Chapter 4
Magnetic Circuit

This chapter introduces and explains magnetic circuits of electrical machines. Basic laws and skills required to analyze magnetic circuits are reinstated and illustrated on examples and solved problems. The terms such as magnetic resistance, magnetomotive force, core flux, and winding flux are recalled and applied. Dual electrical circuit is introduced, explained, and applied in solving magnetic circuits. Basic properties of ferromagnetic materials are recalled, including saturation phenomena, eddy current losses, and hysteresis losses. Laminated magnetic circuits as the means of reducing the iron losses are explained and analyzed.

One of the key operating principles of electromechanical converters based on magnetic field is creation of Lorentz force acting on a current-carrying conductor placed in the magnetic field. Magnetic field can be obtained from a permanent magnet or by using an *electromagnet*. Electromagnet is a system of windings carrying electrical currents that create magnetic field. It is useful in replacing the permanent magnets by coils carrying a relatively small electrical current. For the electromagnet currents to be moderate, it is necessary to employ magnetic circuits made of ferromagnetic material (iron), conducting the magnetic flux in a way similar to copper conductor directing electrical current. As the copper conductor provides a low-resistance path to electrical current, so does the magnetic circuit provide a path to magnetic flux that has a low magnetic resistance. An example of magnetic circuit is shown in Fig. 4.1.

Figure 4.1 shows a magnetic circuit made of iron, a ferromagnetic material with permeability $\mu = B/H$ higher than that of the vacuum (μ_0) by several orders of magnitude. This magnetic circuit has an air gap of size δ. Within the gap, it is possible to place a conductor carrying current in order to obtain Lorentz force and accomplish electromechanical conversion of energy (the conductor is not shown in the figure). Magnetic flux within the magnetic circuit is created by means of the *excitation* winding with N series-connected contours, also called *turns*. Each turn encircles the magnetic circuit. Assuming that there are no losses and that the lines of

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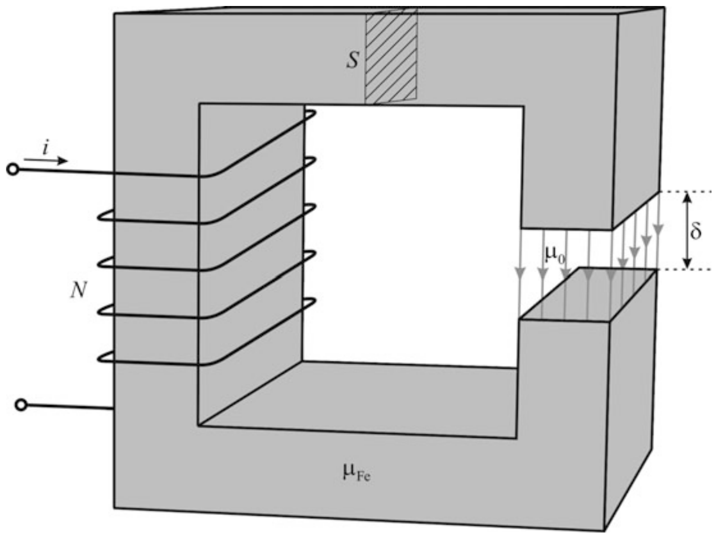


Fig. 4.1 Magnetic circuit made of an iron core and an air gap

the field are parallel, it is concluded that magnetic induction in iron (B_{Fe}) is equal to the magnetic induction in the air (B_0). The strength of the magnetic field in iron is $H_{Fe} = B_{Fe}/\mu_{Fe}$, whereas in the air gap, it is equal to $H_0 = B_0/\mu_0$. Since permeability of iron is much higher, the magnetic field in iron will be considerably lower than the field in the air gap. Ampere law thus reduces to $Ni = H_0\delta$, and the current required for obtaining magnetic field H_0 in the gap is equal to $i = H_0\delta/N$.

In order to obtain magnetic induction B in the air gap, it is necessary to establish the current $i = B\delta/(N\mu_0)$ in the excitation winding. Hence, the required excitation current is proportional to the air gap δ. An attempt to remove the iron part of the magnetic circuit can be represented as an increase of the gap δ to $\delta + l_{Fe}$, where l_{Fe} is the length of the iron part of the magnetic circuit. The required current would increase $1 + l_{Fe}/\delta$ times. Since $l_{Fe} >> \delta$, removal of the iron would result in a multiple increase of the excitation current and the associated losses. Therefore, it is concluded that the magnetic circuit is a key part of electrical machinery. It directs and concentrates the magnetic field to the region where the conductors move and the electromechanical conversion takes place. The presence of an iron magnetic circuit allows the necessary *excitation* to be accomplished with considerably smaller currents and lower losses.

In the preceding section, an analysis of a simple magnetic circuit has been done. In the analysis, certain simplifications have been made. In order to analyze more complex magnetic circuits, a list of the basic laws and usual approximations to simplify the analysis is presented within the next section.

4.1 Analysis of Magnetic Circuits

Magnetic circuit is a domain where magnetic field is created by one or several current circuits or permanent magnets. The laws applicable for analysis of magnetic circuits are:

- The flux conservation law
- Generalized form of Ampere law
- Constitutive relation $B(H)$ which describes a magnetic material

4.1.1 Flux Conservation Law

$$\oint_S \vec{B} \cdot \mathrm{d}\vec{S} = 0. \tag{4.1}$$

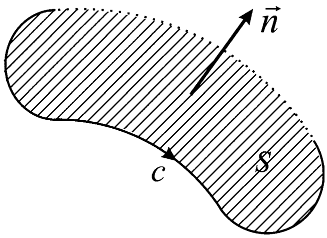
Taking into account ferromagnetic properties of magnetic materials used in making magnetic circuits, a series of simplifications can be introduced in order to facilitate their analysis. One of the assumptions is that there is no leakage of magnetic lines outside magnetic circuit. Neglecting the leakage, it can be shown that the flux remains constant along the magnetic circuit. In other words, the magnetic flux in each cross-section of the magnetic circuit is the same. The flux in each cross-section is also called *the core flux* or *the flux per turn*, meaning the flux in a single turn of the winding encircling the magnetic circuit. The algebraic value of the flux in the cross-section is defined in accordance with the normal to the cross-section surface, and it is denoted by Φ. In most cases, magnetic circuit is encircled by a winding made of N series-connected turns having the orientation. Assuming that there is no flux leakage from the magnetic circuit, the flux in each turn is equal to Φ. Therefore, the flux of the winding is $\Psi = N\Phi$, with the same reference direction as for the flux in one turn.

The windings are connected in electrical circuits. The voltage across a winding is equal to $u = Ri + \mathrm{d}\Psi/\mathrm{d}t$, where R is resistance of the series-connected turns, i is winding current, while $\mathrm{d}\Psi/\mathrm{d}t$ is back electromotive force. It is of uttermost importance to match the reference direction of the electrical circuit (current) with the orientation of the magnetic circuit (flux). As a rule, the reference normal for the flux is determined from the reference direction of the current by the right-hand rule.

By applying the flux conservation law, it can be shown that the flux in one contour (turn) is equal to the flux through any other surface leaning on the same contour. This equality will be used to simplify calculation of the flux in the windings of cylindrical machines.

Commonly used assumption is that the magnetic field is homogeneous over the cross-section of a magnetic circuit and that the length of any magnetic field line is equal to the length of the average representative line of the magnetic circuit.

Fig. 4.2 The reference normal n to surface S which is leaning on contour c



4.1.2 Generalized Form of Ampere Law

Generalized form of Ampere law for fields with stationary electrical currents is given by (4.2). Contour c and surface S are shown in Fig. 4.2:

$$\oint_c \vec{H} \cdot \mathrm{d}\vec{l} = \int_S \vec{J} \cdot \mathrm{d}\vec{S}. \tag{4.2}$$

Electrical currents in electrical machines are not distributed in space, but they exist in conductors forming the turns, windings, and current circuits. The conductors are usually made of copper wires. With a layer of insulating material wrapped around wires, they do not have galvanic contact with other parts. Therefore, the current is directed along wires and does not leak away. Consequently, instead of a surface integral of current density J, one should use the sum of currents in the conductors passing through a surface S, respecting the reference direction determined by the unit vector. Equation 4.2 thus takes the form (4.3):

$$\oint_c \vec{H} \cdot \mathrm{d}\vec{l} = \Sigma I. \tag{4.3}$$

4.1.3 Constitutive Relation Between Magnetic Field H and Induction B

The relation between the vector of magnetic field ***H*** and magnetic induction ***B*** in individual parts of a magnetic circuit is determined by the properties of the magnetic material, and it is given by (4.4):

$$\vec{B} = \vec{B}(\vec{H}). \tag{4.4}$$

In linear media, magnetic induction ***B*** is proportional to magnetic field ***H***. Coefficient of proportionality is a scalar quantity μ called magnetic permeability (4.5). Magnetic permeability in vacuum is $\mu_0 = 4\pi \cdot 10^{-7}$ [H/m]. In ferromagnetic materials like iron, the characteristic $B(H)$ is not linear. It is usually presented graphically or by the corresponding analytical approximation called *the characteristic of magnetization*. For small values of magnetic field, the magnetization characteristic of iron $B(H)$ is linear and has the slope $\Delta B/\Delta H$ which is several thousand times higher than the permeability of vacuum μ_0.

$$\vec{B} = \mu\vec{H} = \mu_0\mu_r\vec{H}. \tag{4.5}$$

4.2 The Flux Vector

Flux through the contour of Fig. 4.2 is a scalar quantity. Flux through surface S, leaning on contour c, is determined by surface integral of the vector of magnetic induction $\boldsymbol{B}$. In the analysis of electrical machines, flux through a contour is often considered as a vector. The *flux vector* is obtained by associating the course and direction with scalar Φ. The spatial orientation is obtained from the unit normal to surface S. In cases with several contours (turns) forming a winding where all of the contours share the same orientation, it is possible to define the flux vector of the winding. This flux has algebraic intensity of $\Psi = N\Phi$ while its course and direction are determined by the unit normal to surface S. The winding can be made of series-connected contours (turns) with different spatial orientation. In such cases, the vector of the winding flux is obtained as a vector sum of flux vectors in individual contours.

4.3 Magnetizing Characteristic of Ferromagnetic Materials

Magnetic circuits of electrical machines and transformers are most frequently made of iron sheets. Iron is ferromagnetic material with magnetization characteristic $B(H)$ shown in Fig. 4.3. The characteristic extends between the two straight lines. The line with the slope $\Delta B/\Delta H = \mu_0$ describes magnetization characteristic of vacuum, while the line with the slope μ_{Fe} corresponds to the first derivative of the function $B(H)$ at the origin. The abscissa of the B-H coordinate system is the external field H, which may be obtained by establishing a current in the excitation winding, while the ordinate is magnetic induction B existing in the ferromagnetic material.

The magnetic properties of iron originate from microscopic Ampere currents within a molecule or a group of molecules. These currents make the origin of the magnetic field of permanent magnets and other ferromagnetic materials. The said currents are the cause of forces acting on ferromagnetic parts brought in a magnetic field. The presence of microscopic currents can be taken into account by treating

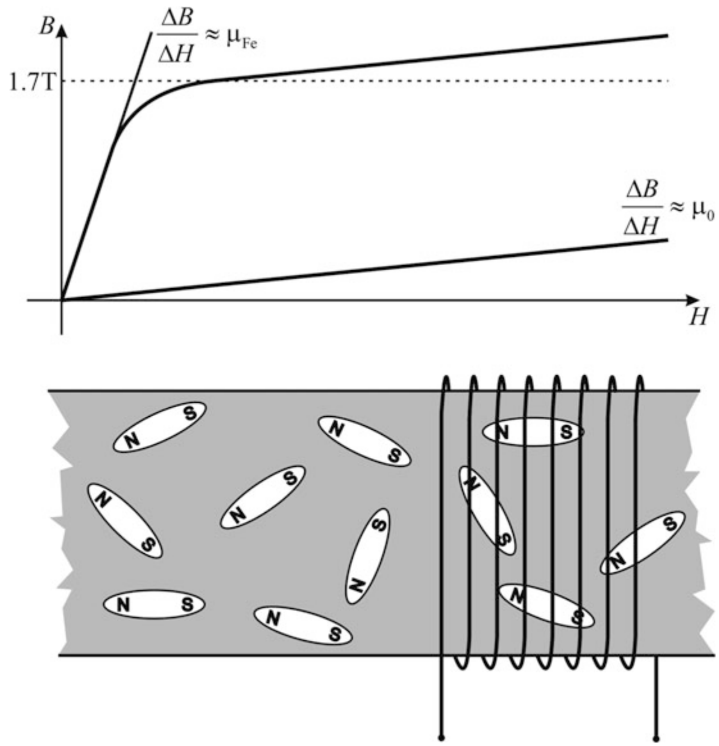


Fig. 4.3 The magnetization characteristic of iron

ferromagnetic materials as a vast collection of miniature magnetic dipoles, as shown in Fig. 4.3. In the absence of external field H, the magnetic dipoles do not have the same orientation. They oscillate and change directions at a speed that depends on the temperature of the material. Therefore, in the absence of an external magnetic field, resulting magnetic induction in the material is equal to zero.

With an excitation current giving rise to magnetic field H, magnetic dipoles turn in an attempt to get aligned with the field. Thermal motion of dipoles prevents them to stay aligned and makes them change the orientation. The higher the field H, the more dipoles get aligned to the field. As a consequence, resulting magnetic induction takes the value $B = \mu_{Fe}H$ which is much higher than the corresponding value in vacuum ($B = \mu_0 H$). In this way, ferromagnetic materials help providing the required magnetic induction B with much smaller excitation current.

When magnetic induction reaches $B_{max} \in [1\ldots2]$ T, all miniature dipoles get oriented in the same direction, aligned with the excitation field H. Any further increase of the field strength H cannot improve the orientation of dipoles, as there are no more disoriented dipoles. This state is called *saturation* of magnetic material. In the region of saturation, further increase of induction is the same as it would have been in vacuum, $\Delta B = \mu_0 \Delta H$. The saturation region is expressed in the right-hand side of the curve in Fig. 4.3.

4.4 Magnetic Resistance of the Circuit

The role of magnetic circuit in electrical machines is to direct the lines of magnetic coupling field to the space where the electromagnetic conversion takes place. The magnetic induction and flux Φ in the magnetic circuit appear under the influence of current in the winding. The strength of the field H is dependent on the product Ni, where N is the number of turns in a winding while i is the electrical current. In a way, the value Ni tends to establish the flux Φ in the magnetic circuit. Therefore, the ratio Ni/Φ is *magnetic resistance of the circuit*. A circuit having smaller magnetic resistance will reach the given flux with smaller currents. A magnetic circuit can have several parts, which can be made of ferromagnetic material, permanent magnets, nonmagnetic materials, or air. Air-filled parts of magnetic circuits are also called *air gaps*. It is of interest to determine magnetic resistance of a magnetic circuit comprising several heterogeneous parts.

Magnetizing characteristics of ferromagnetic parts of magnetic circuit are nonlinear and shown in Fig. 4.3. Operation of a magnetic circuit is usually performed in the vicinity of the origin of $B(H)$ diagram. It is therefore justifiable to linearize the magnetization characteristics and consider that the permeability μ_{Fe} of ferromagnetic (iron) parts is constant. In the linearized ferromagnetic circuits, nonlinearity of ferromagnetic material is neglected, and permeability of every part of the magnetic circuit is considered constant. In addition, it is assumed that there is no leakage of magnetic field outside magnetic circuit. The basic assumptions and steps in the analysis of linearized magnetic circuits are given by the following considerations.

On the basis of (4.3), the line integral of magnetic field $\boldsymbol{H}$ along contour c, indicated in Fig. 4.4, is equal to the product Ni. *Magnetomotive force* $F = Ni$ is equal to the integral of the field $\boldsymbol{H}$ through the closed contour passing through all the parts of the magnetic circuit. Magnetomotive force $\boldsymbol{F}$ is a scalar quantity. Vector of the magnetomotive force is obtained by associating the spatial orientation to the scalar $F = Ni$. The orientation of the magnetomotive force $\boldsymbol{F}$ is determined by the vector $\boldsymbol{H}$. Both the orientations of $\boldsymbol{F}$ and $\boldsymbol{H}$ are related to electrical currents in the winding that encircles the magnetic circuit. In Fig. 4.4, vector of the magnetomotive force F is collinear with the normal n_k related to the reference direction of the electric currents by the right-hand rule.

Surface integral of magnetic induction over surface S is denoted by Φ and is called *flux of the core* or *flux across the cross-section of the magnetic circuit* or *flux in one turn*. Assuming that there is no leakage of magnetic field outside of the magnetic circuit, the line integral of the magnetic field H along contour c is equal Ni for every and each contour passing through the magnetic circuit. Since the basic assumption is that the lengths of magnetic lines are equal to the length of the representative average line of the magnetic circuit, it can be considered that the magnetic field is homogeneous across each cross-section of the magnetic circuit. Thus, the flux through one turn is $\Phi = BS$. *Winding flux* $\Psi = N\Phi$ represents the flux through the winding with N series-connected turns.

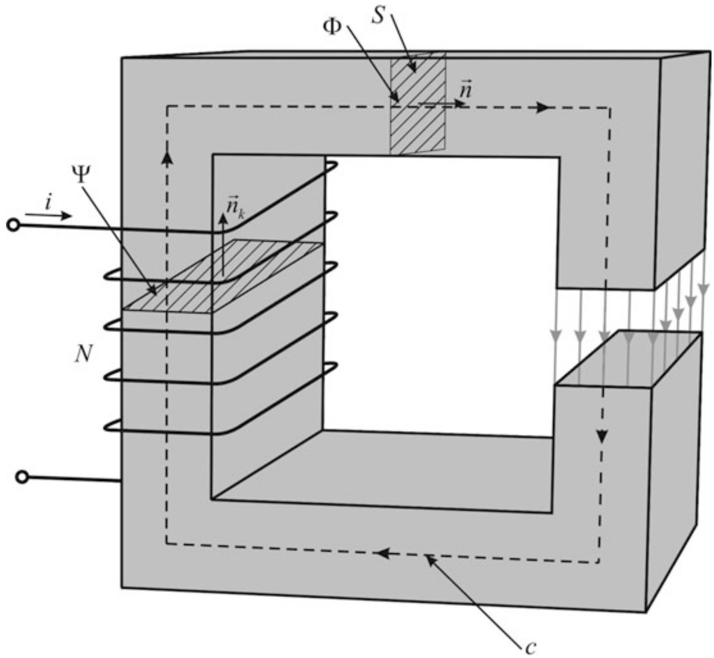


Fig. 4.4 Sample magnetic circuit with definitions of the cross-section of the core, flux of the core, flux of the winding, and representative average line of the magnetic circuit. Magnetic circuit has a large iron core with a small air gap in the right-hand side

Flux through any cross-section of the magnetic circuit is constant. Since $S = S_0 = S_{Fe}$, equality $SB_{Fe} = SB_0$ applies. Therefore, $B_{Fe} = B_0$, where B_0 is magnetic induction in the air gap while B_{Fe} is magnetic induction in the ferromagnetic material (iron). Magnetic field in the air gap is $H_0 = B_0/\mu_0$, whereas the field in the ferromagnetic material is $H_{Fe} = B_{Fe}/\mu_{\mathrm{Fe}}$. Generalized Ampere law results in (4.6), where l is average length of the ferromagnetic circuit and l_0 is length of the air gap:

$$H_{Fe} l + H_0 l_0 = Ni. \tag{4.6}$$

By inserting $H_0 = B_0/\mu_0$, $H_{Fe} = B_{Fe}/\mu_{\mathrm{Fe}} = H_{Fe} = B_0/\mu_{\mathrm{Fe}}$ in (4.6), one obtains (4.7), which gives magnetic induction $B_{Fe} = B_0$

$$\frac{B_0}{\mu_{Fe}} l + \frac{B_0}{\mu_0} l_0 = Ni = F. \tag{4.7}$$

Since flux of the core is $\Phi = BS$, its dependence on magnetomotive force F can be represented by (4.8):

$$\Phi = \frac{Ni}{\frac{l}{\mu_{Fe} S} + \frac{l_0}{\mu_0 S}} = \frac{F}{\frac{l}{\mu_{Fe} S} + \frac{l_0}{\mu_0 S}} = \frac{F}{R_\mu}. \tag{4.8}$$

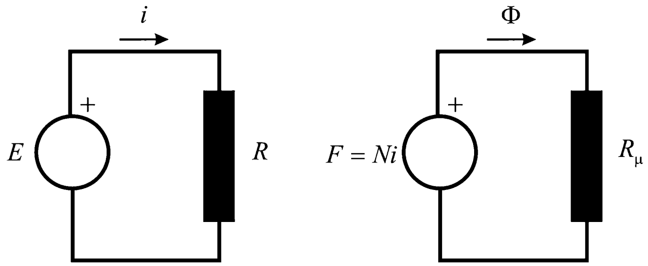


Fig. 4.5 Representation of the magnetic circuit by the equivalent electrical circuit

A dual electrical circuit can be associated with the magnetic circuit, as shown in Fig. 4.5. In this circuit, *electromotive force E* causes the current $i = E/R$ in resistance R. Electromotive force E is equal to the integral of the external electrical field in the electrical generator. In magnetic circuit, *magnetomotive force F* produces the flux $\Phi = F/R_\mu$. The flux Φ in magnetic circuit of magnetic resistance R_μ is dual to the electrical current $i = E/R$ in electrical circuit of resistance R. For this reason, the electrical circuit is an equivalent representation of the magnetic circuit and is therefore called *dual circuit*. A more detailed analysis can show that Kirchhoff laws can be applied to complex magnetic circuits in the same way they apply to electrical circuits.

Magnetomotive force F can be considered as *magnetic voltage* of the considered contour c. By analogy with electrical circuit with $i = E/R$, the flux in magnetic circuit is $\Phi = F/R_\mu$, where R_μ is *resistance of the magnetic circuit* or *reluctance*. Therefore, the flux in a magnetic circuit is obtained by dividing the magnetomotive force Ni by the magnetic resistance R_μ. This applies for linear magnetic circuits with constant permeability μ, with no magnetic leakage, and with constant core flux along the whole magnetic circuit. The last condition stems from the law of conservation of magnetic flux. Quantities Φ, F, and R_μ of a magnetic circuit are duals to quantities i, U, and R of the equivalent electrical circuit. Equation 4.8 represents "Ohm law" for magnetic circuit or Hopkins law.

Magnetic resistance of a uniform magnetic circuit of length l, constant cross-section S, and permeability μ is equal to $R_\mu = l/(S\mu)$. Magnetic circuit may consist of several segments of different dimensions and different magnetic properties. The segments of magnetic circuits are usually connected in series. The equivalent magnetic resistance of the magnetic circuit can be obtained by adding the individual resistances of series-connected segments. For a magnetic circuit with n segments, the equivalent magnetic resistance can be determined by adding resistances $R_{\mu k} = l_k/(S_k\mu_k)$, as shown in (4.9):

$$R_\mu = \sum_{k=1}^{n} \frac{l_k}{\mu_k S_k}. \tag{4.9}$$

The expression (4.9) assumes that the permeability μ does not change within the same segment, that all the segments have the same flux per cross-section, and that the lengths of magnetic lines within each segment are equal to the average length of the segment. In cases when the cross-section S and permeability μ vary continually along magnetic circuit, magnetic resistance is determined by (4.10), where c is oriented representative average line of the magnetic circuit. The cross-section $S(x)$ and permeability $\mu(x)$ are functions of variable x, which represents the path of circulation along the contour c, that is, the path along the average line of the circuit. Considering a tiny slice of the magnetic circuit having the length Δx, the cross-section $S(x)$, and permeability $\mu(x)$, it is reasonable to assume that $S(x) \approx S(x + \Delta x)$ and $\mu(x) \approx \mu(x + \Delta x)$. Therefore, magnetic resistance ΔR_μ of the considered part of magnetic circuit is equal to $\Delta x/(S\mu)$. The equivalent resistance of the magnetic circuit is obtained by adding resistances of all such parts of the magnetic circuit, resulting into integral (4.10). Equation 4.10 is in accordance with the formula for calculating resistance of a resistor with variable cross-section $S(x)$ and variable conductivity $\sigma(x)$:

$$R_\mu = \oint_c \frac{\mathrm{d}x}{\mu(x)S(x)}. \tag{4.10}$$

Magnetic resistance can be used in determining the self-inductance of the winding with N turns encircling the magnetic circuit. Inductance of the winding is equal to the ratio of the flux in the winding $\Psi = N\Phi$ and the electrical current in the winding. On the basis of (4.11), inductance of the winding is equal to the ratio of the squared number of turns and magnetic resistance:

$$L = \frac{\Psi}{i} = \frac{N\Phi}{i} = \frac{N}{i}\frac{Ni}{R_\mu} = \frac{N^2}{R_\mu} = \frac{N^2}{\sum_{i=1}^{k} \frac{l_i}{\mu_i S_i}}. \tag{4.11}$$

4.5 Energy in a Magnetic Circuit

Energy of magnetic field is determined by integration of the spatial energy density w_m within the domain where the magnetic field exists. In a linear ferromagnetic and in air, spatial density of magnetic energy is $BH/2$. In the case of a magnetic circuit with no leakage, magnetic field is present only within the circuit. Therefore, the space V where the integration (4.12) is carried out is limited to the magnetic circuit under the scope:

$$W_e = \int_V w_m \,\mathrm{d}V = \int_V \left(\int \vec{H} \cdot \mathrm{d}\vec{B} \right) \mathrm{d}V = \int_V \left(\frac{1}{2} BH \right) \mathrm{d}V. \tag{4.12}$$

Magnetic circuit can be divided into elementary volumes $\mathrm{d}V = S\mathrm{d}l$, where S is the cross-section of the magnetic circuit and $\mathrm{d}l$ is the length of the elementary volume, measured along the representative average line of the magnetic circuit (contour c). According to the flux conservation law, the flux is the same through any cross-section of the magnetic circuit. Therefore, the surface integral of magnetic induction B is equal to Φ on any cross-section of the circuit. The usual and well-founded assumption is that the magnetic field is homogeneous at every cross-section, namely, that the magnetic induction B across the cross-section does not change. Therefore, it can be concluded that magnetic induction B on each cross-section S is Φ/S. With $\mathrm{d}V = S\mathrm{d}l$, the integral (4.12) can be simplified by substituting $w_m\mathrm{d}V$ by $\frac{1}{2}\Phi H\mathrm{d}l$. The vector $\boldsymbol{H}$ is collinear with the oriented element of contour $\mathrm{d}l$. Therefore, the scalar product of the two vectors can be replaced by the product of their algebraic intensities.

$$W_e = \frac{1}{2}\int_V (BH)\mathrm{d}V = \frac{1}{2}\int_S \mathrm{d}S \oint_c BH\,\mathrm{d}l = \frac{\Phi}{2}\oint_c H\,\mathrm{d}l = \frac{\Phi}{2}\oint_c \vec{H}\cdot\mathrm{d}\vec{l}. \tag{4.13}$$

According to Ampere law, line integral of the magnetic field $\boldsymbol{H}$ along contour c which represents average line of the magnetic circuit is equal to Ni. Therefore, the expression for energy of magnetic field takes the form (4.14). It should be noted that the result (4.14) cannot be applied to magnetic circuits with nonlinear magnetic materials:

$$W_e = \frac{\Phi}{2}\oint_c \vec{H}\cdot\mathrm{d}\vec{l} = \frac{\Phi}{2}Ni = \frac{\Psi i}{2} = \frac{1}{2}Li^2. \tag{4.14}$$

Question (4.1): The magnetic circuit shown in Fig. 4.4 is made of ferromagnetic material whose permeability can be considered infinite. Determine self-inductance of the winding.

Answer (4.1): Assuming that μ is infinite, magnetic resistance of the circuit reduces to $R_\mu = l_0/(S\mu_0)$. Inductance of the winding is $L = \mu_0 S N^2/l_0$.

4.6 Reference Direction of the Magnetic Circuit

Magnetic circuit can have more than one winding around the core. Figure 4.6 shows a magnetic circuit having two windings, N_1 and N_2. *Winding flux* arises in each of the windings. The two windings are coupled by the magnetic circuit. Therefore, the flux in each winding depends on both currents, i_1 and i_2. Reference direction of the winding flux is related by the right-hand rule to the reference direction of the

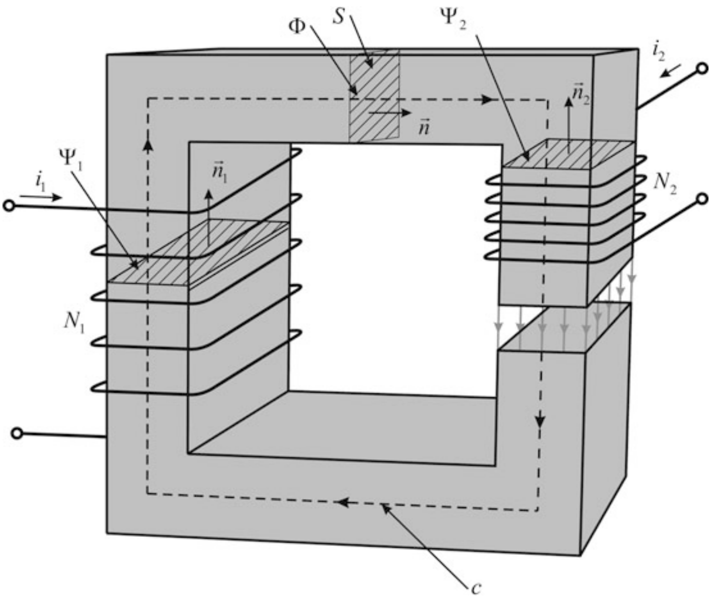


Fig. 4.6 Two coupled windings on the same core

current of the considered winding. Reference direction of flux Ψ_1 is denoted by unit vector n_1 in Fig. 4.5. This direction is in accordance with the adopted direction of circulation around contour c. The unit vector denoting this direction is the normal n. The adopted direction is called *reference direction of the magnetic circuit*. The flux intensity is determined by the product of the core flux and number of turns, thus $\Psi_1 = N_1\Phi$. The reference direction of the flux Ψ_2 in the other winding is denoted by unit vector n_2, and it is opposite to the adopted direction. Since the flux of the core Φ is defined as the flux though cross-section S in the direction of unit vector n, the flux in the second winding is negative, $\Psi_2 = -N_2\Phi$. Choice of the reference direction of magnetic circuit can be arbitrary; therefore, in the analysis of circuits having several windings, each winding should be allocated reference direction according to the right-hand rule and compared with the reference direction of the magnetic circuit.

Relations $\Psi_1 = N_1\Phi$ and $\Psi_2 = -N_2\Phi$ have been obtained under the assumption that there is no leakage of magnetic field, that is, that the flux over cross-section is maintained constant. In the absence of leakage, flux in the turns of winding N_1 is equal to the flux in the turns of winding N_2; therefore, the ratio Ψ_1/Ψ_2 is equal to N_1/N_2, the ratio of the number of turns. The same holds for the ratio e_1/e_2 between the electromotive forces induced in the windings. In real magnetic circuits, a certain amount of flux is leaking away from the magnetic circuit. A small portion of flux in winding N_1 can escape the core before arriving at winding N_2. This flux is called *stray* or *leakage flux* of the first winding. In the same manner, the leakage flux of the second winding encircles the winding N_2, but it leaks away from the core before reaching the winding N_1. In the case when the leakage flux cannot be neglected, ratio$\left|\Psi_1/\Psi_2\right|$ deflects from N_1/N_2. Strength of magnetic coupling between two windings is described by the *coefficient of inductive coupling* $k \le 1$. In the absence of leakage, the coupling coefficient is equal to 1. With $k = 0.9$, the relative amount of leakage flux is 10%.

4.7 Losses in Magnetic Circuits

The energy accumulated in the field of electromechanical converters exhibits a cyclic change. Therefore, magnetic induction in magnetic circuits varies within conversion cycles. In AC current machines and transformers, magnetic induction has a sinusoidal variation. Variations of induction B in ferromagnetic materials cause energy losses. These can be divided into eddy current losses and hysteresis losses. Power of losses per unit mass is also called *specific* power or *loss power density*.

4.7.1 Hysteresis Losses

Variation of magnetic field in a ferromagnetic material implies setting in motion magnetic dipoles and changing their orientation. Rotation of magnetic dipoles requires a certain amount of energy. This energy can be estimated from the surface of *hysteresis curve* of the $B = f(H)$ diagram. When induction B oscillates with a cycle time (period) T, as shown in Fig. 4.7, the operating point in the $B = f(t)$ diagram runs along the trajectory called *hysteresis curve*. The energy consumed by rotation of dipoles within one cycle T is proportional to the surface encircled by the hysteresis curve swept by the (B-H) operating point. The origin of hysteresis losses

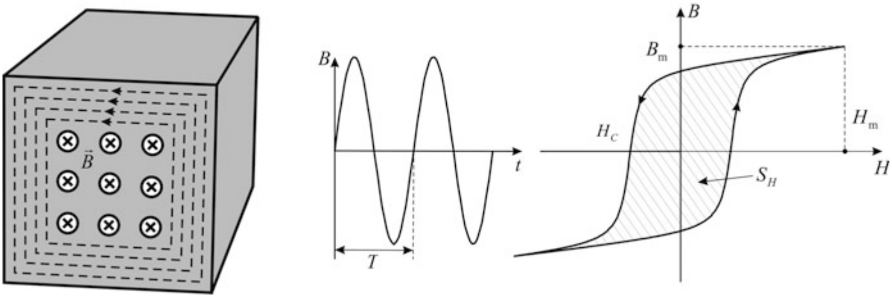


Fig. 4.7 Eddy currents in a homogeneous piece of an iron magnetic circuit (*left*). An example of the magnetization characteristic exhibiting hysteresis (*right*)

is friction between neighboring magnetic dipoles in the course of their cyclic rotation. This internal friction causes consumption of energy which is converted into heat.

Specific power losses due to hysteresis p_H are proportional to operating frequency and to surface encircled by the hysteresis curve in the B-H plane. The energy lost in each operating cycle due to hysteresis in ferromagnetic material of volume V is

$$W_H = V \oint H \, \mathrm{d}B = V \cdot S_H, \tag{4.15}$$

where S_H is surface encircled by hysteresis curve. With the operating frequency f, power loss due to hysteresis is

$$P_H = f V \cdot S_H. \tag{4.16}$$

The specific power losses, that is, losses per unit volume, are

$$p_{H1} = \frac{P_H}{V} = f S_H. \tag{4.17}$$

Surface of the hysteresis curve S_H depends on the shape of the curve and peak values of the magnetic field H_m and induction B_m. The surface is proportional to the product $B_m H_m$. The peak values H_m and B_m are in mutual proportion. Therefore, the surface S_H is also proportional to ${B_m}^2$. Therefore, the losses per unit volume can be expressed as

$$p_{H1} = \sigma_{H1} \cdot f \cdot B_m^2. \tag{4.18}$$

By introducing coefficient σ_H which is equal to the ratio of the coefficient σ_{H1} and specific mass of ferromagnetic material, specific losses due to hysteresis per unit mass are

$$p_H = \sigma_H \cdot f \cdot B_m^2. \tag{4.19}$$

4.7.2 Losses Due to Eddy Currents

Ferromagnetic materials are usually conductive. In parts of magnetic circuit that are made of conductive ferromagnetic, it is possible to envisage toroidal tubes of conductive material and to consider each of them a closed contour capable of carrying electrical currents. Variation of magnetic induction $\boldsymbol{B}$ changes the flux in such contours. As a consequence, electromotive forces are induced in such contours,

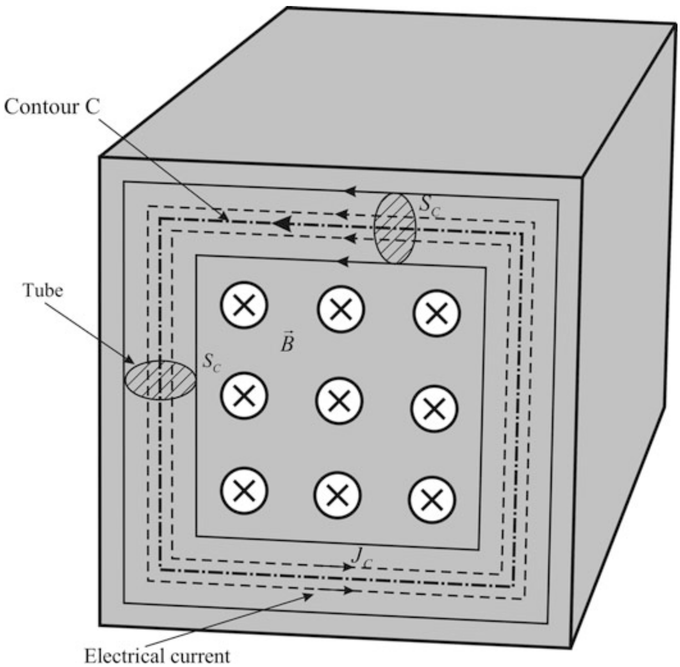


Fig. 4.8 Eddy currents cause losses in iron. The figure shows a tube containing flow of spatially distributed currents

and they produce electrical currents that oppose to the flux changes. A number of conductive contours can be identified within each piece of ferromagnetic. Therefore, the change in magnetic induction causes spatially distributed currents which contribute to losses in magnetic circuits. Such currents are also called *eddy currents*. The losses associated to such currents are called *eddy current losses*.

Figure 4.8 shows a piece of ferromagnetic material with oscillatory induction B of amplitude B_m and angular frequency ω. Lines of magnetic induction are encircled by contour C which is at the same time the average line of the tube having cross-section S_C and length l_C. Since the tube is in a ferromagnetic material of finite conductivity σ, it can be represented by a conductive contour with equivalent resistance $R_C = l_C/(S_C\sigma)$. Changes in inductance B result in flux changes. In turn, flux changes give rise to induced electromotive force in the contour

$$e = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}(-SB_m \sin \omega t) = \omega\, SB_m \cos \omega t, \tag{4.20}$$

where S is the surface encircled by the contour C in Fig. 4.8. Amplitude of the electromotive force induced in the contour is proportional to the angular frequency and magnetic induction B, hence $E \sim \omega\, B_m \sim 2\pi f B_m$. Electrical current established in the conductive contour is proportional to the electromotive force and inversely proportional to contour resistance,[1] $I \sim E/R_C \sim 2\pi f B_m/R_C$. Power losses in the contour are proportional to resistance R_C and square current I_C, as given in (4.21):

$$P_C \sim R_C I_C^2 \sim R_C \left(\frac{\omega B_m}{R_C}\right)^2 \sim \frac{\omega^2 B_m^2}{R_C}. \tag{4.21}$$

Therefore, total losses in magnetic circuit due to eddy currents are proportional to the squared angular frequency and squared magnetic induction. Specific losses due to eddy currents are

$$p_V = \sigma_V \cdot f^2 \cdot {B_m}^2, \tag{4.22}$$

where σ_V is coefficient of proportionality, dependent on the specific conductivity and specific mass of the material.

Question (4.2): The cross-section of magnetic circuit is shown in Fig. 4.8. Dimensions of these cross-sections are $L \times L = S$. Magnetic induction B is equal in all points of this cross-section and perpendicular to the surface S. Make an approximate comparison of eddy current losses at the point which is displaced from the center by $L/2$ and at the point which is displaced from the center by $L/4$.

Answer (4.2): Eddy current can be estimated by considering two contours, the larger one of radius $L/2$ and the smaller one with radius $L/4$. Specific eddy current losses, that is, the losses per unit volume, depend on the square of the induced electrical field E_i, $p_V \sim \sigma {E_i}^2$. Induced electrical field can be estimated by dividing the induced electromotive force E of the contour by the length of the contour. The contour of radius $L/2$ has four times larger surface and, therefore, four times larger flux and electromotive force E. Its length is two times larger than the length of the small contour. Therefore, induced electrical field E_i along the larger contour has twice the strength of the induced electrical field along the small contour. Finally, the eddy current losses at the point further away from the center are four times larger.

4.7.3 Total Losses in Magnetic Circuit

The sum of specific losses due to hysteresis and due to eddy currents is given by (4.23). Specific losses p_{Fe} are expressed in W/kg units. With uniform flux density B, the loss distribution in magnetic circuit is uniform as well. In this case, total magnetic field losses in a magnetic circuit of mass m are $P_{Fe} = p_{Fe} m$.

$$p_{Fe} = p_H + p_V = \sigma_H \cdot f \cdot {B_m}^2 + \sigma_V \cdot f^2 \cdot {B_m}^2. \tag{4.23}$$

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[1] Considered contour has resistance R_C and self-inductance L_C. It has an induced electromotive force E of angular frequency ω. Electrical current in the contour should be calculated by dividing the electromotive force by the contour impedance $\underline{Z}_C = R_C + j\omega L_C$. At lower frequencies where $R_C >> \omega L_C$, reactance ωL_C of the contour can be neglected.

In magnetic circuits with variable cross-section as well as in cases where the circuit comprises parts made of different materials and different properties, specific losses p_{Fe} are not the same in all parts of the circuit. Thus, total losses P_{Fe} are determined by integrating specific losses over the volume of the magnetic circuit.

4.7.4 The Methods of Reduction of Iron Losses

Power losses in magnetic circuits of electromechanical converters reduce their efficiency. In addition, the losses are eventually turned to heat, and they increase temperature of the magnetic circuit. Overheating can result in damage to the magnetic circuit or to other nearby parts of the machine. Therefore, it is necessary to transfer this heat to the environment. In other words, it is necessary to provide the means for proper cooling. Loss reduction simplifies the cooling system, increases conversion efficiency, and reduces the amount of heat passed to the environment.

In iron sheets and other ferromagnetic materials used for making magnetic circuits of electrical machines and transformers, iron losses due to eddy currents prevail over iron losses due to hysteresis. Eddy current losses are larger than hysteresis losses by an order of magnitude. The losses can be reduced by taking additional measures in designing and manufacturing magnetic circuits, thus increasing the efficiency of electrical machines and preventing their overheating.

By adding silicon and other materials of low specific conductivity into iron used for making magnetic circuits, specific conductivity of such an alloy is reduced. The increase of resistance R_C of the eddy current contours reduces the amplitude of such currents (4.21) and reduces eddy current losses.

Another approach to reducing eddy current losses is *lamination*, the process of assembling magnetic circuits out of sheets of ferromagnetic material. The sheets are oriented along direction of the magnetic field, in the way shown in Fig. 4.9. A laminar magnetic circuit is not made of solid iron, but of iron sheets which are electrically isolated from one another.

Since the sheets are parallel with magnetic field, contours of induced eddy currents are perpendicular to the field. Electrical insulation between neighboring layers prevents eddy currents; thus, they can be formed only within individual layers. It can be shown that this contributes to a considerable reduction of eddy current losses.

Iron sheets used for designing magnetic circuits of line-frequency transformers (50 or 60 Hz) and conventional electrical machines are 0.2–0.5 mm thick. Insulation between the sheets is made by inserting thin layers of insulating material (paper, lacquer) or by short-time exposure of iron sheets to an acid which forms a thin layer of nonconductive iron compound (salt).

In contemporary electrical machines used in electrical vehicles, hybrid cars, and alternative power sources, the operating frequency may be in excess of 1 kHz. Magnetic circuits of such machines are made of very thin iron sheets (0.05–0.1 mm) or of amorphous strips based on alloys of iron, manganese, and other metals, as well

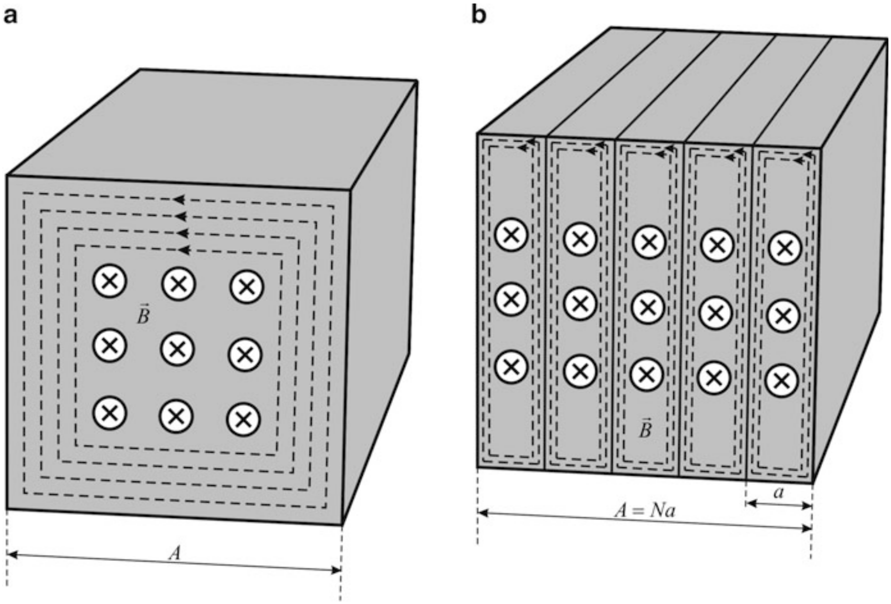

Fig. 4.9 Electrical insulation is placed between layers of magnetic circuit to prevent flow of eddy currents

as of *ferrites*. Ferrite is material obtained from molten iron alloy exposed to an increased pressure and fed to a nozzle with a very small orifice. Expanding at the mouth of the nozzle, the molten alloy is dispersed into small balls with diameter next to 50 μm. Short oxidation of these balls creates a thin layer of insulating oxide. Consequently, miniature balls fall into a cooling oil. By collecting them, one obtains a fine dust made of insulated ferromagnetic balls. Put under pressure (*sintering*), this dust becomes a hard and fragile material called ferrite. Magnetic properties of ferrites are similar to those of iron. At the same time, due to a virtual absence of eddy currents, the losses in ferrites are very low.

4.7.5 Eddy Currents in Laminated Ferromagnetics

Figure 4.10 shows one sheet of iron from the package which is used in making magnetic circuit. Thickness of the sheet is a. Magnetic induction B is directed along the sheet, and it changes in accordance with $B(t) = B_m \sin\omega t$, where B_m is amplitude and ω is angular frequency. Thickness a is very small compared to the height l of the sheet. Within the cross-section of the sheet, a contour C can be identified of width $2x$. Since $x \leq a/2$, one can assume that $x << l$. In Fig. 4.10, reference

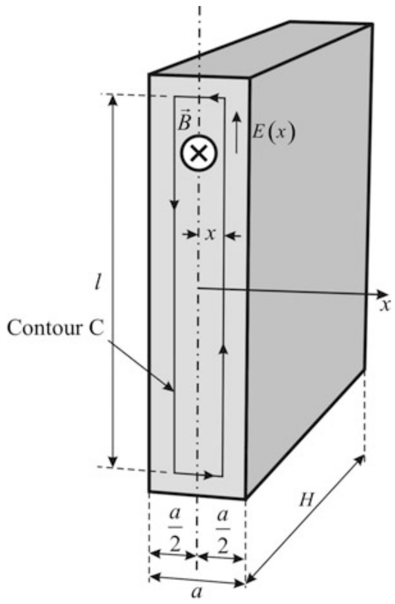

Fig. 4.10 Calculation of eddy current density within one sheet of laminated magnetic circuit

direction of contour C is opposite to the direction of the vector of magnetic induction; thus, flux through the contour is

$$\Phi = -2 \cdot x \cdot l \cdot B_m \cdot \sin \omega t.$$

The electromotive force in the contour is determined by the first derivative of the flux. Its amplitude is determined by the product of the frequency and amplitude of magnetic induction. Within the contour of the width $2x$,

$$e = \oint_C \vec{E} \cdot d\vec{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 2 \cdot x \cdot l \cdot \omega \cdot B_m \cdot \cos \omega t.$$

The sign of the electromotive force e depends on the selected reference direction. It also changes when calculating the counter-electromotive force. When calculating the eddy current losses, the choice of reference direction does not influence the result of the calculation. The losses depend on the square of eddy currents, which in turn depend on e^2. Since $x << l$, the part of the contour integral along short sides of the contour C can be neglected. Therefore, the induced electrical field E along the long sides of the contour C can be determined by (4.24):

$$e = \oint_C \vec{E} \cdot d\vec{l} = 2 \cdot l \cdot E(x) = 2 \cdot x \cdot l \cdot \omega \cdot B_m \cdot \cos \omega t,$$

$$|E(x)| = x \cdot \omega \cdot B_m \cdot \cos \omega t. \tag{4.24}$$

In ferromagnetic material of specific conductivity σ exposed to induced electrical field E, the density of spatial currents is $J = \sigma E$. In the considered sheet of iron, current density is

$$J(x) = \sigma E(x) = \sigma \cdot x \cdot \omega \cdot B_m \cdot \cos \omega t.$$

Spatial currents in material of finite conductivity give rise to power losses also called Joule losses. Specific power of these losses is equal to the product of the current density and algebraic intensity of electrical field,

$$p_{Fe}(x) = \frac{\Delta P}{\Delta V} = \sigma E^2(x) = \frac{J^2(x)}{\sigma} = \sigma \cdot (x \cdot \omega \cdot B_m \cdot \cos \omega t)^2.$$

Total losses $P1_{Fe}$ in a single sheet of ferromagnetic material of dimensions $a \times l \times H$ are obtained by spatial integration and are determined by (4.25):

$$\begin{aligned} P1_{Fe} &= \int_V p_{Fe}(x) \mathrm{d}V = 2 \cdot \int_0^{\frac{a}{2}} H \cdot l \cdot \sigma \cdot (x \cdot \omega \cdot B_m \cdot \cos \omega t)^2 \mathrm{d}x \\ &= \frac{a^3}{12} H \cdot l \cdot \sigma \cdot {B_m}^2 \cdot \omega^2 \cdot (\cos \omega t)^2 = k \cdot a^3 \cdot {B_m}^2 \cdot \omega^2. \end{aligned} \tag{4.25}$$

Coefficient k is dependent on the dimensions H and l, specific conductivity σ, and factor $\cos^2\omega t$, whose average value is 0.5. Result (4.25) can be used in the analysis of the reduction of losses due to splitting magnetic core to layers (sheets) of thickness a.

Figure 4.9a shows a homogenous piece of ferromagnetic material which could be considered as one layer of thickness $a = A$. Starting from the assumption that thickness of the considered part is considerably smaller than the height, it is possible to apply the result (4.25) and determine losses P_{hom} by (4.26):

$$P_{hom} = k \cdot {B_m}^2 \cdot \omega^2 \cdot A^3. \tag{4.26}$$

The considered part of magnetic circuit can be made to consist of N mutually insulated layers (sheets) of thickness $a = A/N$, as shown in Fig. 4.9b. If the layer of electrical insulation between the ferromagnetic sheets is considerably smaller than a, it can be assumed that the cross-section of laminated magnetic circuit is filled with iron. Therefore, magnetic resistance of laminated magnetic circuit is equal to the resistance of magnetic circuit of the same shape, made of homogenous piece of ferromagnetic material, as shown in Fig. 4.9a. Equation 4.25 gives the eddy current losses $P1_{Fe}$ in one sheet (layer), whatever the size. It has been applied (4.26) to homogeneous magnetic circuit in Fig. 4.9a, which is considered as a single sheet of iron, N times wider than the sheets shown in Fig. 4.9b, where the total number of such sheets is assumed to be N. The losses P_{lam} in laminated magnetic circuit of width $A = aN$ are determined by expression (4.27):

$$P_{lam} = N\, k \cdot B_m{}^2 \cdot \omega^2 \cdot \left(\frac{A}{N}\right)^3 = k \cdot B_m{}^2 \cdot \omega^2 \cdot A \cdot a^2 = \frac{P_{\mathrm{hom}}}{N^2}. \qquad (4.27)$$

Result (4.27) indicates that losses due to eddy currents in a part of magnetic circuit of given dimensions decrease N^2 times if the ferromagnetic material is split into N insulated layers (sheets) of equal thickness oriented along the direction of magnetic field. In cases with variable magnetic field perpendicular to the iron sheets, the lamination does not reduce the eddy current losses. In addition, lamination of a magnetic circuit does not reduce the losses due to hysteresis.

In the case of a well-positioned laminar structure of magnetic circuit, losses due to eddy currents are proportional to the squared laminar thickness a, which leads to the conclusion that one should be using iron sheets as thin as possible. Consequently, the question arises, why not use the iron sheets thinner than $0.1 \div 0.2$ mm? Thinner sheets are more difficult to cut and to assemble. At the same time, a decrease in thickness would reduce the equivalent cross-section of iron, decrease the peak flux, and increase the magnetic resistance. Namely, there is an insulating layer between the sheets, made of paper or nonconductive iron compounds. It is several tens of micrometers thick, and it exists on both sides of the sheets. Any further reduction of sheet thickness would reduce the amount of iron in the cross-section of magnetic circuit below reason.

In magnetic circuits made of solid material where eddy currents are considerable, magnetic field does not have homogeneous distribution over the cross-section. An increase in operating frequency leads to significant eddy currents which, in turn, result in uneven distribution of magnetic induction B across the cross-section of the core. Namely, eddy currents create magnetic field which opposes to variations of magnetic induction in the core. Such an effect of eddy currents is more emphasized in the middle of the core, the region which is encircled by all the eddy current contours (see Fig. 4.9a). This phenomenon results in difference between magnetic induction in the center of the core and the induction at the peripheral regions. In magnetic circuits made of iron sheets, these effects are reduced considerably, and there is no significant difference in the field intensity across the cross-section of the core.