Chapter 9
Energy, Flux, and Torque

Magnetic field in the air gap is obtained from electrical currents in stator and rotor windings. Another source of the air gap field can be permanent magnets that may be placed within magnetic circuits of either stator or rotor. The stator and rotor fields in the air gap are calculated in the previous chapter. Interaction of the two fields incites the process of electromechanical conversion.

In this chapter, expressions for the magnetic field in the air gap are used to calculate the field energy, to derive the energy accumulated in the magnetic field, and to calculate the electromagnetic torque caused by the interaction between the stator and rotor fields. In order to simplify the analysis, the flux linkages in one turns and the winding fluxes are represented by flux ***vectors***. The concept of flux ***vector*** is introduced and explained along with magnetic axes of turns and windings. The torque expression is rewritten and expressed as the ***vector*** product of stator and rotor flux ***vectors***. It is pointed out that continuous torque generation requires either stator or rotor windings to create the revolving magnetic field. This chapter ends with the analysis of two-phase windings systems and three-phase winding systems that create revolving magnetic field.

9.1 Interaction of the Stator and Rotor Fields

Electrical machines usually have windings on both stator and rotor. Currents through the windings create stator and rotor fluxes. There are machines which have permanent magnets instead of the stator or rotor winding. Magnetic field in the air gap has its radial and tangential components. The radial component is R/δ times larger than the tangential. It determines the spatial distribution of the magnetic energy of the field, as well as the course and direction of the field lines.

The stator and rotor fields exist in the same air gap and the same magnetic circuit. They add up and make the resulting magnetic field and the resulting flux. Assuming that magnetic circuit is linear ($\mu = \text{const.}$), the resulting field is obtained by superposition of stator and rotor fields. Namely, the strength of the resulting

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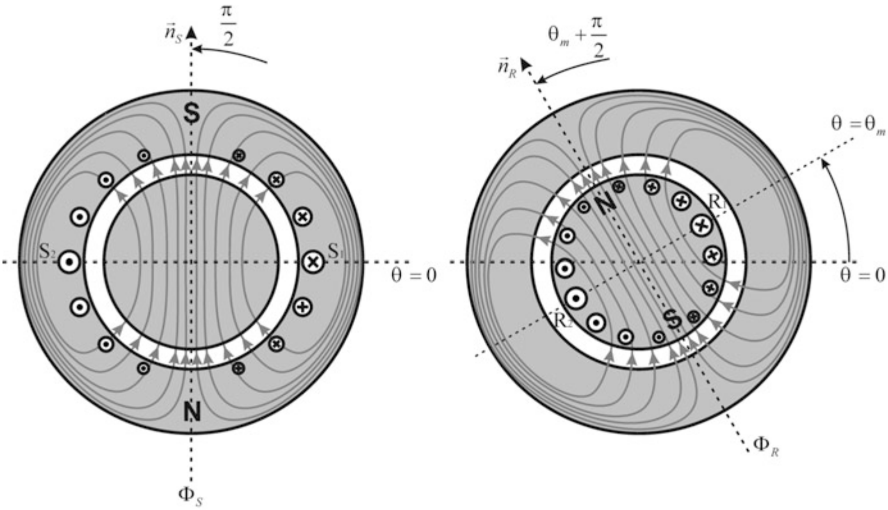

Fig. 9.1 Magnetic fields of stator and rotor

field is obtained by adding the two fields. The lines of the stator and rotor fields are shown in Fig. 9.1, which presents the course and direction of relevant flux ***vectors*** and magnetic axes. It is assumed that both the stator and rotor have a number of sinusoidally distributed conductors. For clarity, Fig. 9.1 shows just a few conductors which denote distributed windings. Conductors S1 and S2 of the stator winding are placed in positions with the maximum density of the stator conductors. The normal of the stator turn S1–S2 is, at the same time, the magnetic axis of the stator winding. In the same way, the axis of the rotor winding is determined by the normal of the rotor turn R1–R2.

In Fig. 9.1, direction of the stator field and flux is determined by normal n_S of the stator turn S1–S2. This normal extends in direction shifted by π/2 with respect to position $\theta = 0$, where the density of the stator conductors reaches its maximum. Direction of the rotor field and flux is determined by normal n_R to the rotor turn R1–R2, which extends in direction shifted by π/2 with respect to position $\theta = \theta_m$, where the density of the rotor conductors reaches its maximum. For this reason, direction of the rotor flux is shifted by $\theta_m + \pi/2$ with respect to position $\theta = 0$. When the stator and rotor conductors have constant currents (DC currents), the stator flux ***vector*** remains in its position (vertical position in Fig. 9.1), while the rotor flux ***vector*** revolves along with the rotor. In this case, the angle between the two flux ***vectors*** is $\Delta\theta = \theta_S - \theta_R = -\theta_m$, due to the rotor displacement of θ_m. In cases where the stator and/or the rotor has two or more windings with alternating currents, the angle between the two flux ***vectors*** can be different than θ_m. Namely, a set of stator (rotor) windings with the proper orientation of their magnetic axes creates the magnetic field and the flux ***vector*** which revolve with respect to their originator.

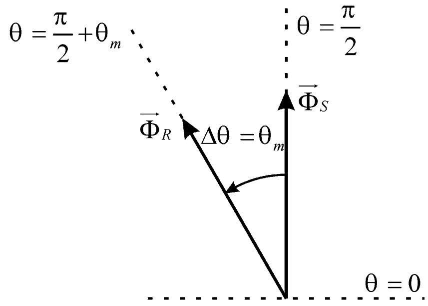

Fig. 9.2 Mutual position of the stator and rotor fluxes

For that to be achieved, the winding currents must have the appropriate frequency and the initial phases. Figure 7.9 shows an example where the two orthogonal stator windings with alternating currents create magnetic field which revolves with respect to the stator. In this case, position of the flux ***vector*** and the angular difference $\Delta\theta$ between the two fluxes depend not only on the rotor position bus also on the supply frequency and the initial phase of the winding currents.

Description and further analysis are facilitated by ***vector*** representation of the stator and rotor fields in the manner shown in Fig. 9.2. The field of the magnetic induction ***B*** can be represented by the flux ***vector***, in accordance with conclusions presented in Sects. 4.4 and 5.5, as well as in Sect. 8.5, formulating the convention of ***vector*** representation of magnetic fields. *Flux* ***vector*** of one turn is obtained by associating the course and direction with scalar Φ. This course and direction is obtained from the unit ***vector*** of the normal to the surface encircled by the relevant turn. Figure 9.2 shows the flux ***vector*** of the turn S1–S2 and the flux ***vector*** of the turn R1–R2. These ***vectors*** represent the magnetic fields of the stator and rotor shown in Fig. 9.1. Scalar value Φ_S represents the flux of the turn determined by the stator conductors S1–S2, placed in the region with maximum density of stator conductors. ***Vector*** Φ_S has the course and direction obtained from the normal to the surface encircled by the turn S1–S2. The same way, scalar value Φ_R represents the flux of the turn determined by the rotor conductors R1–R2, placed in the region with maximum density of rotor conductors. ***Vector*** Φ_R has the course and direction obtained from the normal to the surface encircled by the turn R1–R2.

By interaction of the stator and rotor magnetic fields, electromagnetic torque is created as a mechanical interaction between the stator and the rotor. Since the rotor can revolve, this torque can bring the rotor into rotation or change the speed of the rotor revolutions. The torque is created due to an interaction of the stator and rotor magnetic fields. Therefore, it is also called electromagnetic torque, T_{em}. Considering the force of attraction between different magnetic poles, it can be concluded that the electromagnetic torque tends to move the rotor in a way that brings closer the north magnetic pole of the rotor and the south magnetic pole of the stator. The torque acts toward bringing the two opposite poles one against the other. In terms of the flux ***vectors***, the electromagnetic torque tends to align the stator and rotor flux ***vectors***. It will be shown further on that the electromagnetic torque can be expressed as the ***vector*** product of the stator and rotor flux ***vectors***.

Question (9.1): Assume that stator magnetic poles do not move with respect to the stator. In addition, assume that rotor magnetic poles do not move with respect to the rotor. If the rotor revolves at a constant speed, what is the change of the torque acting on the rotor?

Answer (9.1): In the considered case, the angle $\Delta\theta$ between the flux ***vectors*** of the stator and rotor is equal to the shift θ_m. If the rotor revolves at a constant speed, the variation of the created electromagnetic torque will be sinusoidal.

9.2 Energy of Air Gap Magnetic Field

It is of interest to determine the electromagnetic torque acting on the rotor and stator of a cylindrical machine. This torque can be determined as the first derivative of the energy accumulated in the magnetic (coupling) field in terms of the rotor displacement θ_m. On the basis of the equations given in Sect. 6.9, the increment of mechanical work $\mathrm{d}W_{meh} = T_{em}\mathrm{d}\theta_m$ is equal to the increment of energy of the magnetic field; thus, the torque can be determined as the first derivative of the magnetic field energy in terms of the rotor shift θ_m, $\mathrm{d}W_m/\mathrm{d}\theta_m$. Therefore, it is necessary to determine the energy of the magnetic field in terms of the rotor position, $W_m(\theta_m)$.

The energy of the magnetic field can be calculated by integrating the density of the field energy w_m over the entire domain where the magnetic field exists. The energy density w_m is expressed in J/m^3, and it represents the amount of the field energy comprised within unit volume; thus, $w_m = \Delta W_m/\Delta V = \mathrm{d}W_m/\mathrm{d}V$. Expression $w_m = ½\ \mu H^2$ determines the density of the field energy in a linear medium, where the magnetic permeability does not change. Therefore, the density of the field energy in the air gap is $w_m = ½\mu_0 H^2$.

Magnetic field exists in the magnetic circuits of the stator and rotor which are made of iron, as well as in the air gap. Since the same magnetic flux which passes through the air gap gets into the stator and rotor magnetic circuits, magnetic induction in iron B_{Fe} and in the air gap B_0 is roughly the same. The permeability of iron μ_{Fe} is several orders of magnitude higher than the permeability of the air μ_0. Therefore, the magnetic field in iron $H_{Fe} = B_{Fe}/\mu_{Fe}$ is negligible compared to the field H_0 in the air gap. The same way, the density of the field energy in iron ($½B^2/\mu_{Fe}$) is negligible when compared to the density of the field energy in the air gap ($½B^2/\mu_0$). For this reason, the overall energy of the magnetic field can be determined by integrating the density of the field energy (specific energy) over the whole domain of the air gap.

In the expression for the field energy density $w_m = ½\mu_0 H^2$, the symbol H represents the strength of the resultant magnetic field in the air gap, namely, the sum of the stator and rotor fields. Since the tangential components of the magnetic field are negligible ($\delta \ll R$), the strength H of the resulting magnetic field in the air gap is equal to the sum of radial components of the stator and rotor fields.

The expression for the density of the resulting magnetic field takes the form $w_m = ½\ \mu_0(H_r^S + H_r^R)^2$.

By using (8.18) and (8.30), which give the radial components of the stator and rotor fields in the air gap at position θ, one obtains the function which determines the density of the magnetic field energy as a function of angle θ,

$$w_m(\theta) = \frac{\mu_0}{2}\left(\frac{R}{\delta}\right)^2 [J_{R0}\sin(\theta-\theta_m) + J_{S0}\sin\theta]^2. \tag{9.1}$$

The total energy accumulated in magnetic field is given by expression

$$W_m = \int\limits_V w_m(\theta)\mathrm{d}V,$$

where V is the total volume of the air gap. Since the elementary volume is obtained as

$$\mathrm{d}V = L\,\delta\,R\,\mathrm{d}\theta,$$

the total magnetic field energy in a cylindrical electrical machine of the length L, radius R, and the air gap δ becomes

$$W_m = L\delta R\int\limits_0^{2\pi} w_m(\theta)\mathrm{d}\theta. \tag{9.2}$$

By introducing (9.1) into (9.2), one obtains the expression

$$\begin{aligned} W_m = \frac{\mu_0 R^3 L}{2\delta} &\left[\int\limits_0^{2\pi} J_{R0}^2 \sin^2(\theta-\theta_m)\mathrm{d}\theta \right. \\ &\left. + \int\limits_0^{2\pi} J_{S0}^2\sin^2(\theta)\mathrm{d}\theta + \int\limits_0^{2\pi} 2J_{R0}J_{S0}\sin(\theta-\theta_m)\sin(\theta)\mathrm{d}\theta\right] \\ = \frac{\mu_0 R^3 L}{2\delta} &\left[J_{R0}^2 I_1 + J_{S0}^2 I_2 + 2J_{R0}J_{S0}I_3\right], \end{aligned} \tag{9.3}$$

where J_{R0} represents the maximum value of the line density of the rotor currents while J_{S0} represents the corresponding value for the stator currents. Evaluation of the expression (9.3) requires finding the three integrals of trigonometric integrand functions, I_1, I_2, and I_3. Since

$$\sin^2\theta = \frac{1}{2}[1-\cos(2\theta)], \quad \sin^2(\theta-\theta_m) = \frac{1}{2}[1-\cos(2\theta-2\theta_m)],$$

the integrals I_1 and I_2 take values

$$I_1 = \int_0^{2\pi} \sin^2\theta \mathrm{d}\theta = \int_0^{2\pi} \frac{1}{2}[1 - \cos(2\theta)]\mathrm{d}\theta = \pi,$$

$$I_2 = \int_0^{2\pi} \sin^2(\theta - \theta_m)\mathrm{d}\theta = \int_0^{2\pi} \frac{1}{2}[1 - \cos(2\theta - 2\theta_m)]\mathrm{d}\theta = \pi.$$

By using equation

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)],$$

the integrand function of the third integral becomes

$$\sin(\theta - \theta_m)\sin(\theta) = \frac{1}{2}[\cos(-\theta_m) - \cos(2\theta - \theta_m)] \,.$$

Considering the integral boundaries 0 and 2π,

$$I_3 = \int_0^{2\pi} \sin(\theta - \theta_m)\sin(\theta)\mathrm{d}\theta = \int_0^{2\pi} \frac{1}{2}[\cos(-\theta_m) - \cos(2\theta - \theta_m)]d\theta$$
$$= \int_0^{2\pi} \frac{1}{2}\cos\theta_m \mathrm{d}\theta + \int_0^{2\pi} \frac{1}{2}\cos(2\theta - \theta_m)\mathrm{d}\theta = \pi\cos\theta_m.$$

Finally, the expression for the energy of the magnetic field becomes

$$W_m = \frac{\mu_0 R^3 L\pi}{2\delta} \left[J_{R0}^2 + J_{S0}^2 + 2J_{R0}J_{S0}\cos\theta_m\right]. \tag{9.4}$$

It is important to recall that all previous considerations start with the assumptions that both stator and rotor windings carry DC currents; thus, the angle $\Delta\theta$ between the stator and rotor flux ***vector*** is equal to $-\theta_m$. On the basis of (9.4), the energy of magnetic field has its maximum value in the case when the ***vector*** of the stator flux is collinear with the ***vector*** of the rotor flux, that is, when $\Delta\theta = -\theta_m = 0$.

As already mentioned, the stator and/or rotor may have several windings with their magnetic axes shifted in space. With sinusoidal currents of the corresponding amplitudes, frequencies, and initial phases, it is possible to achieve the resultant magnetomotive force which keeps the amplitude constant while rotating at the speed determined by the frequency of the winding currents. Revolving magnetomotive force creates the revolving magnetic field and flux in the air gap which can be represented by rotating flux ***vector***. One way of creating revolving magnetic field is shown in Fig. 7.9. In machines with alternating currents on the stator and/or rotor, the angle between the stator and rotor flux ***vectors*** depends on the rotor position, but it also depends on instantaneous values of the winding currents. For this reason, relation $\Delta\theta = -\theta_m$ is not valid unless the windings have DC currents, such as in the case shown in Fig. 9.1.

In general, expression for the total energy of magnetic field takes the form

$$W_m = \frac{\mu_0 R^3 L\pi}{2\delta} \left[J_{R0}^2 + J_{S0}^2 + 2J_{R0}J_{S0}\cos(\Delta\theta)\right] \tag{9.5}$$

where $\Delta\theta$ is the angle between the stator and rotor flux ***vectors***.

9.3 Electromagnetic Torque

The energy of the magnetic field in electrical machine shown in Fig. 9.1 is given by expression (9.4). Machine under consideration has one distributed winding on the stator and one distributed winding on the rotor. The windings carry constant (DC) currents. By using the expression for the field energy, it is possible to determine the electromagnetic torque.

The electromagnetic torque is a measure of mechanical interaction between the stator and rotor. The torque of the same amplitude acts on both stator and rotor in different directions. Under conditions when the stator is fixed and does not move, the torque cannot make the stator turn. On the other hand, the rotor has the freedom to turn. Therefore, the torque can make the rotor revolve and/or it can alter the rotor speed. The angle θ_m denotes shift of the rotor with respect to the stator. Under circumstances, the angle θ_m also determines the shift between the two windings as well as the angle $\Delta\theta$ between the stator and rotor flux ***vectors***. Expression for the torque is $T_{em} = +\mathrm{d}W_m/\mathrm{d}\theta_m$. It has positive sign due to the assumption that the windings are connected to corresponding electrical power sources. Hence, considered electrical machine acts as an electromechanical converter connected to the power source, hence the expression $T_{em} = +\mathrm{d}W_m/\mathrm{d}\theta_m$. Moreover, it is assumed that the stator and rotor windings are supplied from controllable current sources. Therefore, electrical currents in the windings do not depend on the rotor position θ_m. For this reason, the line densities of electrical currents J_{R0} and J_{S0} do not depend on the rotor position θ_m, and their first derivatives $\mathrm{d}J_{R0}/\mathrm{d}\theta_m$ and $\mathrm{d}J_{S0}/\mathrm{d}\theta_m$ are equal to zero. Under the circumstances, electrical currents do not change as the rotor moves by $\mathrm{d}\theta_m$. For the purpose of calculating $+\mathrm{d}W_m/\mathrm{d}\theta_m$, electrical currents can be considered constant, resulting in $\Delta\theta = -\theta_m$ and $\cos(\Delta\theta) = \cos(\theta_m)$. Therefore, expression for the electromagnetic torque becomes

$$T = +\frac{\mathrm{d}W_m}{\mathrm{d}\theta_m} = \frac{\mathrm{d}}{\mathrm{d}\theta_m}\left\{\frac{\mu_0 R^3 L\pi}{2\delta}\left[J_{R0}^2 + J_{S0}^2 + 2J_{R0}J_{S0}\cos\theta_m\right]\right\}$$

or

$$T = \frac{\mathrm{d}}{\mathrm{d}\theta_m}\left\{\frac{\mu_0 R^3 L J_{R0} J_{S0}}{\delta}\pi\cos\theta_m\right\}. \tag{9.6}$$

The torque is given by expression (9.7), and it is proportional to the fourth power of machine dimensions and inversely proportional to the air gap δ:

$$T = -\frac{\mu_0 \pi R^3 L}{\delta} J_{R0} J_{S0} \sin\theta_m. \tag{9.7}$$

The sign of the obtained torque is negative. This means that the torque acts in direction which is opposite to the reference counterclockwise direction. In the preceding sections, electrical machine is presented in cylindrical coordinate system where z-axis is directed toward the reader (⊙). From the reader's viewpoint, the reference direction of rotation around this axis is counterclockwise. The torque which supports the motion in counterclockwise direction can be represented as a ***vector*** collinear with z-axis. This association can be supported by the right-hand rule. The counterclockwise direction is adopted as the reference direction for the angular speed and torque. With that in mind, positive torque excites and supports the motion in counterclockwise (positive) direction. While the rotor revolves at a positive angular speed, a positive torque tends to increase the speed. On the other hand, torque of negative value excites and supports the motion in clockwise (negative) direction. While the rotor revolves at a positive angular speed, a negative torque tends to decrease the speed. The system in Fig. 9.1 tends to draw the north pole of the rotor toward the south pole of the stator and, hence, generates a negative torque, acting in clockwise direction.

The torque in (9.7) is proportional to the product of the stator currents, the rotor currents, and the sine of the displacement θ_m. In the case under consideration, the stator and rotor currents are constant, DC currents. Therefore, position of the stator flux Φ_S is determined by position of the stator. In other words, the stator flux does not move. At the same time, the position of the rotor flux Φ_R is determined by the position of the rotor itself. Therefore, the stator and rotor flux ***vectors*** are displaced by θ_m. Hence, the torque is proportional to the sine of the angle between the two fluxes. With that in mind, there are good grounds for expressing the torque ***vector*** in terms of the ***vector*** product of the stator and rotor flux ***vectors***. This statement will be proved in the subsequent sections.

Question (9.2): Assume that the rotor is turning at a constant speed. What is the average value of the torque in the case where the stator and rotor windings both have DC currents?

Answer (9.2): The electromagnetic torque is a sinusoidal function of the angle between the stator and rotor flux ***vectors***. In cases with no change in the relative position of the two fluxes, this angle does not change, neither does the sine of the angle. Therefore, there are conditions for generating a constant, nonzero torque. If the angle between the two fluxes keeps changing at a constant rate, the electromagnetic torque is sinusoidal function of time, and it has an average value equal to zero. In the given case, both windings have DC currents, and they generate the flux ***vectors*** which stay aligned with magnetic axes of corresponding windings. Since the rotor is turning, the rotor flux revolves with respect to the stator flux. Therefore, the average value of the torque will be equal to zero.

9.3.1 The Torque Expression

Equation (9.7) gives the electromagnetic torque of the electrical machine shown in Fig. 9.1, whose windings carry DC currents. In all the cases where the windings have constant (DC) currents, position of the stator flux ***vector*** is determined by the position of the stator itself, while position of the rotor flux ***vector*** tracks the position of the rotor. Therefore, the angle $\Delta\theta$ between the two ***vectors*** is equal to $-\theta_m$.

In cases where the stator (or rotor) has a set of windings with alternating (AC) currents, position of the flux ***vector*** is not uniquely determined by position of the stator (rotor); it also depends on electrical currents in the windings. Under proper conditions, AC currents create rotating magnetic field, that is, the field which revolves with respect to the windings. Creation of rotating magnetic field is analyzed in detail in Section 9.9, *Rotating magnetic field*. It is of interest to calculate the electromagnetic torque in cases where the stator and/or rotor windings have AC currents and create rotating magnetic field.

Starting from Figs. 9.1 and 9.2 and assuming that the windings carry DC currents, position of the stator flux ***vector*** $\theta_{\Psi S}$ and position of the rotor flux ***vector*** $\theta_{\Psi R}$ are

$$\theta_{\Psi S} = \frac{\pi}{2}, \quad \theta_{\Psi R} = \theta_m + \frac{\pi}{2}.$$

In the case when the stator has at least two spatially shifted stator windings with AC currents, and provided that conditions detailed in Section 9.9 are met, the stator flux ***vector*** rotates with respect to the very stator, and its position is

$$\theta_{\Psi S} = \frac{\pi}{2} + \theta_{iS},$$

where the angle θ_{iS} depends on instantaneous values of stator currents. If the rotor as well has a system of windings creating a rotating magnetic field, then angle of the rotor flux ***vector*** is

$$\theta_{\Psi R} = \frac{\pi}{2} + \theta_m + \theta_{iR},$$

where the angle θ_{iR} is determined by instantaneous values of the rotor currents. The angle between the stator flux ***vector*** and the rotor flux ***vector*** is equal to

$$\Delta\theta = \theta_{\Psi S} - \theta_{\Psi R} = -\theta_m + \theta_{iS} - \theta_{iR}.$$

The electromagnetic torque is calculated as the first derivative (9.8) of the energy accumulated in magnetic field. Magnetic field energy is defined by (9.5). When determining the first derivative of the magnetic field energy in terms of the coordinate θ_m, it is assumed that the electrical currents in the windings do not depend on θ_m. Validity of such an assumption is obvious in cases where the windings are supplied from external current sources. Therefore, the first derivative of the sum $-\theta_m + \theta_{\Psi S} + \theta_{\Psi R}$ in terms of θ_m is equal to -1, while the torque expression becomes

$$\begin{aligned} T_{em} &= +\frac{\mathrm{d}W_m}{\mathrm{d}\theta_m} = \frac{\mathrm{d}}{\mathrm{d}\theta_m}\left\{\frac{\mu_0 R^3 L\pi}{2\delta}\left[J_{R0}^2 + J_{S0}^2 + 2J_{R0}J_{S0}\cos(\Delta\theta)\right]\right\} \\ &= \frac{\mu_0 R^3 L\pi}{\delta} J_{R0}J_{S0}\frac{\mathrm{d}}{\mathrm{d}\theta_m}\left[\cos(-\theta_m + \theta_{iS} - \theta_{iR})\right] \\ &= \frac{\mu_0 R^3 L\pi}{\delta} J_{R0}J_{S0}\sin(-\theta_m + \theta_{iS} - \theta_{iR}) \\ &= \frac{\mu_0 R^3 L\pi}{\delta} J_{R0}J_{S0}\sin\Delta\theta. \end{aligned} \tag{9.8}$$

The obtained expression shows that the torque is proportional to the product of amplitudes of the stator and rotor currents and to the sine of the angle between the stator and rotor flux ***vectors***. The torque expression (9.8) holds notwithstanding the AC or DC currents in the machine windings. In order to show that the electromagnetic torque depends on the ***vector*** product of the stator and rotor flux ***vectors***, it is necessary to probe further and clarify the relations between the single turn flux, the winding flux, and the amplitude of the flux ***vector***.

9.4 Turn Flux and Winding Flux

In this section, some more detailed considerations concerning the winding flux and ***vector*** of the resultant flux are given. Algebraic intensity of the flux ***vector*** is calculated by relating the flux ***vector*** to the flux in one turn and the flux in the winding. The goal of these efforts is to represent the electromagnetic torque as the ***vector*** product of the stator and rotor flux ***vectors***.

For the purpose of facilitating the analysis of electrical machines, directed scalars, such as magnetomotive forces and fluxes, can be represented by corresponding ***vectors***. In Sect. 4.4, it is shown that the field of the ***vector*** of magnetic induction B can be represented by ***vector***, thus defining the flux ***vector*** in a single turn (contour). In Sect. 5.5, the winding magnetic axis is introduced and defined, while Sect. 8.5 gives the convention of representing the magnetic field by ***vector***. These results are used here to express the winding flux and the resultant flux.

Magnetic field in electrical machines appears as a consequence of magnetomotive forces established by stator and rotor currents. An example of a machine having one stator and one rotor winding is given in Fig. 9.1. In this example, it is assumed that electrical currents in both windings are constant and that the magnetic circuit is linear. Not all of the conductors are shown in the Fig. 9.1. It is understood that a number of stator and rotor conductors are distributed along the machine circumference and that their line density changes in a sinusoidal manner. The stator magnetic field is caused by the stator currents and shown in the left of Fig. 9.1. The rotor magnetic field is caused by the rotor currents, and it is shown in the right. The resultant flux is obtained by superposition, that is, by adding the stator and rotor fields and fluxes. At any point in the air gap, it is possible to identify the ***vector*** of the magnetic induction $\boldsymbol{B}^S$ created by stator currents and the ***vector*** of the magnetic induction $\boldsymbol{B}^R$ created by rotor currents. Resultant magnetic induction $\boldsymbol{B}^{Res}$ is equal to the ***vector*** sum $\boldsymbol{B}^S + \boldsymbol{B}^R$. The stator flux is calculated as a surface integral of the ***vector*** $\boldsymbol{B}^S$, while the rotor flux is the surface integral of the ***vector*** $\boldsymbol{B}^R$. The resultant flux is the surface integral of the ***vector*** $\boldsymbol{B}^S + \boldsymbol{B}^R$. Therefore, the resultant flux ***vector*** is the ***vector*** sum of the rotor flux ***vector*** and the stator flux ***vector***.

At this point, it is of interest to clarify the terms *stator flux* and *rotor flux*. Within further developments, the references to *stator flux* imply the flux created by magnetomotive forces of stator currents. In cases where the rotor does not have any electrical currents in its windings nor does it comprise permanent magnets, the only flux in electrical machine is the stator flux. In absence of rotor currents, magnetic inductance $\boldsymbol{B}^R$, created in the air gap by means of the rotor currents, is equal to zero. In such conditions, the resultant magnetic induction is equal to $\boldsymbol{B}^S$. The stator flux in one turn is determined by the surface integral of the ***vector*** $\boldsymbol{B}^S$ over the surface S encircled by the turn. In the same way, all the developments within this book consider the *rotor flux* as the surface integral of the magnetic induction $\boldsymbol{B}^R$, wherein the induction $\boldsymbol{B}^R$ is created by the rotor currents and corresponds to the resultant induction in cases where the stator currents are equal to zero. The resultant magnetic induction $\boldsymbol{B}^{Res} = \boldsymbol{B}^S + \boldsymbol{B}^R$ exists in the machine with both the stator and rotor currents. The resultant flux is the surface integral of the ***vector*** $\boldsymbol{B}^{Res}$.

One can consider the term *stator flux* to be the flux in the stator winding, whatever the magnetomotive force incites the flux. Adopting this viewpoint, the stator flux can be created by the stator currents, by the rotor currents, or by the contemporary action of both currents. This meaning of the term is better explained by citing *resulting stator flux*, implying the resultant flux in the stator winding, caused by any magnetomotive force and whatever magnetic inductance. The same holds for the term *rotor flux*.

Flux in one turn (contour) is determined as the surface integral of the ***vector*** of magnetic induction $\boldsymbol{B}$ over the surface encircled by the contour. The reference direction to be respected in the course of integration is determined by the right-hand rule. Placing the right hand so that the four fingers point to direction $\odot$ (Fig. 9.3),

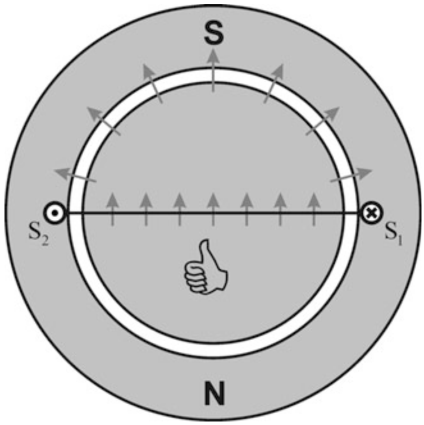

Fig. 9.3 Calculation of the flux in one turn. While the expression for magnetic induction B_{Fe} on the diameter S_1S_2 is not available, the expression $B(\theta)$ for magnetic induction in the air gap is known

while the base of the hand is turned toward ⊗, the thumb will indicate the reference direction of the contour flux. A positive current in the contour will create positive flux. The field lines would extend in the reference course and direction.

In Fig. 9.3, one stator turn with conductors S1 and S2 has electrical current that creates magnetic field in the air gap. The arrows indicate the course and direction of the magnetic induction ***B*** in the air gap. The flux in the turn is determined by calculating surface integral of the ***vector B*** induction over the surface encircled by the turn S1–S2. There is a multitude of different surfaces that are all surrounded by the turn S1–S2. Due to div$\boldsymbol{B} = 0$, the flux of the ***vector B*** on all such surfaces has the same value. Therefore, the flux calculation can be performed by selecting the surface that leads to less difficulty in calculation of the surface integral. This surface may be a rectangle $D \times L$, with one side being the diameter S1–S2 and the other side being the axial length L of the machine. However, analytical expression for the magnetic induction B is unknown along the diameter S1–S2 and within the rotor magnetic circuit. On the other hand, magnetic induction $B(\theta)$ in the air gap is known. For this reason, the integration is carried out over the surface which is passing through the air gap, residing at the same time on the considered contour.

9.4.1 Flux in One Stator Turn

It is of interest to determine the flux in the turn S1–S2 of the stator winding, created by the electrical currents in stator conductors. It is assumed that the stator has sinusoidally distributed conductors creating the stator current sheet. Considered turn S1–S2 is a part of the stator winding. It is connected in series with a multitude of other turns, displaced along the circumference. The turn S1–S2 resides in the position where the density of stator conductors is at the maximum.

The stator conductors with sinusoidal distribution and with DC current in the direction shown in Figs. 9.1 and 9.3 create the stator magnetic field in the air gap. Prevailing radial component of the magnetic field H^S is determined by (8.18),

$$H_r^S(\theta) = \frac{J_{S0} R}{\delta} \sin\theta.$$

The air gap permeability is constant (μ_0). Therefore, the corresponding magnetic induction in the air gap is

$$B_r^S(\theta) = \mu_0 \frac{J_{S0} R}{\delta} \sin\theta.$$

Magnetic induction $B_r^S(\theta)$ is created by action of the stator currents. In cases where the rotor currents are equal to zero, B_r^S determines the resultant magnetic induction in the air gap. The maximum intensity of magnetic induction is $B_m = \mu_0(R/\delta)J_{S0}$, and it is reached in the region of magnetic poles, such as the upper part of the figure, where the field lines leave the air gap and enter into magnetic circuit of the stator. In order to determine the flux in the turn S1–S2, it is necessary to select the surface convenient for the calculation of the surface integral. Since the expression $B_r^S(\theta)$ for the magnetic induction in the air gap is readily available, it is most suitable to adopt the surface which passes through the air gap. Hence, the choice is semicylinder of diameter R and length L. It looks like a rectangle of dimensions L × (πR), folded to make a semicylinder which starts from S1, passes through the air gap, and gets to S2. In Fig. 9.3, the cross section of such semicylinder corresponds to the upper semicircle where the field lines leave the air gap and enter into stator magnetic circuit. The surface S is

$$S = \pi \cdot R \cdot L.$$

The flux Φ_{S1} in the turn S1–S2 is obtained by calculating the surface integral of the magnetic induction over the surface S. The subscript "$S1$" intends that the symbol Φ_{S1} stands for the flux in one (1) turn of the stator (S). With,

$$\Phi_{S1} = \int_S B_r^S(\theta)\, \mathrm{d}S,$$

where

$$\mathrm{d}S = L \cdot R \cdot \mathrm{d}\theta.$$

The flux in one turn is obtained as

$$\Phi_{S1} = \int_0^\pi B_r^S(\theta)\, L \cdot R\, \mathrm{d}\theta = \frac{\mu_0 L R^2}{\delta} J_{S0} \int_0^\pi \sin(\theta)\mathrm{d}\theta$$
$$= \frac{\mu_0 L R^2}{\delta} J_{S0}(-\cos\theta)\Big|_0^\pi = \frac{2\mu_0 L R^2}{\delta} J_{S0}. \tag{9.9}$$

9.4.2 Flux in One Rotor Turn

Preceding analysis calculates the flux in one stator turn. In a similar way, it is possible to obtain the flux in one rotor turn, namely, the flux in the contour R1–R2 of the rotor winding (Fig. 9.1). In calculating the surface integral of the magnetic induction and obtaining the rotor flux, one should take into account the magnetic induction B_r^R, created by the electrical currents in distributed rotor winding. In the expression for radial component of the rotor field

$$B_r^R(\theta) = \mu_0 \frac{J_{R0}R}{\delta}\sin(\theta - \theta_m).$$

J_{R0} represents the maximum line density of the rotor currents, while the angle θ_m represents the rotor position, that is, the rotor displacement with respect to the stator. At the same time, θ_m denotes the angular displacement between the stator and rotor windings (Fig. 9.1). The flux through the contour R1–R2 is

$$\begin{aligned}\Phi_{R1} &= \int_{\theta_m}^{\pi+\theta_m} B_r^R(\theta)\, L \cdot R\, \mathrm{d}\theta = \frac{\mu_0 L R^2}{\delta} J_{R0} \int_{\theta_m}^{\pi+\theta_m} \sin(\theta - \theta_m)\mathrm{d}\theta \\ &= \frac{\mu_0 L R^2}{\delta} J_{R0}[-\cos(\theta - \theta_m)]|_{\theta_m}^{\pi+\theta_m} = \frac{2\mu_0 L R^2}{\delta} J_{R0}.\end{aligned} \tag{9.10}$$

Calculation of the flux in one turn may be done in a shorter way, avoiding the integration. Magnetic inductance B_r passes through the semicylindrical surface $S = \pi RL$ in radial direction. In cases where the magnetic inductance $B_r^S(\theta)$ does not change over the interval $\theta \in [0 .. \pi]$, the flux in one stator turn can be obtained by multiplying the inductance $B_r^S(\theta) = \text{const.}$ and the surface area πRL. Yet, in electrical machine with sinusoidally distributed conductors, magnetic inductance changes along the circumference. Both $B_r^S(\theta)$ and $B_r^R(\theta)$ change as sinusoidal functions of the angle θ. Notwithstanding variable magnetic inductance in the air gap, the surface integration can be avoided in all cases where the average value of $B_r(\theta)$ is known on one semicircle. The flux Φ_{S1} in the stator turn S1–S2 can be calculated as the product of the surface πRL of the semicylinder and the average value of the magnetic induction $B_r^S(\theta)$ over the interval $\theta \in [0 .. \pi]$. With $B_r^S(\theta) = B_{max} \sin(\theta)$, the average value is $\pi/2$ times lower than the maximum value; thus, $B_{av} = (2/\pi)\, B_{max} = 2\mu_0 R J_{S0}/(\delta\pi)$. The result Φ_{S1} is obtained by multiplying the average value B_{av} of the magnetic induction and the surface area $S = \pi RL$, and it is in accordance with (9.9).

It should be noted that the contours S1–S2 and R1–R2 have been selected so as to have their conductors placed in the regions with the highest density of conductors. The stator flux ***vector*** is shown in Fig. 9.1, and it coincides with the normal on the contour. Other turns have their conductors displaced with respect to S1–S2, and their normals are inclined with respect to the stator flux ***vector***. Namely, the lines of the stator magnetic fields pass through the inclined turns at an angle other than $\pi/2$. Therefore, the flux in other turns is smaller than the flux in the turn

S1–S2. As the angular displacement of the turn with respect to S1–S2 increases toward $\pi/2$, the flux decreases.

In the same way, the rotor flux ***vector***, shown in Fig. 9.1, coincides with the normal to the contour R1–R2. Therefore, the flux in the turn R1–R2 has the maximum value of all rotor turns.

The flux in turns that are inclined with respect to S1–S2 (R1–R2) is smaller compared to the values given by (9.9) and (9.10). It is shown hereafter that this flux depends on the cosine of the angle between the flux ***vector*** and the normal on the relevant turn. As an example, the flux is calculated in one stator turn with conductor $\otimes$ in position $\theta = \theta_1$ and conductor $\odot$ in position $\theta = \pi + \theta_1$. The normal on the considered turn is shifted by θ_1 with respect to the normal on the turn S1–S2. The flux $\Phi_S(\theta_1)$ in the inclined turn is determined by calculating the surface integral of the magnetic induction (incited by the stator currents) over the semicylindrical surface reclining on the conductor $\otimes$ in position $\theta = \theta_1$ and reaching the conductor $\odot$ in position $\theta = \pi + \theta_1$:

$$\begin{aligned}\Phi_S(\theta_1) &= \int\limits_{\theta_1}^{\pi+\theta_1} B_r^S(\theta)\,L\cdot R\,\mathrm{d}\theta = \frac{\mu_0 LR^2}{\delta} J_{S0} \int\limits_{\theta_1}^{\pi+\theta_1} \sin(\theta)\mathrm{d}\theta \\ &= \frac{\mu_0 LR^2}{\delta} J_{S0}(-\cos\theta)\Big|_{\theta_1}^{\pi+\theta_1} = \frac{2\mu_0 LR^2}{\delta} J_{S0}\cos\theta_1 = \Phi_{S1}\cos\theta_1. \end{aligned} \tag{9.11}$$

Equation (9.11) shows that flux in the turn shifted by angle θ_1 is cosine function of the angle. When $\theta_1 > \pi/2$, the flux in this turn obtains negative value. With $\theta_1 = \pi$, the turn gets to positions S1 and S2, with directions $\otimes$ and $\odot$ exchanged. The flux in such turn reaches the same absolute value as the flux in the original turn S1–S2, but it has the opposite sign. In the same way, it can be shown that the flux created by the rotor currents in one rotor turn depends on the angle θ_2 between the normal on the considered turn (contour) and the vertical n_R on the turn R1–R2 (Fig. 9.1). The flux in the rotor turn $\Phi_R(\theta_2)$ is calculated from the surface integral of the rotor magnetic induction over the surface encircled by the conductor $\otimes$ in position $\theta = \theta_m + \theta_2$ and the conductor $\odot$ in position $\theta = \pi + \theta_m + \theta_2$ (9.12).

The results (9.11) and (9.12) show that the flux in a single stator or rotor turn depends on the cosine of the angle between the normal on the considered turn and the flux ***vector*** whose amplitude, course, and direction represent the field of magnetic induction.

The method of representing the field of magnetic induction by the flux ***vector*** has been discussed in Subsect. 4.4 and used in Figs. 9.1 and 9.2:

$$\begin{aligned}\Phi_R(\theta_2) &= \frac{\mu_0 LR^2}{\delta} J_{R0} \int\limits_{\theta_2+\theta_m}^{\pi+\theta_2+\theta_m} \sin(\theta-\theta_m)\mathrm{d}\theta \\ &= \frac{\mu_0 LR^2}{\delta} J_{R0}(-\cos(\theta-\theta_m))\Big|_{\theta_2+\theta_m}^{\pi+\theta_2+\theta_m} \\ &= \frac{2\mu_0 LR^2}{\delta} J_{R0}\cos\theta_2 = \Phi_{R1}\cos\theta_2. \end{aligned} \tag{9.12}$$

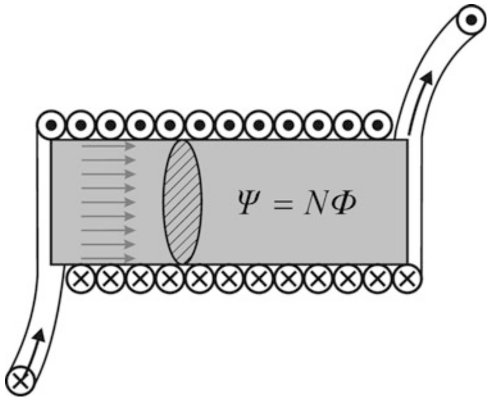

Fig. 9.4 Flux in concentrated winding

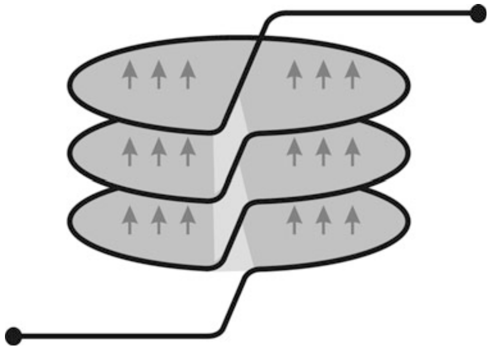

Fig. 9.5 The surface reclining on a concentrated winding with three turns

9.4.3 Winding Flux

Flux in a winding is the sum of fluxes in individual turns constituting the winding. A sample winding consisting of N series connected turns wound around a straight ferromagnetic bar is given in Fig. 9.4. Each turn of the sample winding has the same flux Φ. Therefore, the total flux of the whole winding Ψ is determined as the product of the number of turns N and the flux of one turn Φ; thus, $\Psi = N\Phi$. This is due to the fact that the surfaces encircled by individual turns have equal areas while their normals are oriented in the same direction. A winding where all the turns have the same flux while their normals are collinear is called *concentrated* winding.

Like the flux in one turn, the flux in a winding can be determined as surface integral of the magnetic induction over the surface reclining on the entire winding. While the surface encircled by one turn is easily identified, the *surface of the winding* is more difficult to identify. In Fig. 9.5, an attempt is made to illustrate the surface of a concentrated winding. The three turns making this winding constitute a complex contour. The shadowed area shows the surface encircled by the winding conductors. If distance between the turns is sufficiently small, it is justified to assume that the three turns have the same flux. Therefore, each of the lines of the considered magnetic field passes through the surface of the winding three times. With Φ designating the flux in one turn, the winding flux Ψ is equal to $N\Phi = 3\Phi$.

In a cylindrical machine with distributed windings, the stator and rotor turns are distributed along the circumference. The stator conductors are placed in slots on the inner surface of the stator magnetic circuit, while the rotor conductors are placed in slots on the rotor surface facing the air gap. Two diametrical conductors constitute one turn, that is, one contour. A winding consists of a number of series connected turns. The winding flux is the sum of the fluxes in individual turns. The flux in one turn depends upon its relative position with respect to the magnetic field. In cases where the field lines are perpendicular to the surface of the turn, the flux in the turn has maximum value. The flux becomes zero in cases where the turn surface runs parallel with the lines of the magnetic field.

It is proven that the flux in one turn is proportional to the cosine of the angle between the ***vector*** of magnetic induction and the normal on the turn surface. This normal is called magnetic axis of the turn. In Fig. 9.1, the normal is perpendicular to the straight line connecting the conductors $\otimes$ and $\odot$. Given the orientation of the magnetic field, the flux $\Phi(\theta)$ in each turn can be determined in terms of its angular position θ. Equations (9.11) and (9.12) provide the flux values for one stator and one rotor turn. They are expressed in terms of the angle between the normal of the relevant turn and the flux ***vector*** which represents the magnetic field.

Since the turns that constitute one distributed winding have their axes oriented in different directions, their relevant fluxes will assume different values. For this reason, the total winding flux cannot be obtained by multiplying the flux in one turn by the number of turns.

In general, the winding flux is determined by adding all the contributions of individual turns. In cases where the winding is concentrated, the winding flux ***vector*** has an amplitude of $\Psi = N\Phi$. In cases where the winding is distributed with the conductor line density of $N'(\theta)$, the winding flux is determined by integration. The number of conductors within a tiny segment of angular width $\mathrm{d}\theta$ is

$$\mathrm{d}N = N^{'}(\theta)R\mathrm{d}\theta,$$

where R is diameter of the machine. Each of the conductors positioned on the interval $\theta \in [0 .. \pi]$ completes one turn with its diametrically positioned counterpart on the interval $\theta \in [\pi .. 2\pi]$. The flux in one turn is determined by the angle between the flux ***vector***, representing the magnetic field and the axis of the turn. Eventually, the flux in one turn can be expressed in terms of the position θ of the turn. Contributions of all $\mathrm{d}N$ turns to the total flux of the winding is

$$\mathrm{d}\Psi = N^{'}(\theta)\Phi(\theta)R\mathrm{d}\theta,$$

while the total flux of the winding is

$$\Psi = \int_0^{\pi} N^{'}(\theta)\Phi(\theta)R\mathrm{d}\theta. \tag{9.13}$$

Equation (9.13) can be used for calculation of the winding fluxes of both stator and rotor windings.

An example of practical use of the (9.13) is calculation of the self-inductance of the stator winding. Self-inductance is coefficient that defines effect of winding currents on winding flux. In cases where the winding flux Ψ does not have any external originator and exists due to the winding current I only, the self-inductance can be calculated as $L_S = \Psi/I$. Before using (9.13), it is necessary to calculate the flux in one turn $\Phi(\theta)$. It is calculated as the surface integral of the magnetic induction $B_r^{\ S}$ in the air gap, wherein $B_r^{\ S}$ denotes the radial component of the magnetic inductance created by the stator currents. Dividing the flux in the stator winding by the stator current I gives the self-inductance of the stator winding.

Similar procedure can be used to determine the mutual inductance between the stator and rotor windings. The mutual inductance L_m defines the effect of the rotor currents on the flux in the stator winding. In cases where the rotor currents contribute to resultant magnetic induction in the air gap, they also change the resultant flux in the stator turns and, hence, the flux in the stator winding. The same coefficient defines the effects of the stator currents on the flux in the rotor winding. Calculation of L_m requires the previous procedure to be modified. When calculating the flux in one stator turn $\Phi(\theta)$, it is necessary to replace the magnetic inductance $B_r^{\ S}$, created by the stator currents, by the magnetic inductance $B_r^{\ R}$, created by the rotor currents. In this way, the value of $\Phi(\theta)$ corresponds to the flux that the rotor currents establish in one stator turn. At that time, calculation of the stator flux according to (9.13) results in the flux created in the stator winding by action of the rotor currents. Dividing this value by the rotor current gives coefficient of mutual inductance between stator and rotor windings.

To proceed, the flux in the stator winding of the electrical machine shown in Fig. 9.1 is calculated by using (9.13), assuming that the magnetic field in the air gap is excited by the stator currents. Therefore, the magnetic field in the air gap is calculated assuming that the rotor currents are equal to zero. Distributed winding of the stator can be considered as a set of $N_T = N_C/2$ contours, where N_C denotes the number of conductors in the stator winding while N_T is the number of turns. According to (8.2), the number of turns is equal to $N_T = 2R\, N'_{Smax}$, where N'_{Smax} is the maximum line density of the stator conductors, which exists at positions of conductors S1 and S2 in Fig. 9.1.

In order to calculate the winding flux, it is necessary to calculate the flux in one turn. For a turn with conductor $\otimes$ in position θ and with conductor $\odot$ in position $\pi + \theta$, the flux $\Phi_S(\theta)$ is determined from (9.11):

$$\Phi_S(\theta) = \Phi_{S1} \cos\theta$$

Since the line density of the stator conductors is

$$N'(\theta) = N_{S\max} \cos\theta,$$

Equation (9.13) becomes

$$\begin{aligned}\Psi_S &= \int_0^{\pi} N_{S\max} \cos\theta \cdot \Phi_{S1} \cos\theta \cdot R \cdot \mathrm{d}\theta \\ &= N_{S\max} \Phi_{S1} R \int_0^{\pi} \cos^2\theta \, \mathrm{d}\theta \\ &= \frac{\pi}{2} N_{S\max} \Phi_{S1} R = \frac{\pi}{4} N_T \Phi_{S1}.\end{aligned} \tag{9.14}$$

With $\pi/4 < 1$, it is concluded from (9.14) that the flux in a distributed winding is smaller than the flux in a concentrated winding having the same number of turns. A concentrated winding would be obtained by placing all the conductors in places S1 and S2 (Fig. 9.1). The flux in the concentrated winding is obtained by multiplying the number of turns N_T and the flux in one turn Φ_{S1}.

9.4.4 Winding Flux Vector

The winding flux can be represented by a ***vector*** denoting the course, direction, and amplitude of the flux. The winding flux ***vector*** can be obtained by adding the flux ***vectors*** representing the fluxes in individual turns. In cases where the flux ***vectors*** of individual turns have different orientations, it is necessary to determine their sum and find the course and direction for the flux ***vector*** of the winding.

The convention of representing the flux in one turn by ***vector*** is presented in Section 4.4. The course and direction of the flux ***vector*** in one turn are determined from the normal to the surface reclining on the relevant turn. In Sect. 5.5, magnetic axis of a winding has been defined on the basis of the course and direction of the lines representing the magnetic field created by electrical currents in the winding itself. The course of the winding axis is determined by positions of magnetic poles created in the magnetic circuit due to the winding currents. The convention of representing the magnetic field of a distributed winding by flux ***vector*** is given in Sect. 8.5. Once again, the course and direction of the flux ***vector*** are determined from spatial orientation of the magnetic field, that is, from positions of the magnetic poles. Practical example of calculating the course and direction of the flux ***vector*** is given for the machine presented in Fig. 9.1. It starts with determining the spatial distribution of the magnetic field created by the currents in distributed winding. Direction of the winding flux ***vector*** is determined on the basis of direction of the

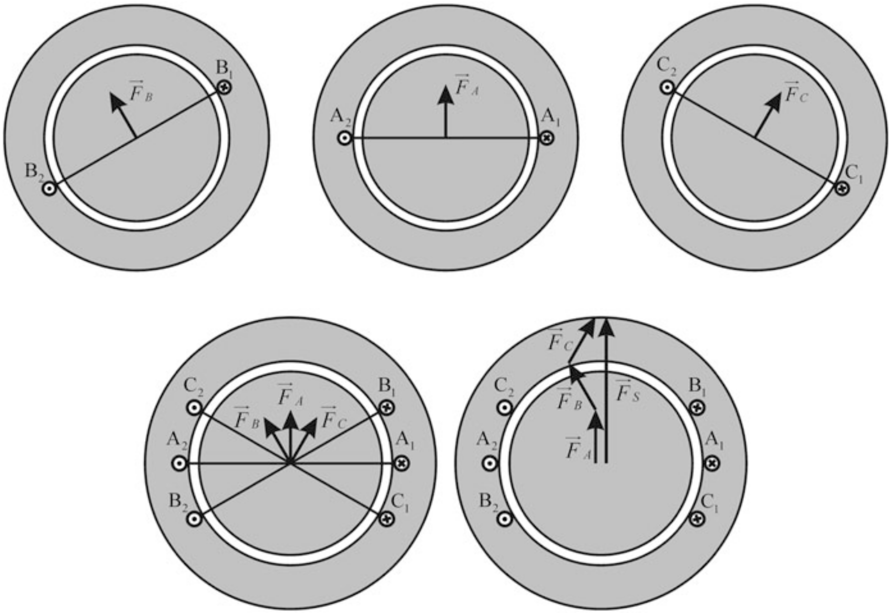

Fig. 9.6 Vector addition of magnetomotive forces in single turns and magnetic axis of individual turns

field lines, while the course is determined by the magnetic poles, wherein the poles are identified as diametrically positioned zones of the magnetic circuit where the magnetic induction reaches maximum values. By using this procedure, magnetic fields of the stator and rotor have been represented by flux ***vectors*** given in Fig. 9.2.

Derivation of the course of the flux ***vector*** and the magnetic axis of the winding can be performed otherwise, by ***vector*** addition of magnetomotive forces created by individual turns or by ***vector*** addition of their flux ***vectors***. Figure 9.6 shows a stator winding comprising three turns, A1–A2, B1–B2, and C1–C2. The upper part of the figure shows individual ***vectors*** of magnetomotive forces for each turn. These magnetomotive forces are denoted by ***vectors*** F_A, F_B, and F_C. They are determined by the normals of corresponding turns. Magnetomotive force F_A would determine the course and direction of the resultant winding flux if the turns B and C did not exist. The resultant magnetomotive force F_S of the stator winding comprising all the three turns is shown in the lower part of the figure. The ***vector*** of the resultant magnetomotive force is obtained by ***vector*** addition of F_A, F_B, and F_C. Since each flux is determined by dividing the corresponding magnetomotive force by the magnetic resistance, the course and direction of the ***vector*** representing the flux in the winding are determined from the resultant magnetomotive force. As it is shown in the figure, the flux ***vector*** of the winding is collinear with the normal of the middle turn A1–A2.

9.5 Winding Axis and Flux Vector

Previous considerations provided details on determining the axis of a winding, the course of the flux ***vector*** in one turn, and the course of the flux ***vector*** in a winding comprising several turns. A brief survey of the conclusions is presented hereafter, aimed to be used in the subsequent considerations. The survey is illustrated by Fig. 9.7.

The figure presents a distributed winding with sinusoidal change of the conductor line density. The maximum line density is reached at positions where the conductors A1 and A2 are placed. The maximum value Φ_{S1} of the flux in one turn is reached in the turn A1–A2.

The course of the flux Φ_{S1} in the turn A1–A2 is determined by the normal n_{S1}. The normal n_{S1} is a unit ***vector*** perpendicular to the flat surface reclining on the contour A1–A2.

The course of the stator winding flux ***vector*** $\boldsymbol{\Psi}_S$ is determined by the unit ***vector*** n_S, representing the winding axis (magnetic axis of the winding). Therefore, in the case of a distributed winding with sinusoidal distribution of conductors, the flux ***vector*** and the winding axis have the same course and direction as the flux ***vector*** Φ_{S1} in the turn A1–A2, wherein the conductors A1 and A2 reside at positions where the conductor line density is maximum.

9.6 Vector Product of Stator and Rotor Flux Vectors

Figure 9.1 shows the lines representing magnetic field of the stator and magnetic field of the rotor in a cylindrical electrical machine with DC currents in both windings. Electromagnetic torque is generated by interaction of the two magnetic

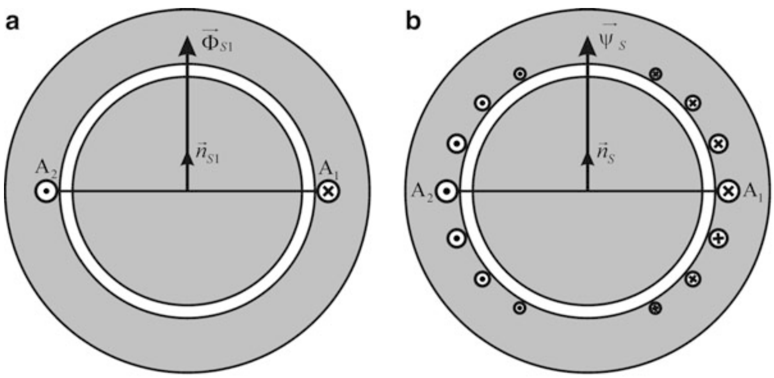

Fig. 9.7 Spatial orientation of flux vector of one turn (**a**), axis of the winding (**b**), and flux vector of the winding

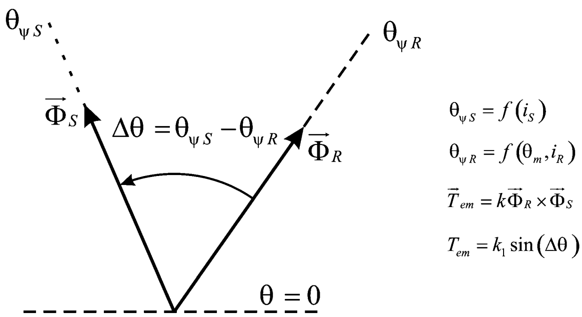

Fig. 9.8 Spatial orientation of the stator flux vector. Spatial orientation of the rotor flux vector. The electromagnetic torque as the vector product of the two flux vectors

fields, and it is expressed by (9.7), where the angle θ_m represents the position of the rotor relative to the stator. With DC currents in the windings, the angle θ_m defines as well the angle between the stator and rotor flux ***vectors***.

In general, both stator and rotor may comprise several windings carrying DC or AC currents. Alternating currents may create rotating magnetic field which can be represented by a revolving flux ***vector***. Such field revolves relative to the windings that give rise to the magnetomotive forces that originate the field. In cases where AC currents in the stator windings create rotating magnetic field, the orientation of the stator flux ***vector*** $\theta_{\Psi S}$ depends on instantaneous values of electrical currents in the stator windings. In cases where AC currents in the rotor create rotating magnetic field, instantaneous values of the rotor currents determine the position of the rotor flux ***vector*** with respect to the rotor. Hence, the orientation of the rotor flux ***vector*** $\theta_{\Psi R}$ with respect to the stator is determined by the rotor position θ_m and by instantaneous values of electrical currents in the rotor winding. Therefore, in general, the angle $\Delta\theta = \theta_{\Psi S} - \theta_{\Psi R}$ between the stator flux ***vector*** and the rotor flux ***vector*** is dependent on the rotor position θ_m and also on instantaneous values of electrical currents in the machine windings, as indicated in Fig. 9.8. On the basis of (9.6), expression for the electromagnetic torque assumes the form

$$T_{em} = \frac{\mu_0 \pi R^3 L}{\delta} J_{R0} J_{S0} \sin\Delta\theta. \tag{9.15}$$

The maximum value of the torque is reached in cases where the angular difference $\Delta\theta = \theta_{\Psi S} - \theta_{\Psi R}$ between the stator and rotor fluxes is equal to π/2.

$$T_{\max} = \frac{\mu_0 \pi R^3 L}{\delta} J_{R0} J_{S0}. \tag{9.16}$$

The electromagnetic torque can be represented by ***vector*** product of the stator flux ***vector*** Φ_{S1} and the rotor flux ***vector*** Φ_{R1}. Fluxes Φ_{S1} and Φ_{R1} exist in the turns S1–S2 and R1–R2 of the electrical machine shown in Fig. 9.1. They are representative turns of stator/rotor distributed windings, and their conductors reside at positions with the maximum line density of stator/rotor conductors. The flux Φ_{S1} is also the resultant flux in the turn S1–S2 in cases when the electrical machine has only the stator currents, while the flux Φ_{R1} is also the resultant in the turn R1–R2 in cases where only the rotor currents exist in the machine. On the basis of (9.9) and (9.10), amplitudes of the two ***vectors*** are determined by expressions

$$\Phi_{S1} = \frac{2\mu_0 L R^2}{\delta} J_{S0}, \qquad \Phi_{R1} = \frac{2\mu_0 L R^2}{\delta} J_{R0}.$$

Since the torque is expressed by function sin $\Delta\theta$, it can be calculated as the ***vector*** product of the stator and rotor flux ***vectors***. Expression for the torque can be represented in the form

$$\frac{\mu_0 \pi R^3 L}{\delta} J_{R0} J_{S0} \sin\Delta\theta = \left(\frac{\pi\delta}{4\mu_0 L R}\right)\left(\frac{2\mu_0 L R^2}{\delta} J_{S0}\right)\left(\frac{2\mu_0 L R^2}{\delta} J_{R0}\right)\sin\Delta\theta$$

which comprises the amplitude of the ***vector*** product of the stator and rotor flux ***vectors***, given by (9.17),

$$\vec{T}_{em} = \left(\frac{\pi\delta}{4\mu_0 L R}\right)\cdot\left[\vec{\Phi}_R \times \vec{\Phi}_S\right] = k\left[\vec{\Phi}_R \times \vec{\Phi}_S\right] \tag{9.17}$$

where the constant k is

$$k = \frac{\pi\delta}{4\mu_0 L R}.$$

Since the flux ***vectors*** Φ_{S1} and Φ_{R1} of the turns S1–S2 and R1–R2 have the same spatial orientation as the flux ***vectors*** of the stator and rotor windings, respectively, the electromagnetic torque can be expressed as the ***vector*** product $\Psi_S \times \Psi_R$ of the flux ***vector*** Ψ_S in the stator winding and the flux ***vector*** Ψ_R in the rotor winding. On the basis of (9.14), the amplitudes of ***vectors*** Ψ_S and Ψ_R are determined by expressions

$$\Psi_S = \frac{\pi}{4} N_{TS} \Phi_{S1}, \qquad \Psi_R = \frac{\pi}{4} N_{TR} \Phi_{R1},$$

where N_{TS} and N_{TR} denote the number of turns of the stator and rotor windings, respectively. Since the flux ***vectors*** of the representative turns are collinear with the flux ***vectors*** of respective windings, (9.17) takes the form

$$\begin{aligned}\vec{T}_{em} &= \left(\frac{\pi\delta}{4\mu_0 L R}\right)\cdot\left[\vec{\Phi}_R \times \vec{\Phi}_S\right] \\ &= \left(\frac{\pi\delta}{4\mu_0 L R}\right)\cdot\left(\frac{4}{\pi N_{TS}}\right)\left(\frac{4}{\pi N_{TR}}\right)\left[\vec{\Psi}_R \times \vec{\Psi}_S\right] \\ &= \left(\frac{4\delta}{\mu_0 \pi L R N_{TS} N_{TR}}\right)\cdot\left[\vec{\Psi}_R \times \vec{\Psi}_S\right] \\ &= k_1 \cdot \left[\vec{\Psi}_R \times \vec{\Psi}_S\right],\end{aligned} \tag{9.18}$$

where k_1 is constant equal to

$$k_1 = \frac{4\delta}{\mu_0 \pi L R N_{TS} N_{TR}}.$$

Equations (9.17) and (9.18) give the ***vector*** of electromagnetic torque. The course of the obtained ***vector*** is determined by axis z of the cylindrical coordinate system, that is, by the axis of rotor revolutions. Direction of the torque ***vector*** is in accordance with the reference direction of z-axis. The torque of a positive value acts toward increasing the rotor speed Ω_m and moves the rotor toward increasing the angle θ_m, which corresponds to the movement in counterclockwise direction. The torque amplitude is determined by equation

$$T_{em} = k\left|\vec{\Phi}_R \times \vec{\Phi}_S\right|.$$

Question (9.3): Expression for the torque (9.7) has the leading minus sign. Should the torque of a negative value be expected in the case represented in Fig. 9.1? Why (9.15) does not include negative sign?

Answer (9.3): The electromagnetic torque is determined by the function sin $\Delta\theta$, where $\Delta\theta$ is the angle equal to $\theta_{\Psi S} - \theta_{\Psi R}$. The angles $\theta_{\Psi S}$ and $\theta_{\Psi R}$ determine the course of the stator and rotor flux ***vectors***. In Fig. 9.1, courses of the two flux ***vectors*** are shown assuming that the windings carry a constant DC current. Then $\theta_{\Psi S} = \pi/2$, while $\theta_{\Psi R} = \pi/2 + \theta_m$, resulting in $\Delta\theta = -\theta_m$. In the considered case, the torque is proportional to function $\sin \Delta\theta = -\sin \theta_m$, which gives the minus sign in (9.7).

9.7 Conditions for Torque Generation

The electromagnetic torque can be calculated from the ***vector*** product of the flux ***vector*** Φ_{S1} and the flux ***vector*** Φ_{R1}, wherein the former is created by the stator currents in the stator turn S1–S2 while the latter is created by the rotor currents in the turn R1–R2 (Fig. 9.1). The expression for the electromagnetic torque is given by (9.17). Equation (9.18) reformulates the expression by introducing the ***vector*** product of the stator flux ***vector*** and the rotor flux ***vector***.

The torque is proportional to the sine of the angle $\Delta\theta$ between the relevant flux ***vectors***. With DC currents in both the stator and the rotor windings, the stator flux does not move with respect to the stator while the rotor flux does not move with respect to the rotor. In this case, the stator flux is leading with respect to the rotor flux by $\Delta\theta = -\theta_m$.

With the rotor revolving at a constant angular speed Ω_m, and with the initial rotor position $\theta_m(0) = 0$, the rotor position changes as $\theta_m(t) = \Omega_m t$. Therefore, the angle between the stator flux ***vector*** and the rotor flux ***vector*** is $\Delta\theta = -\theta_m = -\Omega_m t$.

Hence, electrical machine with both stator and rotor windings carrying DC current develops electromagnetic torque proportional to sin($\Omega_m t$). The average value of this torque and the average value of the corresponding conversion power $P_{em} = T_{em}\,\Omega_m$ are both equal to zero. Therefore, an electrical machine with DC currents in the stator and in the rotor windings cannot provide an average power other than zero.

A nonzero average value of the torque is obtained in cases where the angle $\Delta\theta$ between the stator and rotor flux ***vectors*** is constant. For the given winding currents, the electromagnetic torque has the maximum value with $\Delta\theta = \pi/2$.

Condition $\Delta\theta =$ const. cannot be achieved with both the stator and rotor windings carrying DC currents. It will be proven later on that at least one of the windings on either stator or rotor must have AC currents and create the revolving magnetic field. There are three distinct cases when the constraint $\Delta\theta =$ const. is fulfilled. These three methods of accomplishing constant relative position of the two flux ***vectors*** have been shown in Fig. 7.8. These examples are reinstated hereafter and explained again in terms of the flux ***vectors***.

It is of interest to notice that a nonzero average value of the electromagnetic torque is achieved only in cases where the angle $\Delta\theta = \theta_{\Psi S} - \theta_{\Psi R}$ is constant. Namely, the stator and rotor flux ***vectors*** must retain their relative position. Whether they revolve or stay firm, the angle between flux ***vectors*** representing the stator and rotor magnetic fields must not change. There are three cases where this condition is fulfilled:

(a) Stator field is still with respect to stator. Rotor field rotates with respect to rotor in the opposite direction of rotation of rotor; thus, rotor field does not move with respect to stator.
(b) Stator field rotates with respect to stator. Rotor field rotates with respect to rotor. Sum of rotor speed relative to stator and rotor field speed relative to rotor is equal to speed of rotation of stator field relative to stator.
(c) Stator field rotates with respect to stator at speed equal to rotor speed. Rotor field does not move with respect to rotor.

Generation of stator or rotor magnetic field which does not rotate with respect to originating windings can be done by one or more windings with DC currents. Generation of the field that revolves with respect to originating windings requires at least two windings of different spatial orientation and with AC currents of the appropriate frequency and initial phase. Conditions for creation of a rotating field have been discussed in the section devoted to rotating field.

Case (a) corresponds to direct current machines (DC machines), and it is shown in Fig. 7.8a. ***Vector*** of the stator flux does not move with respect to the stator because the stator conductors have constant DC currents. The torque generation requires the rotor flux to retain its relative position to the stator flux. This means that the rotor flux ***vector*** in Fig. 7.8a should not move either. For the rotor flux ***vector*** to remain still while the rotor windings revolve, electrical currents in rotor conductors should create magnetic field which rotates with respect to the rotor at the speed $-\Omega_m$. In this case, the rotor revolves in positive direction at the speed $+\Omega_m$, while the rotor flux revolves with respect to the rotor in the opposite direction. Therefore, the rotor flux remains still with respect to the stator. For the rotor windings to create revolving field, the rotor must have AC currents, the frequency of which is determined by the speed of rotation. Yet, DC machines are supplied from DC power sources. In order to convert DC supply currents into AC rotor currents, DC machines have a mechanical commutator, device with brushes attached to the stator and collector attached to the rotor. Collector has a number of isolated segments, connected to the rotor conductors. When the rotor revolves, the segments slide under the brushes, altering electrical connections and changing the way of injecting the supply current into the rotor winding. In such way, commutator directs DC current of the electrical power source into rotor conductors in such way that the rotor conductors have AC currents. The frequency of these currents is determined by the speed of rotation. The commutator will be explained in more detail in the chapter on *DC machines*. Thanks to the commutator, the flux ***vector*** Φ_{R1} remains still with respect to flux ***vector*** Φ_{S1}; thus, the angle $\Delta\theta$ remains constant.

Case (b) shown in Fig. 7.8b corresponds to asynchronous machines. Windings of the stator and rotor have AC currents of angular frequency ω_s and ω_k, respectively. The stator flux ***vector*** rotates with respect to the stator at the speed Ω_s, determined by the angular frequency ω_s. The rotor flux ***vector*** rotates with respect to the rotor at the speed of Ω_k, determined by the angular frequency ω_k. Difference $\omega_s - \omega_k$ in angular frequency determines the rotor speed Ω_m. The flux ***vectors*** of the stator and rotor rotate at the same speed (Ω_s). Consequently, their mutual position $\Delta\theta$ does not change.

Case (c) shown in Fig. 7.8c corresponds to synchronous machines. There are rotor windings with constant DC currents producing the rotor flux. Alternatively, the rotor does not have any windings. Instead, there are permanent magnets mounted on the rotor. In either case, the rotor flux ***vector*** Φ_{R1} rotates at the same speed as the rotor does. The stator of the machine has a system of windings with two or more phases carrying AC currents. Angular frequency ω_s of the stator currents creates the stator magnetic field which revolves at the speed Ω_s, determined by the angular frequency ω_s of the stator currents. In synchronous machines, the stator frequency ensures that the field rotates at the same speed as the rotor, that is, $\Omega_s = \Omega_m$. Therefore, the flux ***vectors*** Φ_{R1} and Φ_{S1} rotate at the same speed, and their mutual position $\Delta\theta$ does not change.

Question (9.4): Derive the expression for the torque acting on a contour with electrical current in a homogenous magnetic field, with the normal to the contour being inclined with an angle θ with respect to the ***vector*** B, as shown in Fig. 3.6. The contour is circular, with diameter D and with electrical current I. Assume that the contour revolves around the axis which is orthogonal to the direction of the field. The speed of rotation is known and constant. Determine the instantaneous and average value of the torque acting on the contour. Assuming that the magnetic field cannot be changed, but it is possible to have an arbitrary current in the contour, determine the current $i(t)$ which would result in a nonzero average value of the torque.

With the assumption that the induction $B(t)$ is variable while the electrical current $i(t) = I$ is constant, determine one solution for $B(\theta)$ which results in a nonzero average value of the torque.

Answer (9.4): The electromagnetic torque T_{em} acting on the contour is equal to $I{\cdot}B{\cdot}S{\cdot}\sin\theta$. When the contour rotates at angular speed Ω while both the current and magnetic inductance are constant, the torque varies according to function $\sin(\Omega\, t)$, and its average value is zero. In cases where the magnetic induction is constant while the current $i(t) = I \sin(\omega t)$ changes with angular frequency ω equal to the speed of rotation Ω, the torque is proportional to $(\sin(\omega t))^2$ and has a nonzero average value. If the contour has a constant current I, nonzero average of the torque can be obtained in cases when the magnetic induction changes as $B(t) = B_m{\cdot}\sin(\omega t)$, where the angular frequency ω corresponds to the speed of rotation Ω.

9.8 Torque–Size Relation

Expression for electromagnetic torque (9.16) shows that the torque is proportional to R^3L, that is, to the axial length of the machine L and third power of its diameter D. Diameter and axial length are linear dimensions of the machine, and common notation l can be used for both. Therefore, the electromagnetic torque is proportional to fourth power of the linear dimensions l of the machine, $T \sim l^4$. Volume of the machine is proportional to the third power of linear dimensions, $V \sim l^3$. Therefore, the torque is proportional to $T \sim V^{4/3}$.

Electrical machines are made of iron and copper, materials of known specific masses.[1] Therefore, the mass m of an electrical machine is determined by the electromagnetic torque for which it has been designed. The mass m and torque T are related by $T \sim m^{4/3}$. As an example, a new machine with all the three dimensions doubled with respect to the original machine develops electromagnetic torque increase $2^4 = 16$ times.

Relation $T \sim m^{4/3}$ can be verified in another way. It has been proven that the torque can be expressed as ***vector*** product of two flux ***vectors***. The flux amplitude depends on the surface ($S \sim l^2$) and magnetic induction ($B < B_{max}$), the latter being limited by magnetic saturation of the ferromagnetic material and not exceeding 1.5 . 1.7 T. The product of two fluxes depends on the fourth power of linear dimension l. Hence, the product of stator and rotor flux ***vectors*** depends on l^4. Hence, the electromagnetic torque available from electrical machine is proportional to l^4.

Power of electromechanical conversion in an electrical machine depends on the torque and speed of rotation Ω; therefore, $P \sim V^{4/3}\Omega$. Considering two machines with the same dimensions and different speeds, the one with the higher speed delivers more power. In cases requiring a constant power of electromechanical conversion P, while the speed of rotation of the electrical machine can be arbitrarily chosen, it is beneficial to select the machine with higher speed, resulting in a lower torque $T = P/\Omega$ and consequently smaller dimensions of the machine due to $T \sim l^4$. An example where the required load speed can be achieved with different machine

[1] $\gamma_{Fe} = \Delta m_{Fe}/\Delta V = 7{,}874\ \mathrm{kg/m^3}$, $\gamma_{Cu} = \Delta m_{Cu}/\Delta V = 8{,}020\ \mathrm{kg/m^3}$.

speeds is the case where the load and the machine are coupled by gears. While designing the system, the gear ratio can be selected so as to result in a higher speed of the machine. This will reduce the size and weight of the machine.

The torque expression (9.16) suggests that the torque is inversely proportional to the air gap width δ. With constant stator and rotor currents, a decrease in the air gap results in an increase in electromagnetic torque. The expression suggests that the torque can be increased with no apparent limits, provided that the air gap δ can get sufficiently small. This conclusion is incorrect as it overlooks the phenomenon of magnetic saturation. The expression for the torque has been derived as a result of an analysis where magnetic saturation in iron is neglected. The stator and rotor magnetic circuits are made of iron sheets of very high permeability μ_{Fe}, making the magnetic field in iron H_{Fe} negligible. The torque expression (9.16) is based on such an assumption. It holds in all the conditions with no magnetic saturation in iron parts of the magnetic circuit. With excessive values of B_{Fe} resulting in magnetic saturation, the value of H_{Fe} cannot be neglected, and this invalidates the (9.16). The preceding analysis finds the magnetic induction B_0 in the air gap inversely proportional to the air gap δ. Disregarding the slots, the magnetic induction in iron is roughly the same, $B_{Fe} \approx B_0$. Therefore, progressive decrease of the air gap leads to increased magnetic induction. As the magnetic induction B reaches $B_{max} = 1.5$ .. 1.7 T, the iron gets saturated, permeability μ_{Fe} drops, and the magnetic field H_{Fe} assumes considerable value that cannot be neglected. At this point, (9.15) and the consequential results, obtained by neglecting saturation, are not valid and cannot be used.

The air gap of electrical machines is designed to be as small as possible, in order to obtained the desired magnetic induction B_0 with smaller electrical currents and, consequently, smaller copper losses. However, there are limitations of mechanical nature which prevent the air gap of smaller machines from getting much below one millimeter. The air gap of large electrical machines is at least several millimeters. A lower limit of the air gap is required to prevent the revolving rotor from touching the stator. Undesired touching and scratching can happen due to finite tolerances in manufacturing the stator and rotor surfaces. Elastic deformation of the shaft in radial direction can result in rotor touching the stator. These phenomena prevent the use of electrical machines with very small air gaps.

Question (9.5): The expressions for the electromagnetic torque and power of electrical machine give values inversely proportional to the air gap δ. Based on these expressions, the power and torque can be increased with no apparent limits, keeping the electrical currents constant and reducing the air gap. There are reasons that invalidate such conclusion. Provide two reasons which indicate that such expectations are not realistic.

Answer (9.5): The expression for the electromagnetic torque suggests that reduction of the air gap δ results in higher torque and higher power of electromechanical conversion. Apparently, very high torque can be achieved with an adequate reduction of the air gap.

This conclusion overlooks the phenomenon of magnetic saturation. The torque expression comes from an analysis that starts with an assumption that the magnetic field H_{Fe} in iron is negligible. This assumption holds only in cases where the flux density B_{Fe} does not reach the saturation limit of $B_{max} = 1.5 .. 1.7$ T. Namely, given a magnetomotive force $Ni = 2H\delta$, the flux density $B = \mu_0 Ni/(2\delta)$ grows as the air gap decreases. As the flux density B reaches the saturation limit B_{max}, any further increase of B_{Fe} is determined by expression $\Delta B = \mu_0 \Delta H$. In other words, differential permeability $\Delta B/\Delta H$ of the saturated ferromagnetic material is close to μ_0. Considering the flux changes, magnetic saturation is equivalent to removing the iron parts of the stator and rotor magnetic circuits. Due to $\Delta B/\Delta H \approx \mu_0$, saturated iron behaves like air. Therefore, the magnetic saturation can be considered as a very large increase in the air gap. Therefore, the initial projections of the air gap reduction leading to large torque gains are not realistic. It is of interest to notice that most electrical machines are designed so as to get the most out of their magnetic circuits. For this to achieve, the flux density levels are close to saturation limits. Therefore, there is no margin to accommodate any further increase in B.

Another reason that prevents the torque increase is the fact that the air gap cannot be decreased below certain limits, roughly 1 mm, imposed by mechanical conditions.

9.9 Rotating Magnetic Field

According to previous considerations, conditions for generating a nonzero average electromagnetic torque include a constant relative position $\Delta\theta$ of the stator and rotor flux ***vectors***. In DC machines, both flux ***vectors*** remain still with respect to the stator. In alternating current machines, whether asynchronous or synchronous, both flux ***vectors*** rotate at the same speed.

In order to meet the above condition and due to rotor revolution, at least one of the two fluxes (Φ_S or Φ_R) has to rotate with respect to the winding that originates the magnetomotive force resulting in the relevant flux. The magnetic field which rotates with respect to the originating windings is also called *rotating magnetic field*. In this section, it is shown that rotating magnetic field requires a system with at least two separate windings with appropriate spatial displacement of their magnetic axes. Alternating currents in the windings should have the same frequency and amplitude. Their initial phases are to be different and should correspond to the spatial displacement of the magnetic axes. In this case, the system of windings creates magnetomotive force and flux that revolve at the speed determined by the frequency of AC currents.

9.9.1 System of Two Orthogonal Windings

Electromagnetic torque is determined by the ***vector*** product of the stator and rotor flux ***vectors***, and it depends on sine of the angle $\Delta\theta$ between the two ***vectors***. A continuous conversion of energy with constant torque and constant power

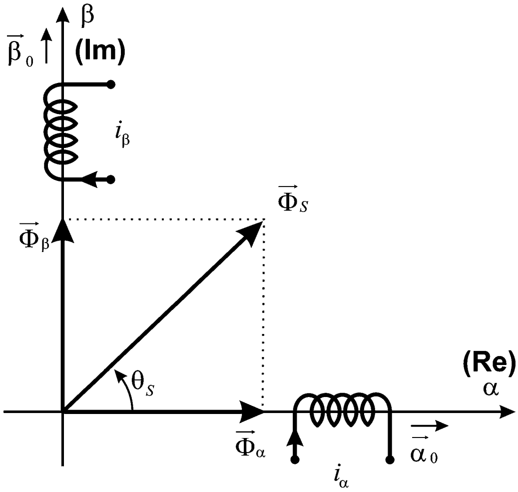

Fig. 9.9 A system with two orthogonal windings

requires that the angle $\Delta\theta$ is constant. For this reason, it is necessary that the stator and/or rotor windings create rotating magnetic field.

Figure 9.9 shows a stator with two windings, α and β. Each winding has N turns. The winding conductors could be either concentrated or distributed. Their construction does not affect the subsequent analysis and conclusions. For brevity, it will be considered that windings α and β are concentrated. Flux in one turn of a concentrated winding is denoted by Φ. It is equal to the ratio of the magnetomotive force $F = Ni$ and the magnetic resistance R_μ. Due to high permeability of iron, magnetic field H_{Fe} can be neglected. Considering concentrated winding with electrical current i, the field strength in the air gap is obtained from relation $Ni = 2\delta H_0$, while magnetic induction B in the air gap is equal to $B_0 = \mu_0 H_0$. Surface S_1 is encircled by one turn of the considered concentrated winding. Assuming that the surface passes through the air gap, it represents one half of a cylinder and it has the surface area $S_1 = \pi LR$. Therefore, the flux in one turn is

$$\Phi = B_0 S_1 = \mu_0 H_0 \pi LR = \frac{\mu_0 \pi LR}{2\delta} Ni.$$

This expression can be verified by calculating the flux by dividing the magnetomotive force and the magnetic resistance, $\Phi = F/R_\mu$. Magnetic resistance R_μ is calculated considering that $H_{Fe} = 0$, and taking into account that each field line passes twice through the air gap. Therefore,

$$R_\mu = \frac{1}{\mu_0}\frac{2\delta}{S_1} = \frac{1}{\mu_0}\frac{2\delta}{\pi LR},$$

where $\pi LR = S_1$ represents the surface area of the cross section of considered magnetic circuit while μ_0 is the permeability in the air gap. Quantity 2δ represents the length of the magnetic circuit where the field strength H assumes considerable values and the line integral of the field results in *magnetic voltage drop*. The air gap is passed twice, where the field lines enter and exit the rotor (or stator) magnetic circuit, that is, next to the north and south magnetic poles. Expression $\Phi = F/R_\mu$ becomes

$$\Phi = \frac{F}{R_\mu} = \frac{Ni}{\left(\frac{1}{\mu_0}\frac{2\delta}{\pi LR}\right)} = \frac{\mu_0 \pi LR}{2\delta} Ni. \tag{9.19}$$

Magnetic axes of the windings α and β reside on the abscissa and ordinate of the orthogonal coordinate system shown in Fig. 9.9. Axis of the winding α is horizontal, along the course defined by the unit ***vector*** $\boldsymbol{\alpha}_0$. By establishing electrical current i_α, magnetomotive force $F_\alpha = Ni_\alpha$ is produced along the course and direction of the unit ***vector*** $\boldsymbol{\alpha}_0$. The flux in one turn is obtained by dividing the magnetomotive force and the magnetic resistance, $\Phi_\alpha = Ni_\alpha/R_\mu$. It is assumed that the windings are concentrated; thus, the flux Ψ_α in the winding is equal to $N\Phi_\alpha = N^2 i_\alpha/R_\mu$. The axis of the winding β is orthogonal with respect to the α winding, and it extends along the course defined by the unit ***vector*** $\boldsymbol{\beta}_0$. The magnetomotive force F_β and the flux Φ_β of this winding are oriented in accordance with the ordinate axis β, and it is proportional to the winding current $\boldsymbol{i}_\beta$. By using the unit ***vectors*** of the two axes, fluxes in the winding turns can be represented by expressions

$$\vec{\Phi}_\alpha = \frac{N}{R_\mu} i_\alpha \cdot \vec{\alpha}_0, \quad \vec{\Phi}_\beta = \frac{N}{R_\mu} i_\beta \cdot \vec{\beta}_0.$$

Electrical currents in windings α and β are alternating currents of the same amplitude I_m and the same angular frequency ω_s. The symbol ω_s denotes the angular frequency of electrical currents in the stator windings. The initial phases of the two currents are different. The current in winding α leads by π/2, the angle that corresponds to the spatial shift between α and β magnetic axes. Variation of currents in the windings is given by (9.20):

$$\begin{aligned} i_\alpha &= I_m \cos(\omega_S t) = I_m \cos\theta_S, \\ i_\beta &= I_m \cos\left(\omega_S t - \frac{\pi}{2}\right) = I_m \sin(\omega_S t) = I_m \sin\theta_S. \end{aligned} \tag{9.20}$$

The resultant magnetomotive force F_S of the stator winding and the stator flux $\boldsymbol{\Phi}_S$ are obtained by summing their α and β components. If the orthogonal windings α and β in Fig. 9.9 have electrical currents as given by (9.20), the magnetomotive force ***vector*** F_S is created, determined by expression (9.21), resulting in the flux ***vector*** given in (9.22). Since α and β components are proportional to functions cosθ_s and sinθ_s, where the angle θ_s changes as $\omega_s t$, the latter equation describes the flux ***vector*** which rotates at the speed of ω_s and has an amplitude which is constant. Equation (29.22) gives the resultant flux corresponding to one turn. Since a concentrated winding is in consideration, the resultant ***vector*** of the flux of the windings is obtained by multiplying the flux in one turn by the number of turns, as defined in (9.23):

$$\vec{F}_S = R_\mu \vec{\Phi}_S = NI_m(\vec{\alpha}_0 \cos\theta_s + \vec{\beta}_0 \sin\theta_s) = NI_m\left[\vec{\alpha}_0 \cos(\omega_s t) + \vec{\beta}_0 \sin(\omega_s t)\right], \quad (9.21)$$

$$\vec{\Phi}_S = \frac{NI_m}{R_\mu}(\vec{\alpha}_0 \cos\theta_s + \vec{\beta}_0 \sin\theta_s) \quad (9.22)$$

$$\vec{\Psi}_S = N\vec{\Phi}_S = \frac{N^2 I_m}{R_\mu}(\vec{\alpha}_0 \cos\theta_s + \vec{\beta}_0 \sin\theta_s). \quad (9.23)$$

The flux components Φ_α and Φ_β are the projections of the stator flux ***vector*** Φ_S (9.22) on the axes of the α–β coordinate system defined by the unit ***vectors*** α_0 and β_0. Projections Φ_α and Φ_β of the ***vector*** Φ_S on α- and β-axes represent, at the same time, the fluxes in one turn of the respective α and β windings. Axes of the windings shown in Fig. 9.9 are mutually orthogonal. Therefore, the currents in α winding do not cause variations of the flux in β winding. The same way, the currents in β winding do not affect the flux in α winding.

Since the revolving ***vector*** has α and β components of the flux, it can be concluded that creation of a rotating field requires the existence of at least two spatially displaced windings.

In (9.22), components Φ_α and Φ_β of the flux are accompanied by unit ***vectors*** α_0 and β_0. Written presentation can be simplified by substituting the plane α–β with the plane representing complex numbers, with α-axis being the real axis and β-axis being the imaginary axis. Formal translation of equations from α–β coordinate system into the complex plane is done by substituting the unit ***vector*** $\boldsymbol{\alpha}_0$ with 1 and substituting the unit ***vector*** $\boldsymbol{\beta}_0$ with imaginary unit j. In this way, (9.22) changes into

$$\underline{\Phi}_S = \Phi_\alpha + \mathrm{j}\Phi_\beta = \frac{NI_m}{R_\mu}(\cos\theta_s + \mathrm{j}\sin\theta_s) = \frac{NI_m}{R_\mu} e^{\mathrm{j}\theta_s}. \quad (9.24)$$

On the basis of the preceding analysis, it is concluded that a system of two orthogonal, mutually independent windings can create a rotating magnetic field. In cases when the windings carry sinusoidal currents of the same angular frequency ω_s, the same amplitude I_m, and with their initial phases shifted by π/2, the consequential magnetomotive force and flux in the machine revolve. Therefore, these quantities can be represented by rotating ***vectors***. The ***vectors*** rotate at the speed Ω_S which is determined by the angular frequency ω_s. In the course of rotation, there is no change in amplitude of these ***vectors***. For the system of two windings shown in Fig. 9.9, the speed Ω_S is equal to the angular frequency ω_s.

Question (9.6): Consider the stator winding shown in Fig. 9.9 and assume that the amplitudes of the two stator currents are equal. The difference of initial phases of currents i_α and i_β is denoted by φ.

- Determine and describe the stator flux ***vector*** in cases where $\varphi = 0$.
- Determine and describe the stator flux ***vector*** for $\varphi = \pi/2$.
- Show that in cases with $0 < \varphi < \pi/2$, the ***vector*** of the stator flux can be represented by the sum of two flux ***vectors***, one of them rotating at the speed $\Omega_S = \omega_s$ and maintaining a constant amplitude while the other pulsating back and forth along the same course.

Answer (9.6): Equation (9.19) allows the flux in one turn to be calculated as the ratio of magnetomotive force $F = Ni$ and magnetic resistance R_μ,

$$\Phi = \frac{\mu_0 \pi L R}{2\delta} Ni = \frac{Ni}{\left(\frac{1}{\mu_0}\frac{2\delta}{\pi L R}\right)} = \frac{Ni}{R_\mu}.$$

When currents i_α and i_β are known, components of the stator flux are determined by expressions

$$\vec{\Phi}_\alpha = \frac{N}{R_\mu} i_\alpha \cdot \vec{\alpha}_0, \quad \vec{\Phi}_\beta = \frac{N}{R_\mu} i_\beta \cdot \vec{\beta}_0.$$

In cases with $\varphi = 0$, the instantaneous values of electrical currents $i_\alpha(t)$ and $i_\beta(t)$ are equal; thus, the resultant flux ***vector*** is

$$\vec{\Phi}_S = \frac{N}{R_\mu} I_m \cos(\omega_S t) \cdot \left(\vec{\alpha}_0 + \vec{\beta}_0\right).$$

Therefore, with $\varphi = 0$, the flux ***vector*** does not revolve, and it pulsates along the line inclined by $\pi/4$ with respect to the abscissa. The algebraic value of the ***vector*** oscillates at the angular frequency ω_s.

In cases where $\varphi = \pi/2$, the ***vector*** of the magnetomotive force is determined by (9.21), while the resultant flux ***vector*** of one turn is

$$\vec{\Phi}_S = \frac{N I_m}{R_\mu}\left[\vec{\alpha}_0 \cos(\omega_s t) + \vec{\beta}_0 \sin(\omega_s t)\right].$$

In general, electrical current in winding β can be written in the form

$$i_\beta = I_m \cos(\omega_S t - \phi) = I_m \cos(\omega_S t)\cos\phi + I_m \sin(\omega_S t)\sin\phi,$$

while the current in winding α can be written as the sum

$$i_\alpha = I_m \cos(\omega_S t) = I_m \cos(\omega_S t)(1 - \sin\phi) + I_m \cos(\omega_S t)\sin\phi.$$

The flux ***vector*** can be represented by the sum of two ***vectors***:

$$\vec{\Phi}_S = \vec{\Phi}_{SO} + \vec{\Phi}_{SP},$$

where the elements of the sum are determined by

$$\vec{\Phi}_{SO} = \frac{NI_m}{R_\mu} \cdot \sin\phi \cdot \left[\vec{\alpha}_0 \cos(\omega_s t) + \vec{\beta}_0 \sin(\omega_s t)\right],$$
$$\vec{\Phi}_{SP} = \frac{NI_m}{R_\mu} \cdot \cos(\omega_s t) \cdot \left[\vec{\alpha}_0 (1 - \sin\phi) + \vec{\beta}_0 \cos\phi\right].$$

Vector Φ_{SO} represents a rotating field which revolves at the speed $\Omega_s = \omega_s$. The amplitude of this ***vector*** does not change in the course of rotation, and it is proportional to the sine of the angle φ. Therefore, with $\varphi = 0$, the rotating field of the machine does not exist. The ***vector*** Φ_{SP} has a course which does not change. This course is determined by the angle φ. With $\varphi = 0$, the course of the pulsating field Φ_{SP} is $\pi/4$ with respect to the abscissa. The flux Φ_{SP} does not rotate but pulsates instead along the indicated course.

9.9.2 System of Three Windings

In most cases, asynchronous and synchronous motors are fed from voltage sources providing a symmetrical three-phase system of voltages and currents. When operating as generators, the machines convert the mechanical work into electrical energy and produce a system of three-phase voltages available to electrical loads at the stator terminals. For this reason, the stator windings of asynchronous and synchronous machines usually have three phases. That is, there are three separate, spatially displaced windings on the stator. The three separate stator windings are called the phases and assigned letters a, b, and c. In three-phase machines, the axes of the phase windings are spatially displaced by $2\pi/3$. Figure 9.10 shows a machine with phase windings a, b, and c carrying sinusoidal currents of equal amplitudes I_m, equal angular frequency ω_s, and with difference in initial phases of $\pm 2\pi/3$. The phase shift of the phase currents corresponds to the spatial displacement between the magnetic axes of the phase windings.

The magnetomotive forces F_a, F_b, and F_c of the windings have amplitudes Ni_a, Ni_b, and Ni_c. Their orientation is determined by magnetic axes of respective windings, and their courses can be expressed in terms of unit ***vectors*** α_0 and β_0,

$$\vec{a}_0 = \vec{\alpha}_0, \qquad \vec{b}_0 = -\frac{1}{2}\vec{\alpha}_0 + \frac{\sqrt{3}}{2}\vec{\beta}_0, \qquad \vec{c}_0 = -\frac{1}{2}\vec{\alpha}_0 - \frac{\sqrt{3}}{2}\vec{\beta}_0\,. \tag{9.25}$$

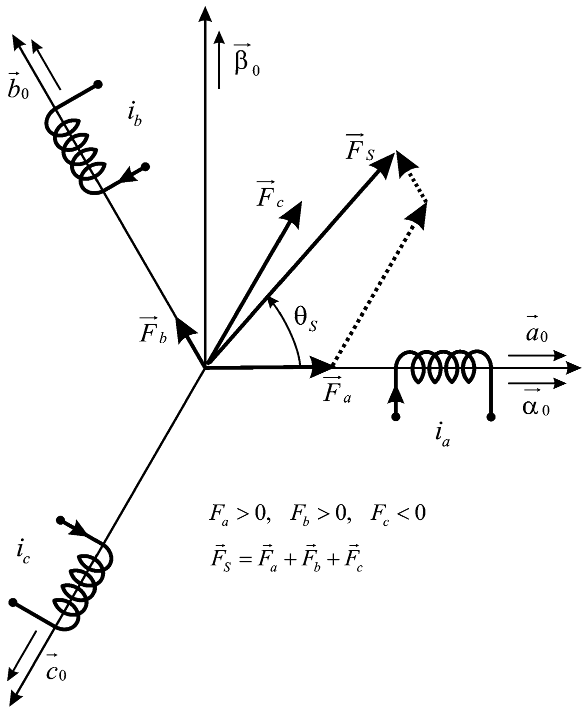

Fig. 9.10 Positions of the vectors of magnetomotive forces in individual phases, position of their magnetic axes, and unit vectors of the orthogonal coordinate system

Currents of the phase windings are determined by equations

$$\begin{aligned} i_a &= I_m \cos \omega_S t, \\ i_b &= I_m \cos(\omega_S t - 2\pi/3), \\ i_c &= I_m \cos(\omega_S t - 4\pi/3), \end{aligned} \tag{9.26}$$

in such way that ***vectors*** of the magnetomotive forces of individual phases become

$$\vec{F}_a = N i_a \vec{a}_0, \qquad \vec{F}_b = N i_b \vec{b}_0, \quad \vec{F}_c = N i_c \vec{c}_0.$$

By using relation (9.25), the ***vectors*** representing magnetomotive forces in individual phases can be expressed in terms of unit ***vectors*** α_0 and β_0,

$$\begin{aligned} \vec{F}_a &= N i_a \vec{\alpha}_0, \\ \vec{F}_b &= N i_b \left(-\frac{1}{2}\vec{\alpha}_0 + \frac{\sqrt{3}}{2}\vec{\beta}_0\right), \\ \vec{F}_c &= N i_c \left(-\frac{1}{2}\vec{\alpha}_0 - \frac{\sqrt{3}}{2}\vec{\beta}_0\right). \end{aligned} \tag{9.27}$$

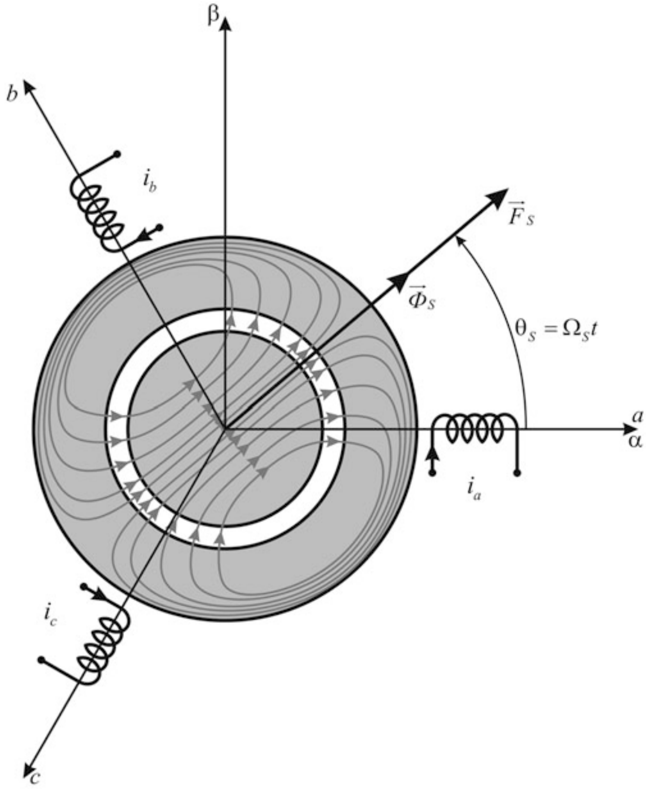

Fig. 9.11 Field lines and vectors of the rotating magnetic field

The resultant magnetomotive force of the stator windings is obtained by ***vectors*** summation of magnetomotive forces in individual phases, and it is given by equation

$$\vec{F} = \vec{F}_a + \vec{F}_b + \vec{F}_c = \frac{3}{2} N I_m \left[\vec{\alpha}_0 \cos \omega_S t + \vec{\beta}_0 \sin \omega_S t \right]. \tag{9.28}$$

Summing the individual magnetomotive forces of the three phases, one obtains the rotating ***vectors*** of the resultant magnetomotive force with an amplitude of $3/2NI_m$. Projections of this ***vectors*** on axes α and β are proportional to functions cos $(\omega_s t)$ and $\sin(\omega_s t)$, proving that the ***vectors*** revolves at the speed of $\Omega_s = \omega_s$ and that it has a constant amplitude. The ratio of the magnetomotive force F_S of the stator windings and the resistance R_μ of the magnetic circuit gives the flux ***vectors*** Φ_S of the stator which rotates at the same speed as the ***vectors*** F_S. Hence, the system of sinusoidal currents in three-phase stator winding results in a revolving magnetic field with the speed of rotation determined by the angular frequency of the phase currents, while the field magnitude depends on the maximum value I_m of the phase currents.

As already emphasized in the introduction, it is necessary to distinguish between the speed of rotation of the rotor, being mechanical quantity expressed in rad/s, and the angular frequency of electrical currents and voltages, appertaining to electrical circuits and being expressed in rad/s as well. Throughout this book, mechanical speed is denoted by Ω, while the angular frequency of voltages and currents is denoted by ω.

Figure 9.11 shows ***vectors*** of the resultant magnetomotive force and the resultant flux in electrical machine with three-phase system of stator windings.