flux is getting the flux in the turn placed at position θ . This flux is calculated according to expression (10.26),

$$\Phi(\theta) = \int_{\theta}^{\theta+\pi} B(\xi) L \cdot R d\xi,$$

where function $B(\xi)$ determines distribution of the magnetic induction in the air gap.

The total flux is obtained by summing the fluxes of all individual turns. In the case of a distributed winding, this summing is performed by integration. In the case where line density of conductors is $N'(\theta) = N_m \cos \theta$, the flux of the winding is calculated according to expression

$$\Psi_S = \int_{-\pi/2}^{\pi/2} N_m \Phi(\theta) R \cos \theta \cdot d\theta.$$

From the obtained results, the expression for flux of the winding takes the form

$$\Psi_S = \int_{-\pi/2}^{\pi/2} N_m \left\{ \int_{\theta}^{\pi+\theta} L R B(\xi) d\xi \right\} R \cos \theta \cdot d\theta. \quad (10.31)$$

In general, line density of conductors constituting the winding may have an arbitrary distribution of conductors, described by the function $N'(\theta)$. The total flux of the winding is then calculated by using expression

$$\Psi_S = \int_{-\pi/2}^{\pi/2} N'(\theta) \left\{ \int_{\theta}^{\pi+\theta} LRB(\xi) d\xi \right\} R \cdot d\theta. \quad (10.32)$$

Expression (10.32) can be applied for calculation of the flux in a distributed winding for an arbitrary field distribution $B(\xi)$ and an arbitrary line density of conductors $N'(\theta)$.

10.5.2 Individual Harmonics

The calculation carried out in this section is focused on deriving the electromotive force induced in the stator winding which has sinusoidal distribution of its conductors along the machine circumference. The electromotive force is determined for the case where permanent magnets of the rotor generate the air gap field with non-sinusoidal distribution $B(\theta)$, comprising higher harmonics. It is started with the expression for magnetic induction in the air gap (10.25), which contains the first, fundamental harmonic but also all odd harmonics. Since this non-sinusoidal distribution of the magnetic induction is symmetrical, the function $B(\theta - \theta_m)$ does not comprise even harmonics. In the considered case, the line density of stator conductors is $N'(\theta) = N_m \cos \theta$, while the variation of the magnetic induction in the air gap is given in (10.25). This expression represents a development of the function $B(\theta)$ into a series comprising only odd members. The first element of the series has $i = 1$, and it represents the fundamental harmonic of the spatial distribution of magnetic induction. For $i > 1$, elements of the series represent higher harmonics of the spatial distribution of the field. On the basis of expression (10.25), spatial harmonic of function $B(\theta)$ of the order $(2i - 1)$ is equal to

$$\frac{4}{\pi} \frac{B_m}{2i - 1} (-1)^{i+1}.$$

By introduction of the latter into expression (10.31), one obtains the quantity Ψ_S^{2i-1} which represents the contribution of the harmonic $(2i - 1)$ to the total winding flux. The value of Ψ_S^{2i-1} is related to $(2i - 1)$ th harmonic of the spatial distribution of magnetic induction.

$$\Psi_S^{2i-1} = \frac{4N_m LR^2 B_m (-1)^{i+1}}{\pi(2i-1)} \times \int_{-\pi/2}^{\pi/2} \left\{ \int_{\theta}^{\pi+\theta} \cos[(2i-1)(\xi - \theta_m)] d\xi \right\} \cos \theta \cdot d\theta. \quad (10.33)$$

By integration of the function $\cos[(2i-1)(\xi - \theta_m)]$ in terms of ξ one obtains

$$\int \cos[(2i-1)(\xi - \theta_m)] d\xi = \frac{1}{2i-1} \sin[(2i-1)(\xi - \theta_m)].$$

The obtained result can be used in expression (10.33) in order to calculate definite integral of the integrand in terms of ξ , within limits from θ up to $\theta + \pi$. The calculation results in

$$\frac{1}{2i-1} \sin[(2i-1)(\xi - \theta_m)] \Big|_{\theta}^{\theta+\pi} = -\frac{2}{2i-1} \sin[(2i-1)(\theta - \theta_m)].$$

By introducing developed results into previous expression, it becomes

$$\Psi_S^{2i-1} = \frac{8N_m LR^2 B_m (-1)^i}{\pi(2i-1)^2} \int_{-\pi/2}^{\pi/2} \sin[(2i-1)(\theta - \theta_m)] \cos \theta \cdot d\theta. \quad (10.34)$$

Since

$$\begin{aligned} \sin[(2i-1)(\theta - \theta_m)] \cos \theta &= \frac{1}{2} \sin[2i\theta - (2i-1)\theta_m] \\ &\quad + \frac{1}{2} \sin[(2i-2)\theta - (2i-1)\theta_m], \end{aligned}$$

the obtained result can be separated into two definite integrals, I_A and I_B

$$\begin{aligned} \Psi_S^{2i-1} &= \frac{4N_m LR^2 B_m (-1)^i}{\pi(2i-1)^2} \int_{-\pi/2}^{\pi/2} \sin((2i-2)\theta - (2i-1)\theta_m) \cdot d\theta \\ &\quad + \frac{4N_m LR^2 B_m (-1)^i}{\pi(2i-1)^2} \int_{-\pi/2}^{\pi/2} \sin(2i\theta - (2i-1)\theta_m) \cdot d\theta = I_A + I_B. \end{aligned} \quad (10.35)$$

Since index i varies from 1 to $+\infty$, the integral I_B is equal to zero since the interval $[-\pi/2 .. \pi/2)$ comprises an integer multiple of periods of the function $\sin(2i\theta)$.

The same conclusion applies for the integral I_A if $i > 1$. Therefore, a nonzero value of flux Ψ_S^{2i-1} exists only for the first (fundamental) harmonic, namely, for $i = 1$. With any $i > 1$, the integrand function is a sine wave with its period comprised an integer number of times within the integration domain $[-\pi/2 .. \pi/2]$.

On the basis of the obtained results, it is concluded that a sinusoidal distribution of the winding conductors along the machine circumference prevents all the higher harmonics of $B(\theta)$ from affecting the winding flux. For that reasons, the induced electromotive force of the winding remains unaffected by higher harmonics in $B(\theta)$ waveform.

Hence, sinusoidal distribution of conductors results in a winding which eliminates all the higher harmonics of the induced electromotive force, retaining only the fundamental harmonic.

10.5.3 Peak and rms of Winding Electromotive Force

On the basis of the obtained results, the following passages provide the expressions for the instantaneous, peak and rms values of the electromotive force induced in the considered stator winding. Expressions (10.37), (10.38), and (10.39) apply for windings with distribution of conductors $N'(\theta) = N_m \cos\theta$ and for distribution of the magnetic induction in the air gap shown in Fig. 10.8 and described by (10.25).

10.5.3.1 Suppression of Higher Harmonics

Results obtained so far indicate that sinusoidal distribution of conductors eliminates higher harmonics of the electromotive force induced in a winding. In a winding with an ideal, sinusoidal distribution of conductors, a sinusoidal electromotive force is induced notwithstanding the higher harmonics in the spatial distribution of magnetic induction $B(\theta)$. Any harmonic of the order $(2i - 1)$ in the spatial distribution of $B(\theta)$ gives its contribution of Ψ_S^{2i-1} to the total flux of the winding, and this contribution is given in expression (10.35). For any $i > 1$, contribution Ψ_S^{2i-1} is equal to zero. Therefore, the winding performs the role of the spatial filter which eliminates the effects of higher harmonics of the spatial distribution of $B(\theta)$, and it passes only the fundamental harmonic.

10.5.3.2 Winding Flux

Only the fundamental harmonic of distribution $B(\theta)$ contributes to the winding flux. The winding flux is determined by the following equation:

$$\Psi_S = \Psi_S^1 = 4B_m N_m L R^2 \sin \theta_m. \quad (10.36)$$

10.5.3.3 Electromotive Force

If the rotor revolves at a constant speed Ω_m , position of the rotor varies according to the law $\theta_m = \Omega_m t$. The flux in the stator winding is then a sinusoidal function of time with angular frequency ω_S determined by the angular speed Ω_m of the rotor. With $\theta_m = \Omega_m t$, variation of the flux is given by equation

$$\Psi_S(t) = 4B_m N_m L R^2 \cdot \sin \Omega_m t = \Psi_{S\max} \sin \Omega_m t.$$

A sinusoidal (counter) electromotive force is induced in the winding, and it is equal to the first derivative of the flux,

$$e_S(t) = \frac{d}{dt} \Psi_S = 4\Omega_m B_m N_m L R^2 \cos \Omega_m t = \Psi_{S\max} \Omega_m \cos \Omega_m t. \quad (10.37)$$

The maximum value of this sinusoidal electromotive force of the stator winding is

$$e_{\max} = \Psi_{S\max} \Omega_m = 4\Omega_m B_m N_m L R^2, \quad (10.38)$$

while its rms value is equal to

$$e_{\text{rms}} = \frac{1}{\sqrt{2}} \Psi_{S\max} \Omega_m = 2\sqrt{2} \Omega_m B_m N_m L R^2.$$

The maximum and rms values of the electromotive force are expressed in terms of N_m , the maximum density of the stator conductors along the machine circumference. Instead, they can be expressed as functions of the number of turns N_N . For the winding with sinusoidally distributed conductors, the expression (8.2) relates the number of turns N_N to the maximum line density of its conductors N_m ,

$$N_T = 2R \cdot N_m;$$

thus, the maximum value of the electromotive force can be calculated from

$$e_{\max} = 2\Omega_m B_m N_T L R,$$

while the rms value of the electromotive force in the winding can be calculated from

$$e_{\text{rms}} = \sqrt{2} \Omega_m B_m N_T L R. \quad (10.39)$$

Performed analysis shows that the electromotive force induced in a winding with sinusoidal distribution of conductors does not contain higher harmonics, which proves that such a winding performs the role of a spatial filter.

Question (10.4): Consider the electrical machine which is the subject of the previous analysis, shown in Fig. 10.8. Assume that the number of conductors does not change but that they are concentrated at positions $\theta = 0$ and $\theta = \pi$. Determine the shape and amplitude of the electromotive force induced in the stator winding.

Answer (10.4): Electromotive force $e_{1C} = LvB(\theta)$ is induced, in each conductor, where L is the machine axial length, $B(\theta)$ is magnetic induction at angular position θ of the conductor placement, while $v = R\Omega_m$ is the peripheral rotor speed. The speed v reflects the *relative* movement of conductors with respect to the field. The spatial distribution of magnetic induction $B(\theta)$ is shown in Fig. 10.8. Therefore, a complex periodic electromotive force is induced with rectangular shape and with the period and frequency determined by the rotor speed. The conductors of the concentrated winding are placed at positions $\theta = 0$ and $\theta = \pi$. Since $B(0) = -B(\pi)$, the electromotive forces induced in diametrical conductors are of the opposite signs. The way of connecting a pair of conductors into one turn leads to subtracting of the respective electromotive forces. Subtracting the two values of the same amplitude and of the opposite sign results in electromotive force in one turn e_{1T} which is twice larger than e_{1C} . Hence, the electromotive force induced in one turn is equal to $e_{1T} = 2 e_{1C} = 2 LR\Omega_m B(0)$. For a concentrated winding with N turns, the electromotive force of the winding is equal to $e_w = 2 NLR\Omega_m B(0)$. It has the shape of a train of rectangular pulses with an amplitude of

$$e_w^{\max} = 2\Omega_m B_{\max} N_k LR.$$

Using the relations expressed in (10.25), the maximum value of the first harmonic of this train of rectangular pulses is $4/\pi$ times higher than the amplitude of the pulses. Therefore, the rms value of the first harmonic of the electromotive force induced in the concentrated winding is equal to

$$\begin{aligned} e_w^{\text{rms}} &= \frac{4}{\pi} \cdot \frac{1}{\sqrt{2}} 2\Omega_m B_{\max} N_T LR \\ &= \frac{4\sqrt{2}}{\pi} \Omega_m B_{\max} N_T LR \approx 1,8 \Omega_m B_{\max} N_T LR. \end{aligned}$$

It is of interest to compare this result with the rms value of the electromotive force obtained in the winding with the same number of turns but with sinusoidal distribution of the conductors. On the basis of (10.39), this value is

$$e_{\text{rms}} = \sqrt{2}\Omega_m B_{\max} N_T LR \approx 1,41 \Omega_m B_{\max} N_T LR.$$

It can be concluded that the rms value of the induced electromotive force in the winding with distributed conductors is $4/\pi$ times smaller compared to the electromotive force in the winding with concentrated conductors. The former amounts approximately 78.5% of the latter.

Question (10.5): Consider electrical machine where permanent magnets on the rotor create magnetic induction with sinusoidal distribution in the air gap, $B(\theta) = B_m \cos(\theta - \theta_m)$, where θ_m is displacement of the rotor with respect to the stator. The stator winding has N_T turns, hence $2N_T$ conductors. The winding can be realized in two ways. The first way is forming a concentrated winding having conductors located at positions $\theta = 0$ and $\theta = \pi$. The other way of making the winding is to have sinusoidal distribution of conductors, with the conductor density $N'(\theta) = N_{sm} \cos(\theta)$ and with the total number of conductors being $2N_T$. Determine the maximum value of the stator flux $\Psi_S(\theta_m)$ in both cases. Geometry of the machine is the same as the one considered in Question 10.4.

Answer (10.5): Relation between the peak conductor density N_{sm} and total number of turns N_T is

$$N_T = \int_{-\pi/2}^{+\pi/2} |N'_S(\theta)| R d\theta = 2RN_{sm},$$

where R denotes the radius of the rotor. The maximum flux achievable in one turn is equal to the product of the average value of magnetic induction wave $B(\theta)$, $B_{av} = 2B_m/\pi$, and the surface area $S = \pi RL$ of the semicircular surface encircled by the turn, wherein the turn is made of two diametrical conductors. The maximum value of the flux is obtained as $\Phi_m = 2B_m LR$.

In cases where the stator winding is concentrated, the stator flux reaches the maximum value of $\Psi_1 = 2B_m LR N_T$. This value is achieved with rotor in position $\theta_m = \pi/2$, when the vector of the rotor flux gets collinear with the magnetic axis of the stator winding.

If the turns of the stator winding are distributed, the stator flux is denoted by $\Psi_2(\theta_m)$. Conductor density is denoted by $N'(\theta) = N_{sm} \cos(\theta)$, and the stator flux Ψ_2 with rotor in position $\theta_m = \pi/2$ is calculated from

$$\begin{aligned} \Psi_2 &= \int_{-\pi/2}^{+\pi/2} \Phi(\theta) dN = \int_{-\pi/2}^{+\pi/2} \left\{ \int_{\theta}^{\theta+\pi} RLB_m \cos(\xi - \pi/2) d\xi \right\} RN'_S(\theta) d\theta \\ &= \int_{-\pi/2}^{+\pi/2} \left\{ \int_{\theta}^{\theta+\pi} RLB_m \sin(\xi) d\xi \right\} RN_{S\max} \cos(\theta) d\theta \\ &= \frac{RN_{S\max}}{2} \int_{-\pi/2}^{+\pi/2} \left\{ \int_{\theta}^{\theta+\pi} \Phi_m \sin(\xi) d\xi \right\} \cos(\theta) d\theta \\ &= \frac{RN_{S\max} \Phi_m}{2} \int_{-\pi/2}^{+\pi/2} \{2 \cos(\theta)\} \cos(\theta) d\theta \\ &= R\Phi_m \frac{N_T}{2R} \frac{\pi}{2} = \frac{\pi}{2} B_m LR N_T = \frac{\pi}{4} \Psi_1. \end{aligned}$$

Based on the above calculation, conclusion is drawn that, all the remaining conditions being equal, the machine with sinusoidally distributed stator winding has the peak stator flux which is $\pi/4$ times lower than the peak stator flux in the machine with concentrated stator winding.