Chapter 8
Magnetic Field in the Air Gap

This chapter presents an analysis of the magnetic and electrical fields in the air gap of a cylindrical machine. It is assumed that the fields come as a consequence of electrical current in the windings. The magnetic field in the air gap is created by the currents in both stator and rotor, which generate the corresponding stator and rotor magnetomotive forces.

Conductors of the stator winding are placed in the grooves made on the inner surface of the stator magnetic circuit, while conductors of the rotor winding are placed in the grooves made on the outer surface of the rotor magnetic circuit. The grooves are called slots, and they are opened toward the air gap (Fig. 8.1). Thus, the conductors are placed near the air gap.

It is also assumed that conductors that make up a winding are many and that they are series connected. They are not located in the same slot. Instead, the conductors are distributed along the circumference of the air gap. Conductor density can be determined by counting the number of conductors distributed along one unit length of the circumference. To begin with, it is assumed that the windings are formed with sinusoidal distribution of conductor density. Namely, the number of conductors placed in the fragment $R \cdot \Delta\theta$ of the circumference (Fig. 8.2) is determined by the function $\cos\theta$, where the angle θ determines the position of the observed fragment. When electrical currents are fed into the winding, they create a sinusoidal distributed current sheet, also called *sinusoidally distributed current sheet*. With these assumptions, the subsequent analysis determines expressions for radial and tangential components of the magnetic field in the air gap, for magnetomotive forces of the stator and rotor windings, and for fluxes per turn and the winding fluxes. The subsequent passages also introduce the notation aimed to simplify the presentation of the windings, magnetomotive forces, and fluxes. At the same time, the energy of the magnetic field in the air gap and electromagnetic torque are calculates as well, the torque being a measure of mechanical interaction between the stator and rotor. Further on, relation between the torque and machine dimensions is analyzed. Eventually, conditions for creating rotating magnetic field in the air gap are studied and specified.

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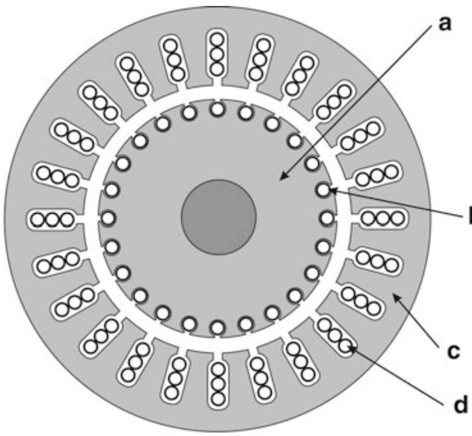


Fig. 8.1 Cross section of the magnetic circuit of an electrical machine. Rotor magnetic circuit (*a*), conductors in the rotor slots (*b*), stator magnetic circuit (*c*), and conductors in the stator slots (*d*)

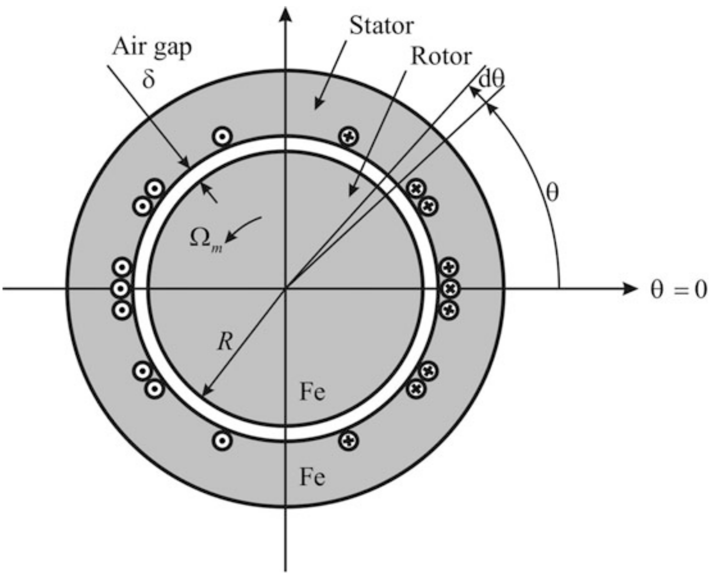


Fig. 8.2 Simplified representation of an electrical machine with cylindrical magnetic circuits made of ferromagnetic material with very large permeability. It is assumed that the conductors are positioned on the surface separating ferromagnetic material and the air gap

On the basis of the analysis of magnetomotive forces, the merits of sinusoidal spatial distribution of conductors are given a rationale. The analysis of electromotive forces in the concentrated windings and windings having periodic, non-sinusoidal spatial distribution is carried out in Chapter 10, *Electromotive Forces*.

8.1 Stator Winding with Distributed Conductors

Electrical machines are usually of cylindrical shape. An example of the cross section of a cylindrical machine is shown in Fig. 8.1. The magnetic circuit is made of iron sheets in order to reduce iron losses. The sheets forming magnetic circuits of the stator and rotor are coaxially placed, and they have shapes shown in Fig. 8.1. Stator has a form of a hollow cylinder. Rotor is a cylinder with slightly smaller diameter than the internal diameter of the stator. Distance δ between the stator and rotor is of the order of one millimeter and is called air gap. The air gap is considerably smaller than radius of the rotor cylinder R, $\delta \ll R$. The sheets are made of iron, ferromagnetic material with permeability much higher than μ_0; thus, the intensity H_{Fe} of magnetic field in iron is up to thousand times lower compared to the intensity H_0 of magnetic field in the air gap. Therefore, H_{Fe} can be neglected in most cases. Due to $\delta \ll R$, the changes of H_0 along the air gap δ can be neglected. For this reason, the value of the contour integral of magnetic field $\boldsymbol{H}$ in an electrical machine is reduced to the sum of products $H_0\delta$, also called *magnetic voltage drop* across the air gap.

Conductors of the stator and rotor are laid along the axis of the cylinder and placed next to the surface which separates the magnetic circuit and the air gap. They can be on both stator and rotor sides. Figure 8.2 shows conductors of the stator. The sign ⊗ represents a conductor carrying current away from the reader, while the sign ⊗ represents a conductor carrying current toward the reader. One pair of conductors connected in series makes up one *contour* or one *turn*. Conductors making one turn are usually positioned on the opposite sides of the cylinder, at an angular displacement of π (*diametrically positioned conductors*).

The conductors are positioned along circumference of the cylinder so that their line density (number of conductors per unit length $R{\cdot}\Delta\theta$) varies sinusoidally as function of angular displacement θ (i.e., $\cos\theta$). In cases where the function $\cos\theta$ suggests a negative number, it is understood that the number of actual conductors is positive, but direction of the current in these conductors is changed (diametrically positioned conductors are denoted by ⊗ and ⊙).

Line density of conductors in the stator winding, shown in Fig. 8.3, changes sinusoidally, and it can be modeled by function

$$N'_S(\theta) = N'_{S\,\max} \cdot \cos\,\theta \tag{8.1}$$

Function $N'_S(\theta)$ gives the number of conductors per unit length along the internal circumference of the stator magnetic circuit. If a very small segment $\mathrm{d}\theta$ is considered, the corresponding fraction of the circumference length is $\mathrm{d}l = R\,\mathrm{d}\theta$, while the number of conductors within this fraction is

$$\mathrm{d}N_S = N'_S(\theta)\,\mathrm{d}l = N'_S(\theta)R\,\mathrm{d}\theta = N'_{S\,\max} \cdot \cos\,\theta\, \cdot R\,\mathrm{d}\theta$$

In the example given in the figure, the density of conductors carrying current of direction ⊗ is the highest at $\theta = 0$, and it amounts $N'_S\,(0) = N'_{S\,\max}$. The highest

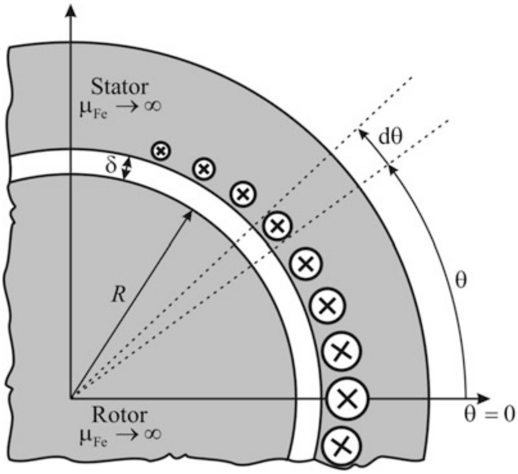


Fig. 8.3 Sinusoidal spatial distribution of conductors of the stator winding

density of conductors carrying current in the opposite direction (⊙) corresponds to position $\theta = \pi$. According to Fig. 8.2, over the interval from $\theta = -\pi/2$ up to $\theta = \pi/2$, there are conductors with reference direction ⊗, while from $\theta = \pi/2$ up to $\theta = 3\pi/2$, there are conductors with reference direction ⊙.

One pair of diametrically placed conductors (⊗ and ⊙) forms one turn or one contour. The considered winding is obtained by connecting several turns in series. The total number of *turns* N_T can be determined by counting conductors having reference direction ⊗, that is, by integrating the function $N_S{}'(\theta)$ over the span extending from $\theta = -\pi/2$ up to $\theta = \pi/2$:

$$N_T = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} N'_S(\theta)\, R\, \mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} N'_{S\,\max} \cos\,\theta\, R\, \mathrm{d}\theta$$
$$= N'_{S\,\max} R \cdot \sin\,\theta\Big|_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} = 2R \cdot N'_{S\,\max}\,. \tag{8.2}$$

Total number of *conductors* of the considered winding N_C is twice the number of turns; thus, $N_C = 2N_T = 4R\, N_S{}'_{\max}$.

The number of conductors can be obtained by calculating the integral of the function $|N_S{}'(\theta)|$ over the whole circumference of the machine, that is, over the interval starting with $\theta = 0$ and ending at $\theta = 2\pi$. This calculation implies counting all conductors, irrespective of their reference direction. Integration of the absolute value of density of conductors takes into account the conductors having reference direction from the reader ⊗ and also the conductors having reference direction toward the reader ⊙:

$$N_C = \int_0^{2\pi} \left|N'_S(\theta)\right| R\, \mathrm{d}\theta = RN'_{S\,\max} \int_0^{2\pi} |\cos\theta|\, \mathrm{d}\theta = 4RN'_{S\,\max} \tag{8.3}$$

8.2 Sinusoidal Current Sheet

Electrical current in series-connected, spatially distributed stator conductors forms a current sheet on the inner surface of the stator cylinder. Current direction from the reader ⊗ extends in the interval $-\pi/2 < \theta < \pi/2$, while the direction toward the reader ⊙ extends over the interval $\pi/2 < \theta < 3\pi/2$. Distribution of current over this surface is shown in Fig. 8.2.

The considered current sheet has the line density of surface currents dependent on the line density of conductors. The line density of the current sheet over the inner surface of the stator cylinder is denoted by $J_S(\theta)$, and it is function of the angular displacement θ. It is determined by the density of conductors $N'_S(\theta)$ and the current strength in a single conductor. Since the stator winding is formed by connecting the conductors in series, all the conductors carry the same current $i_1(t)$, also called the stator current. Current through conductors is determined by the reference direction, shown in Fig. 8.2, and algebraic intensity $i_1(t)$ of the current supplied to the winding at the two winding ends, also called *terminals*. Line density of the surface currents is determined by (8.4):

$$J_S(\theta) = N'_S(\theta) \cdot i_1 = (N'_{S\,\max} \cdot i_1) \cos\theta \tag{8.4}$$

If the maximum line current density is denoted by

$$J_{S0} = N'_{S\,\max} \cdot i_1$$

one obtains

$$J_S(\theta) = J_{S0} \cos\theta \tag{8.5}$$

Considering a small segment $d\theta$, the corresponding part of the circumference is $dl = R\, d\theta$, and the total current within this segment is

$$di = J_S(\theta) R\, d\theta$$

Electrical currents in axially placed conductors create magnetic field within the machine. By considering the boundary surface between the air gap and magnetic circuit made of iron (ferromagnetic), it can be noted that the magnetic flux entering ferromagnetic material from the air gap does not change its value; thus, the orthogonal components of magnetic induction $\boldsymbol{B}$ in the air (B_0) and the ferromagnetic material (B_{Fe}) are equal. Since permeability μ_{Fe} of the ferromagnetic material is considerably higher than permeability μ_0 of the air, it is justifiable to neglect the field H_{Fe} in the ferromagnetic material and consider that field H exists only in the air gap.

Question (8.1): In cases where current sheet density is zero, is it possible that the tangential component of the field H exists in the air next to the inner surface of the magnetic circuit of the stator?

Answer (8.1): It is necessary to consider magnetic field in the immediate vicinity of the surface separating the air gap and magnetic circuit of the stator. In the absence of electrical currents, the tangential component of the magnetic field in the air must be equal to the tangential component of the magnetic field in iron. Since permeability of iron is so high that intensity of the field H in iron can be neglected, the tangential component of the field H in iron is considered to be zero. Therefore, the tangential component of the magnetic field in the air is zero as well.

8.3 Components of Stator Magnetic Field

It is required to determine the components of the magnetic field $\boldsymbol{H}$ created in the air gap by the sheet of stator currents. The air gap is of cylindrical shape; therefore, it is convenient to adopt the cylindrical coordinate system. The unit vectors of this system, indicating the radial ($\boldsymbol{r}$), axial (z), and tangential ($\boldsymbol{\theta}$) directions, are presented in Fig. 8.4. Axis (z) is directed toward the reader (⊙). For the purpose of denoting individual components of the magnetic field, magnetic induction, and induced electrical field in the air gap, the following rules are adopted:

- Components of the field originated by the stator currents are denoted by superscript "S" ($\boldsymbol{H}^S$), while components of the field created by the rotor currents are denoted by superscript "R" ($\boldsymbol{H}^R$).
- Radial components of the field are denoted by subscript "r" ($\boldsymbol{H}_r$), tangential by subscript "θ" (H_θ), and axial by subscript "z" ($\boldsymbol{H}_z$).

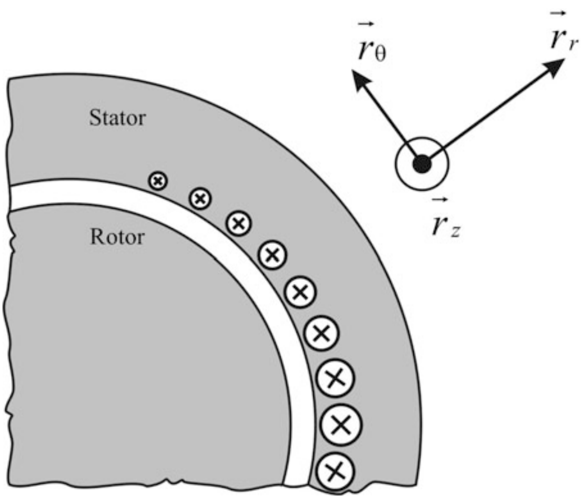


Fig. 8.4 Unit vectors of cylindrical coordinate system. Unit vectors $\boldsymbol{r}_r$, $\boldsymbol{r}_z$ and $\boldsymbol{r}_\theta$ determine the course and direction of the radial, axial, and tangential components of magnetic field

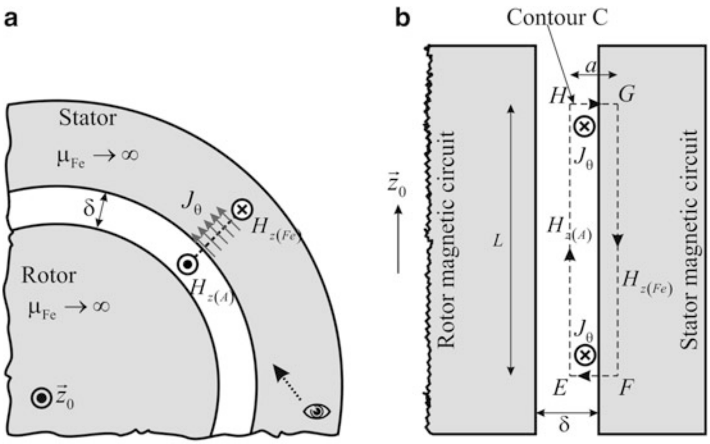


Fig. 8.5 Cross section (a) and longitudinal cross section (b) of a narrow rectangular contour C positioned along axis z. Width a of the contour EFGH is considerably smaller than its length L. Signs ⊙ and ⊗ in the left-hand part of the figure indicate reference direction of the contour and do not indicate direction of the magnetic field. Reference directions of the magnetic field are indicated in Fig. 8.2.

Thus, the radial component of the magnetic field created by the stator winding is denoted by $\boldsymbol{H}_r^S$, while the axial component of the magnetic field created by the rotor winding is denoted by $\boldsymbol{H}_z^R$.

8.3.1 Axial Component of the Field

In electrical machines having magnetic circuits of cylindrical shape and with conductors positioned in parallel with the cylinder axis, that is, axis z of the cylindrical coordinate system, axial component of magnetic field is equal to zero. This statement can be confirmed by considering Fig. 8.5.

Figure 8.5 shows the front and side views of closed rectangular contour C. It has the length L and the width a. The two longer sides of the contour are positioned along the axis z. The longer sides of the contour are denoted by ⊙ and ⊗ in the cross section of the machine, shown on the left side in Fig. 8.5. One of the two sides (⊗) passes through magnetic circuit of the stator which is made of iron. The axial component of magnetic field in iron is denoted by $H_{z(Fe)}$. The other side of the rectangular contour (⊙) passes through the air gap. The axial component of magnetic field in the air gap is denoted by $H_{z(A)}$.

In most general case, electrical machine may have electrical currents in all the three directions: radial, tangential, and axial. Tangential current would be represented by a circular path on the left side of the figure, while on the right side of the figure, their direction is from the observer into the drawing. Assuming that the machine comprises conductors with electrical currents in tangential direction and that these conductors are placed on the inner side of the stator, they can be modeled as the current sheet with line density J_θ, as shown in Fig. 8.5. Surface integral of J_θ over the surface S which is encircled by the contour C is equal to the line integral of the magnetic field along the contour. Each of the four sides of the contour makes its own contribution to the integral. In cases where the course of circulation around the contour does not correspond to the reference direction for radial and axial components of the field, then the corresponding contributions assume a negative sign:

$$\int_S \vec{J}\,\mathrm{d}\vec{S} = \int_S J_\theta\,\mathrm{d}S = \int_C \vec{H}\,\mathrm{d}\vec{l}$$

$$\int_C \vec{H}\,\mathrm{d}\vec{l} = \int_E^H H_{z(A)}\,\mathrm{d}l + \int_H^G H_r\,\mathrm{d}l - \int_G^F H_{z(Fe)}\,\mathrm{d}l - \int_F^E H_r\,\mathrm{d}l.$$

It is assumed that the contour is very long and narrow; hence, $a \ll L$. Longer sides are positioned close to the surface which separates the air gap from the stator magnetic circuit. The other two sides of the rectangle are much shorter. Therefore, the integral of the radial component of the magnetic field along sides FE and HG can be neglected; thus, the line integral along contour C is reduced to the integral along sides GF and EH:

$$\int_S J_\theta\,\mathrm{d}S = -\int_G^F H_{z(Fe)}\,\mathrm{d}l + \int_E^H H_{z(A)}\mathrm{d}l.$$

Since permeability of iron is very high and the magnetic induction in iron B_{Fe} has finite value, the magnetic field strength $H_{Fe} = B_{Fe}/\mu_{Fe}$ in iron is very low. It can be considered equal to zero. Therefore, line integral along the contour shown in Fig. 8.5 is reduced to the integral of magnetic field along side EH:

$$\int_S J_\theta\,\mathrm{d}S = \int_E^H H_{z(A)}\,\mathrm{d}l \qquad (8.6)$$

Electrical currents in rotating electrical machines exist in insulated copper conductors. These conductors are placed in slots, carved on the inner surface of the stator magnetic circuit and along the rotor cylinder. The slots extend axially, they are parallel to the axis of the cylinder and also parallel to z axis. Hence, in cylindrical electrical machines, only z component of electrical currents can exist. Thus, the density of tangential currents J_θ is equal to zero. Therefore, the value of the integral of the axial component of the magnetic field along the side EH is also zero. Under assumption that $H_{z(A)}$ remains constant, $J_\theta = 0$ proves that $H_{z(A)} = 0$. Yet, there is no proof at this point that $H_{z(A)}$ remains constant along the machine length.

The contour C can be chosen in such way that its length L is considerably smaller than the overall axial length of the machine. In such case, there are no significant variations of the field $H_{z(\mathrm{A})}$ along the side EH, and the expression (8.6) assumes the value:

$$\int_S J_\theta \, \mathrm{d}S = 0 = \int_E^H H_{z(A)} \, \mathrm{d}l \approx H_{z(A)} L \tag{8.7}$$

which leads to conclusion that $H_{z(\mathrm{A})} = 0$. There is also another way to prove that the axial component of the field is equal to zero. Statement $H_{z(\mathrm{A})} = 0$ can be proved even if the contour length L is longer and becomes comparable to the axial length of the machine. The integral in (8.6) is equal to zero for an arbitrary choice of points H and E, and this is possible only if the axial component of magnetic field in the air gap $H_{z(\mathrm{A})}$ is equal to zero at all points along the axis z. This statement can be supported by the following consideration.

The contour C (EHGF) can be slightly extended by moving the side FE into position $\mathrm{F_1E_1}$, wherein the points E and $\mathrm{E_1}$ are very close. In such way, the contour C_1 is formed, defined by the points $\mathrm{E_1HGF_1}$. In the absence of electrical currents in tangential direction (J_θ), the line integral of the field H along the contour C is equal to zero. The same holds for the contour C_1. For the reasons given above, the line integral along the contour C reduces to the integral along the side EH, while the line integral along the contour C_1 reduces to the integral along the side $\mathrm{E_1H}$. Both integrals are equal to zero. Therefore, the line integral of the field H along the side $\mathrm{EE_1}$ has to be equal to zero as well. The point $\mathrm{E_1}$ can be placed next to the point E, so that the changes in the field strength H from E to $\mathrm{E_1}$ become negligible. At this point, the line integral along the side $\mathrm{EE_1}$ reduces to the product of the path length $\mathrm{EE_1}$ and the field strength $H_{z(\mathrm{A})}$ at the point E, leading to $H_{z(\mathrm{A})} = 0$. This statement applies for arbitrary choice of points E and $\mathrm{E_1}$. This proves that the axial component of the magnetic field in the air gap is equal to zero. Notice that all the above considerations start with the assumption that the machine cylinder is very long and that the field changes at the ends of the cylinder are negligible.

Magnetic circuit of electrical machines has the stator hollow cylinder and the rotor cylinder, both made of iron sheets. At both ends of the cylinder, the air gap opens toward the outer space. Considering the windings, each turn has two diametrical conductors. The ends of these conductors have to be tied by the end turns, denoted by D in Fig. 5.6. The end turns are found at both the front and the rare side of the cylinder. Electrical current in end turns extends in tangential direction. Due to the air gap opening toward the outer space and due to end turns, there is local dispersion of the flux at both ends of the machine in the vicinity of the air gap opening. Therefore, a relatively small z component of the magnetic field may be established toward the ends of cylindrical machines. Above-described *end effects* and parasitic axial field are neglected throughout this book. It should be mentioned that the above-mentioned effects should be considered in the analysis of machines with an unusually small axial length L and with diameter $2R$ considerably larger than the axial length L.

8.3.2 Tangential Component of the Field

The analysis carried out in this subsection determines the tangential component of the magnetic field $H_\theta{}^S$ in the air gap, produced by electrical currents in the stator winding. Tangential component of the field is calculated in the air gap, next to the inner side of the stator. Namely, the observed region is close to the boundary surface separating the magnetic circuit of the stator and the air gap.

Boundary conditions for the magnetic field at the surface separating two different media are studied by electromagnetic. In the case with no electrical currents over the surface, tangential components of vector $\boldsymbol{H}$ are equal at both sides of the surface. By considering the surface separating the stator magnetic circuit and the air gap (Fig. 8.6), it can be stated that tangential component of the magnetic field in iron is equal to zero ($H_{Fe} = B_{Fe}/\mu_{Fe}$). This is due to magnetic induction B_{Fe} in iron being finite and permeability μ_{Fe} of iron being very high. Therefore, it is possible to conclude that the tangential component of magnetic field $H_\theta{}^S$ in the air, next to the inner stator surface, is equal to zero in all cases where the stator winding does not carry electrical currents.

In the example considered above, the magnetic field in the air gap is analyzed as a consequence of the stator currents. Besides these currents, the machine can also have electrical currents in rotor conductors. With the stator currents equal to zero ($J_S = 0$), the field $H_\theta{}^S$ against the inner stator surface is equal to zero, notwithstanding the rotor currents. Hence, the rotor currents do not have any influence on tangential component of the magnetic field in the air gap region next to the stator surface. Moreover, tangential components of magnetic field in the air gap are not the same against the inner surface of the stator and against the outer surface of the rotor.

It is known that in close vicinity of a plane which carries a uniform sheet of surface currents with line density σ, there is magnetic field of the strength $H = \sigma/2$, wherein the field is parallel to the plane and orthogonal to the current, while the plane resides in air or vacuum. In cases where the surface currents exist in the plane separating high-permeability ferromagnetic material and the air, the field in the air is $H = \sigma$. This statement can be proved with the help of Fig. 8.6. The figure shows the plane separating a space filled with air (left) from a space filled by ferromagnetic material (right). The boundary plane carries a uniform current sheet of line density σ. Closed contour EFGH is of the length L and width a, considerably smaller than the length. Line integral of the magnetic field along the closed rectangular contour is equal to $L\sigma$, and it sums all the currents passing through the contour. Since magnetic field in the ferromagnetic material is very low, the integral along side FG can be neglected. Because $a \ll L$, integral of the magnetic field along the closed contour is reduced to the product of side HE length and the field strength H_A. Since $L\sigma = LH_A$, it is shown that the magnetic field strength in the air is equal to the line current density σ. In the same way, it can be concluded that tangential component of the magnetic field H_θ in the air gap of a cylindrical machine in the vicinity of the inner side of the stator will be equal to the line density of stator currents, while the field H_θ

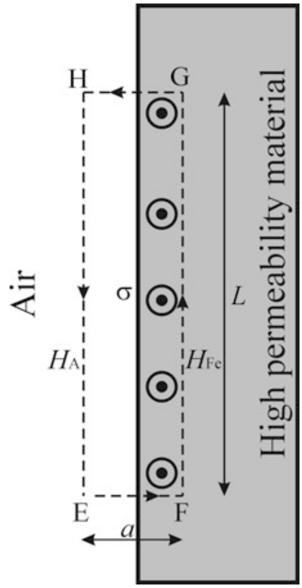


Fig. 8.6 Magnetic field strength in the vicinity of the boundary surface between the ferromagnetic material and air is equal to the line density of the surface currents

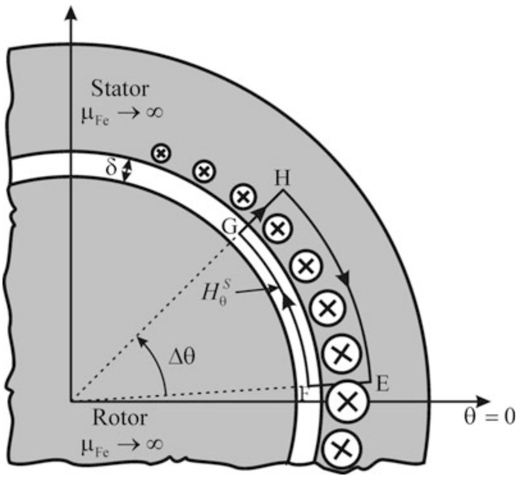


Fig. 8.7 Calculation of the tangential component of magnetic field in the air gap region next to the boundary surface between the air gap and the stator magnetic circuit

close to the rotor will be equal to the line density of rotor currents. First of the two statements will be proved by using Fig. 8.6.

It is of interest to consider the closed contour EFGH, having very short sides EF and GH, with circular arcs FG and HE having roughly the same lengths $R\Delta\theta$, where R is internal radius of the stator. Circular arc HE passes through ferromagnetic stator core, while circular arc FG passes through the air in the immediate vicinity of the stator inner surface.

Line integral of the magnetic field along closed contour EFGH is equal to the sum of all currents flowing through the surface leaning on the contour. In the considered case, there are surface currents of the stator with line density $J_S(\theta)$. If a relatively narrow segment is considered, such that $\Delta\theta \ll \pi$, it is justified to assume that the line current density $J_S(\theta)$ does not change over the arc FG, and the line integral of the magnetic field along the contour becomes

$$\int_{EFGHE} \vec{H} \cdot \mathrm{d}\vec{l} = \int_{\theta_{FE}}^{\theta_{GH}} J_S(\theta) R \,\mathrm{d}\theta \approx J_S(\theta) \cdot R \cdot \Delta\theta. \tag{8.8}$$

Since the strength of the magnetic field in iron is very small and sides EF and GH are very short, the line integral along the closed contour reduces to the integral of the component $H_\theta{}^S$ of the magnetic field in the air along the arc FG. With the assumption $\Delta\theta \ll \pi$, it is justified to consider that the field strength $H_\theta{}^S$ does not change along the considered circular arc and that the integral is

$$\int_{EFGHE} \vec{H} \cdot \mathrm{d}\vec{l} = \int_{\theta_{FE}}^{\theta_{GH}} H_\theta^S(\theta) R \,\mathrm{d}\theta \approx H_\theta^S(\theta) \cdot R \cdot \Delta\theta. \tag{8.9}$$

On the basis of expressions (8.8) and (8.9), in the region close to the inner surface of the stator, tangential component of the magnetic field in the air gap is equal to the line current density of the stator current sheet:

$$H_\theta^S(\theta) = J_S(\theta) = J_{S0} \cos\ \theta. \tag{8.10}$$

8.3.3 *Radial Component of the Field*

Calculation of radial component of the magnetic field in the air gap relies on the line integral of the field along the closed contour EFGH shown in Fig. 8.8. Side EF of the contour is positioned along radial direction at position $\theta = 0$. It starts from the stator magnetic circuit, passes through the air gap in direction opposite to the reference direction of the radial component of the field (inside-out), and ends up in the rotor magnetic circuit. Side GH is positioned radially at $\theta = \theta_1$. It starts from the rotor magnetic circuit, passes through the air gap in the reference direction of the radial component of the field, and comes back into the stator magnetic circuit. The contour has two circular arcs, FG and HE. They have approximately equal length $R\theta_1$, and they pass through magnetic circuits of the rotor (FG) and stator (HE).

Due to a very high permeability of iron, the strength $H_{Fe} = B_{Fe}/\mu_{Fe}$ of the magnetic field is negligible in these segments of the contour which pass through

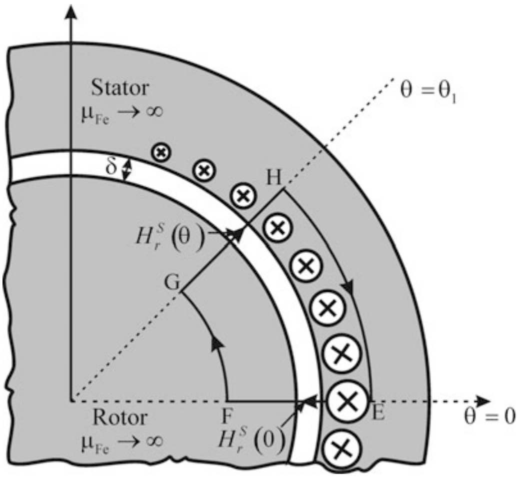


Fig. 8.8 Calculation of the radial component of magnetic field in the air gap

iron. Therefore, it can be considered that the magnetic field exists only along segments EF and GH passing through the air gap. These segments are of length δ, considerably smaller compared to the radius of the machine ($\delta \ll R$). It is thus justified to assume that intensity of the radial component of the magnetic field along sides EF and GH in the air gap does not change along this short path δ through the air gap. At position $\theta = 0$, the field strength is $H_r^S(0)$, while at position $\theta = \theta_1$, the field strength is $H_r^S(\theta_1)$. With these assumptions, line integral of the magnetic field along the contour (circulation) becomes

$$\int_C \vec{H} \cdot \mathrm{d}\vec{l} = +\delta \cdot H_r^S(\theta_1) - \delta \cdot H_r^S(0). \tag{8.11}$$

Negative sign in front of $H_r^S(0)$ in the preceding expression indicates that the direction along the side EF of the contour is opposite to the reference direction for the radial component of the magnetic field, as defined in the cylindrical coordinate system.

Circulation of vector $\boldsymbol{H}$ along the closed contour EFGH is equal to the sum of all currents passing through the surface leaning on the contour, that is, to the integral of the surface currents of line density $J_S(\theta)$ between the limits $\theta = 0$ and $\theta = \theta_1$ (8.12). By comparing Fig. 8.8 to Figs. 8.2 and 8.3, it can be concluded that the highest line density of the stator surface currents takes place at $\theta = 0$. The line density of stator currents is determined by (8.1):

$$\int_0^{\theta_1} J_S(\theta) R \,\mathrm{d}\theta = \int_0^{\theta_1} J_{S0} \cos\theta \cdot R \cdot \,\mathrm{d}\theta = R \cdot J_{S0} \cdot \sin\theta_1 \tag{8.12}$$

In position θ_1, the radial component of the air gap magnetic field caused by the stator currents is equal to

$$H_r^S(\theta_1) = H_r^S(0) + \frac{J_{S0}R}{\delta} \sin\theta_1 \tag{8.13}$$

In order to calculate radial component of the field, it is necessary to determine the constant $H_r^S(0)$.

In cases when the stator currents are absent ($J_{S0} = 0$), expression (8.13) reduces to $H_r^S(\theta) = H_r^S(0)$. With $J_{S0} = 0$, the field caused by the stator currents should be zero as well. This can be proved by the following consideration. If constant $H_r^S(0)$ is positive while $J_{S0} = 0$, radial component of the magnetic field in the air gap does not change along the machine circumference, and it is directed from rotor toward stator. On these grounds, it is possible to show that constant $H_r^S(0)$ has to be equal to zero.

In courses on Electrical Engineering Fundamentals and Electromagnetics, it is shown that the flux of the vector ***B*** which comes out of a closed surface S must be equal to zero. An example of the closed surface S can be the one enveloping the rotor of an electrical machine. This surface has three parts, cylindrical surface passing through the air gap and the two flat, round parts at both machine ends, representing the bases of the cylinder. The flux of the vector ***B*** through the surface S is called *the output flux,* and it is calculated according to

$$\oint_S \vec{B} \cdot \mathrm{d}\vec{S} = 0$$

Differential form of the preceding statement is

$$\mathrm{div}\,\vec{B} = 0,$$

and it represents one out of four Maxwell equations. Divergence is a spatial derivative of a vector which can be used for establishing the relation between the surface integral (2D) of the vector over a closed surface S and the space integral (3D) of the spatial derivative of the same vector within the domain encircled by the closed surface S. Therefore, the information on the divergence of vector ***B*** in domain V, encircled by surface S, can be used to calculate the output flux of the vector ***B***:

$$\oint_S \vec{B} \cdot \mathrm{d}\vec{S} = \int_V \mathrm{div}\,\vec{B}\ \mathrm{d}V.$$

As a consequence of div ***B*** $= 0$, the surface integral of vector ***B*** over the close surface S is equal to zero:

$$\oint_S \vec{B} \cdot \mathrm{d}\vec{S} = 0. \tag{8.14}$$

The law given by (8.14) can be used to prove that the constant $H_r^S(0)$ equals zero. It is necessary to note a closed surface S of cylindrical form, enveloping the rotor in the way that the cylindrical part S_1 passes through the air gap while the two flat round parts (basis) stay in front and at the rare of the rotor.

Equation 8.7 shows that axial component of the magnetic field H_z in electrical machines is zero. Due to $B_z = \mu_0 H_z$ in the air, the same holds for the magnetic induction; hence, $B_z = 0$. As a consequence, the flux of the vector B through the front and rear basis of the closed cylindrical surface S is equal to zero. In accordance with the law (8.14), the flux through the cylindrical surface S_1 passing through the air gap must be equal to zero as well.

Relation $B = \mu_0 H$ connects the magnetic field strength H and the magnetic induction B in the air. Since the permeability μ_0 does not vary, flux of the vector H through the cylindrical surface S_1 residing in the air gap can be obtained by dividing the flux of vector $\boldsymbol{B}$ through the same surface by the permeability μ_0. Therefore, the flux of the vector H through the same surface must be equal to zero as well as the flux of the vector $\boldsymbol{B}$. In the case when $J_{S0} = 0$ and $H_r^S(\theta) = H_r^S(0)$, the flux of the magnetic field H through the cylindrical surface S_1 is equal to $2\pi RL$ $H_r^S(0)$, where R is the radius and L is the length of the machine, which completes the proof that constant $H_r^S(0)$ in (8.13) has to be equal to zero. Having proved that $H_r^S(0) = 0$, one can obtain the expression for the radial component of the magnetic field in the air gap.

In Fig. 8.8, position θ_1 of side GH of the contour EFGH is arbitrarily chosen. Therefore, all previous considerations are applicable at any position θ_1. Thus, it can be concluded that radial component of the magnetic field created in the air gap by the stator currents is equal to

$$H_r^S(\theta) = \frac{J_{S0}R}{\delta}\sin\theta, \tag{8.15}$$

where the above expression defines the strength of the stator magnetic field H_r at the position θ within the air gap. The expression is applicable in cases where only the stator windings carry electrical currents and when these currents can be represented by surface currents with sinusoidal distribution around the machine circumference.

Question (8.2): Consider a closed surface which partially passes through the air and partially through ferromagnetic material such as iron. Is it possible to prove that the output flux of the field H though this closed surface is equal to zero? Is it possible to prove that the output flux of induction B through this closed surface is equal to zero?

Answer (8.2): According to (8.14), the output flux of the vector of magnetic induction through any closed surface S is equal to zero. This law is applicable in homogeneous media, where permeability does not change, but also in the media with variable permeability, as well as the media comprising parts of different permeability. Therefore, the output flux of magnetic induction is also equal to zero through the closed surface passing through the air in one part and through iron in the other part.

Equation 8.14 deals with magnetic induction $\boldsymbol{B}$. It is applicable to magnetic field H only in cases where the permeability $\mu = B/H$ does not change over the integration domain. Therefore, if surface S passes through media of different permeability, it cannot be stated that output flux of the vector $\boldsymbol{H}$ through a closed surface is equal to zero.

8.4 Review of Stator Magnetic Field

The subject of the preceding analysis is cylindrical electrical machine of the length L, with the rotor outer diameter $2R$. The rotor is placed in hollow, cylindrical stator magnetic circuit so that an air gap $\delta \ll R$ exists between the stator and rotor cores.

The magnetic field is created in the air gap by electrical currents in the stator winding. The stator windings have a sinusoidal distribution of their conductors along the circumference. Therefore, the stator currents can be replaced by a sheet of surface currents extending in axial direction, with a sinusoidal change of their density around the machine circumference. This current sheet is located on the inner side of the stator magnetic circuit, facing the air gap. The line density of the surface currents (8.4) is determined by the conductor density (8.1) and the electrical current i_1 in stator winding. As a consequence of the stator magnetomotive force, magnetic field is established in the air gap, with its axial, radial, and tangential components discussed above. Due to a very high permeability of iron, it is correct to assume that the magnetic field strength in iron is negligible.

In cylindrical coordinate system, the axial component of the field H in the air gap is equal to zero, while the tangential and radial components are given by expressions (8.17) and (8.18):

$$H_z^S(\theta) = 0 \tag{8.16}$$

$$H_\theta^S(\theta) = J_{S0}R \cdot \cos\theta \tag{8.17}$$

$$H_r^S(\theta) = \frac{J_{S0}R}{\delta}\sin\theta \tag{8.18}$$

Since $\delta \ll R$, the radial component is considerably higher compared to the tangential component. Difference in intensities between the radial and tangential components is up to two orders of magnitude.

Question (8.3): Consider a cylindrical machine of known dimensions having the stator winding with only one turn made out of conductors A1 and A2. Conductor A1 carries electrical current in direction away from the reader (⊗), and its position is at $\theta = 0$. The other conductor (A2) is at position $\theta = \pi$, and it carries current in direction toward the reader (⊙). Conductors A1 and A2 are connected in series, and they are fed from a current source of constant current I_0. Determine the radial component of magnetic field $H_r^S(\theta)$ in an arbitrary position θ. If the rotor revolves, what is the form of the electromotive force that would be induced in a single rotor conductor axially positioned on the surface of the rotor cylinder? What is the form of this electromotive force in cases where radial component of the stator field changes according to 8.18?

Answer (8.3): It is necessary to envisage a contour which passes through both stator and rotor magnetic circuits. This contour has to pass through the turn A1–A2, encircling one of the conductors. Such contour is passing across the air gap two times, both passages extending in radial direction. The circulation of the vector $\boldsymbol{H}$ (i.e., the line integral of $\boldsymbol{H}$ around the closed contour) is equal to the current strength I_0. Thus, the radial component of the magnetic field in the air gap is $H_m = I_0/(2\delta)$. Direction of the radial field depends on the position along the circumference. Along the first half of the circumference, starting from the conductor A1 and moving clockwise toward the conductor A2, direction of the magnetic field is from the stator toward the rotor, while in the remaining half of the circumference, direction of the field is from the rotor toward the stator. Therefore, variation of the magnetic field in the air gap can be described by the function $H_r^S(\theta) = H_m \operatorname{sgn}(\sin\theta)$. In the case when the rotor revolves at a speed Ω, position of the rotor conductor changes as $\theta = \theta_0 + \Omega t$, where θ_0 denotes the position of the rotor conductor at $t = 0$. The electromotive force induced in the conductor is $e = LvB$, where L is the length of the conductor and $v = R\Omega$ is the peripheral velocity, while $B = \mu_0 H_r^S$ is algebraic intensity of the vector $\boldsymbol{B}$ around the conductor. Therefore, the change of the electromotive force is determined by the function $H_r^S(\theta) = H_r^S(\theta_0 + \Omega t)$. In the example given above, the electromotive force would change as sgn(sin $(\theta_0 + \Omega t)$. In cases where the field $H_r^S(\theta)$ changes in a sinusoidal manner, the electromotive force induced in rotor conductors would be sinusoidal as well.

8.5 Representing Magnetic Field by Vector

The subject of the previous analysis was the magnetic field created by the stator winding. Figure 8.10 shows the lines of the radial field. The stator conductors are not shown in this figure, neither is the detailed representation of sinusoidally distributed sheet of stator currents. Instead, direction of electrical currents and position of the maximum current density are denoted by placing symbols $\otimes$ and $\odot$. The field lines shown in the figure correspond to sinusoidal change of the magnetic field H and magnetic induction B along the machine circumference, in accordance with (8.18). The regions on the inner surface of the stator magnetic circuit with the highest density of the fields B and H are denoted as the north (N) and south (S) magnetic pole. In the region of the north pole of the stator magnetic circuit, the field lines come out of the stator core and enter the air gap, while in the zone of the south magnetic pole, the field lines from the air gap enter the ferromagnetic core (Fig. 8.9).

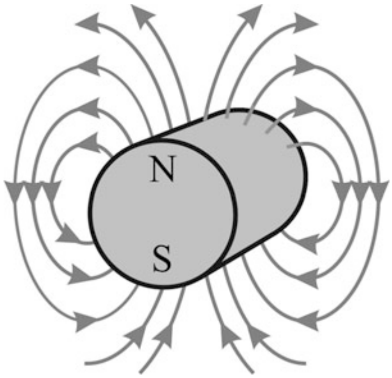


Fig. 8.9 Closed cylindrical surface *S* envelops the rotor. The lines of the magnetic field come out of the rotor (surface *S*) in the region called north magnetic pole of the rotor, and they reenter in the region called south magnetic pole

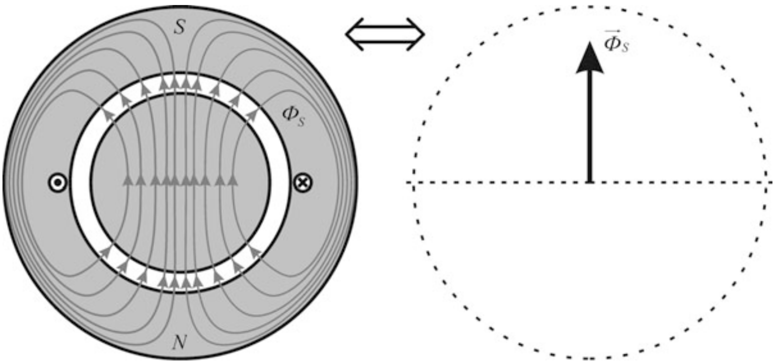


Fig. 8.10 Convention of vector representation of the magnetic field and flux

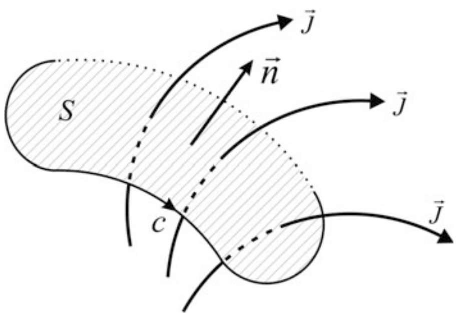
The previous analysis and Fig. 8.10 represent the magnetic field produced by only one stator winding. An electrical machine has a number of stator and rotor windings. The resulting magnetic field comes as a consequence of several magnetomotive forces. The magnetomotive force of each winding creates the field represented by the field lines similar to those in Fig. 8.10. An effort of presenting several such fields in a single drawing would be rather difficult to follow, let alone getting useful in making conclusions and design decisions.

In further analyses, the magnetic field produced by single winding can be represented in a concise way by introducing the *flux vector* of the winding. Magnetic flux is an integral of the vector ***B*** over the given surface *S*. The result of such integration is a scalar. Yet, the flux in an electrical machine is tied to the normal *n* on the surface *S*, and it depends on spatially oriented field of ***B***. Therefore, the flux is also called *directed scalar*. Considering the magnetic field created by a single turn, it is possible to calculate the flux as a scalar quantity and to define the *flux vector* by associating the course and direction to the scalar value. In Sect. 4.4, the flux in a single turn is represented by the flux vector, wherein the spatial orientation and reference direction are determined from the normal to the surface S defined by the contour C, made out by the single-turn conductors.

In most cases, a winding consists of a number of turns connected in series. All the turns may not share the same spatial orientation. Therefore, the normals on the surfaces, leaning on individual turns, may not be collinear. Hence, there is a need to clarify the course and direction of the winding flux. In cases where the winding is concentrated, all the conductors reside on only two diametrical slots, and all the turns have the same orientation. Therefore, their normals coincide and define the spatial orientation of the winding flux vector. Yet, the same approach cannot be applied in cases where the winding conductors and its turns are distributed along the machine circumference.

The flux shown in Fig. 8.10 is created by the currents in conductors that are sinusoidally distributed along the inner surface of the stator. A pair of diametrical conductors constitutes one contour, that is, a single turn. The normals on individual turns are obviously not collinear. Yet, the winding flux can be represented by a vector[1] collinear with the *winding axis*. Determination of the windings axes is

[1] Interpretation of magnetic flux as a vector can be understood as a convention and a very suitable engineering tool in the analysis of complex electromagnetic processes taking place in electrical machines. Nevertheless, magnetic flux is a scalar by definition. It may be called *directed scalar*, as it is closely related to the spatial orientation of relevant turn or winding, and it depends on the course and direction of the vector of magnetic induction. Magnetic flux Ψ can be compared to the strength I of spatially distributed electrical currents, which describe the phenomenon of moving electrical charges. The following illustration shows spatial currents passing through the surface S which is leaning on the contour c:



The vector of current density J gives direction of the current I through the contour. Its integral over surface S (the flux of spatial currents) gives the current intensity I. In the case when the vector of spatial currents J is of the same orientation at all points of surface S (homogeneous), the current intensity can be determined by the following expression:

$$I = \int_S \vec{J} \cdot \mathrm{d}\vec{S} = \int_S J \cos(\vec{J}, \vec{n})\, \mathrm{d}S = J \cos(\vec{J}, \vec{n})\, S$$

described in Sect. 5.5. A more elaborated definition of the winding axis in cases with spatially distributed conductors is presented further on.

The flux vector is determined by its course, direction, and amplitude. The vector presented in the right-hand side of Fig. 8.10 represents the field of magnetic induction B, distributed sinusoidally over the air gap and shown in the left-hand side of the figure. Direction of the flux is determined by the course of the field lines, which start from the north magnetic pole (N) of the stator, pass through the air gap, enter into the rotor magnetic circuit, then pass for the second time through the air gap, and enter into the stator magnetic circuit in the region of the south pole (S). Direction of the flux is determined by direction of the magnetic field $\boldsymbol{H}$ and induction $\boldsymbol{B}$.

In linear ferromagnetic and in the air gap, the vectors $\boldsymbol{B}$ and $\boldsymbol{H}$ have the same course and direction due to $\boldsymbol{B} = \mu\boldsymbol{H}$. Spatial distribution of the field lines representing magnetic induction $\boldsymbol{B}$ can be represented by the flux vector $\boldsymbol{\Phi}$**.** The flux amplitude Φ and the magnetomotive force F are related by $F = R_\mu \Phi$, where R_μ is magnetic resistance encountered along the flux path, that is, magnetic resistance of the magnetic circuit. It is of interest to notice that the magnetomotive force F can be represented by vector $\boldsymbol{F}$, which represents the spatial distribution of the field $\boldsymbol{H}$. Due to $\boldsymbol{B} = \mu\boldsymbol{H}$, such vector is collinear with $\boldsymbol{\Phi}$**,** while its amplitude is $F = R_\mu \Phi$, and it is equal to the circulation of the vector $\boldsymbol{H}$ along the flux path.

The amplitude of the flux vector $\boldsymbol{\Phi}_S$ is the surface integral of the vector $\boldsymbol{B}$ over the surface leaning on one turn of the stator winding. It is possible to define the

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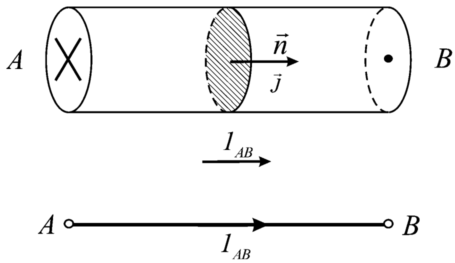
For the line conductor shown in the next figure, the unit vector of normal n on surface S represents *reference direction* of the current, or reference direction of a branch of an electrical circuit:



The sign of the current I in the section AB of the conductor corresponds to the direction of the vector J. For this reason, the current intensity I can be called *directed scalar*. By replacing the spatial current density J and the current intensity (strength) I by the magnetic induction B and magnetic flux Φ, the previous considerations can be used to establish the magnetic flux as a directed scalar. Flux vector through a contour c has direction of the normal on surface S and its algebraic intensity, determined by the integral of magnetic induction over the surface S.

vector of the total winding flux Ψ as the sum of flux vectors representing the flux in individual turns.[2]

Question (8.4): Consider Fig. 8.10, where symbols ⊗ and ⊙ denote direction of current in conductors of the stator winding. There are $2N_T$ conductors, sinusoidally distributed along the machine circumference, all of them carrying electrical current I. Derive the expression for the maximum value of the radial component H_r^S of the air gap field which is achieved in the regions of the north and south magnetic poles (use the previously obtained expressions and the relation between the maximum line density of conductors N_{Smax} and the total number of conductors, $N_T = 2RN_{Smax}$, $H = N_T I/(2\delta)$). Determine the amplitude of the stator magnetomotive force F.

Answer (8.4): It is necessary to determine the line integral of the field H along the closed contour starting from the north pole of the stator, going vertically toward the south magnetic pole, and closing through the stator magnetic circuit. Circulation of the vector H is $N_T I = 2\delta H$. Intensity of the magnetic field is $H_{max} = N_T I/(2\delta)$. Magnetomotive force F is equal to the circulation of the vector $\boldsymbol{H}$, $F = N_T I$.

Question (8.5): Assume now that the number of stator conductors does not change and that stator current is the same, but the conductors are grouped at the places designated by ⊗ and ⊙ in Fig. 8.10. Instead of being distributed, the conductors are concentrated in diametrical slots. Such winding is called *concentrated winding*. What is, in this case, the value of the line integral of the magnetic field H? Are there any changes in the maximum intensity of the field H below the north and south poles? What is the amplitude of the stator magnetomotive force F?

Answer (8.5): The magnetic field strength H_{max} and the magnetomotive force F are equal as in the preceding case, $H = N_T I /(2\delta)$, $F = N_T I$.

Question (8.6): Compare the field distribution $H(\theta)$ for concentrated and distributed winding.

Answer (8.6): On the basis of the previous expressions, magnetic field of the winding with sinusoidally distributed conductors has a sinusoidal distribution of the magnetic field in the air gap. In the case when the conductors are concentrated, radial component of magnetic field $H_r^S(\theta) = H_m\ \mathrm{sgn}(\sin\theta)$ has a constant amplitude along the circumference, and its direction is positive over one half and negative over the other half of the circumference. In both cases, maximum intensity of the field is $H = N_T I /(2\delta)$.

Question (8.7): Determine the flux through a contour made of two conductors denoted by ⊗ and ⊙ in Fig. 8.10 in the case when the winding is concentrated and has N_T conductors. All conductors of the considered winding directed toward the

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[2] Total flux Ψ of the stator winding with N turns, with sinusoidal distribution of conductors along circumference of the stator, and with flux Φ_S in one of the turns is not equal to $N\Phi_S$ because the fluxes of individual turns are not equal. Flux Φ_S in a single turn (contour) is function of position θ.

reader are in position denoted by ⊙. The remaining conductors of the opposite direction are in position denoted by ⊗.

Answer (8.7): It is necessary to note that the magnetic field strength in the air gap is $H = +N_T I/(2\delta)$ over the interval $\theta \in [0\ldots\pi]$ and $H = -N_T I/(2\delta)$ over interval $\theta \in [\pi\ldots2\pi]$. The flux through the contour is obtained by calculating the integral of the magnetic induction B over the surface leaning on the contour. Since the surface integral of magnetic induction over a closed surface is equal to zero (div $\boldsymbol{B} = 0$), the surface integral of $\boldsymbol{B}$ through all the surfaces leaning on the same contour is the same. Therefore, there is a possibility of selecting the proper surface that would facilitate the calculation. For the surface in the air gap, the expression for magnetic induction B is known. Over the interval $[0\ldots\pi]$, the magnetic induction in the air gap has radial direction and intensity $B = +\mu_0 N_T I/(2\delta)$. The surface leaning on the contour can be specified by the semicircular banded rectangle which leans on conductor ⊗, passes through the air gap over the arc interval $[0\ldots\pi]$, and leans on conductor ⊙, which is positioned at $\theta = \pi$ in Fig. 8.10. The considered surface has the length L, width πR, and surface area $S = L\pi R$. In all parts, the vector of magnetic induction is vertical to the surface; thus, the flux through the surface, that is, the flux through the contour, is equal to $\Phi = BS = \mu_0\, \pi LR\, N_T\, I\, /(2\delta)$.

Question (8.8): Determine the flux of a contour consisting of two conductors denoted by ⊗ and ⊙ in Fig. 8.8 in the case when the winding has a sinusoidal distribution of conductors.

Answer (8.8): It is necessary to note that in the zones of magnetic poles, at positions $\theta = \pi/2$ and $\theta = 3\pi/2$, the magnetic induction in the air gap is equal to the one in the preceding case ($B_{max} = +\mu_0 N_T I/(2\delta)$), but the field changes along the circumference. As in the preceding case of Question 8.7, the flux through the contour can be obtained by calculating the surface integral of the magnetic induction over the semicircular banded rectangle of the length L and width πR, which passes through the air gap and leans on conductors ⊗ and ⊙. The area of the considered surface is $S = L\pi R$. The flux cannot be calculated as $B_{max}S$, as the magnetic induction exhibits sinusoidal changes over the surface. The flux through the contour is equal to the product $B_{av}S$, where B_{av} is the average value of the magnetic induction in the air gap over the interval $\theta \in [0\ldots\pi]$. It is well known that the function $\sin(\theta)$ has an average value of $2/\pi$ on the interval $\theta \in [0\ldots\pi]$. Therefore, $B_{av} = 2/\pi B_{max}$. The flux through the contour is $\Phi = B_{av}S = \mu_0\, LR\, N_T\, I\, /\delta$.

Question (8.9): By using the results obtained in previous two questions, specify how do the magnetomotive force of the winding and the flux in one contour change by converting a concentrated winding into winding with sinusoidal distribution of conductors. Are there any reasons in favor of using distributed windings?

Answer (8.9): If the two windings have the same current in their conductors and the same number of conductors, the maximum strength H_{max} of the magnetic field in the air gap of the machine and the magnetomotive force $F = 2\delta H_{max}$ are the same.

For the concentrated winding, the field strength retains the same value along the circumference, while for the distributed winding, the field varies in accordance with $\sin(\theta)$. For this reason, the flux in one turn is smaller for the distributed winding. The ratio of the fluxes in one turn obtained in two considered cases is $2/\pi$. Even though the flux of the distributed winding is smaller, there are reasons in favor of using the windings with sinusoidally distributed conductors. It has to do with the harmonics of the induced electromotive force. With sinusoidal distribution of the conductors along the circumference, the electromotive force induced in the winding is sinusoidal, unspoiled with harmonics, and with no distortion even in cases where the change of the magnetic field along the circumference is non-sinusoidal and when the function $B(\theta)$ comprises significant amount of harmonics. In the later case, a concentrated winding will have an electromotive force waveform which resembles $B(\theta)$. Therefore, a winding with sinusoidal distribution of conductors has the properties of a filter. A proof of this statement will be presented in Chap. 10.

8.6 Components of Rotor Magnetic Field

In addition to stator windings, electrical machines usually have windings on the rotor as well. Rotor could have several windings. The following analysis will consider magnetic field produced by one rotor winding. Conductors of the considered winding are placed on the surface of the rotor magnetic circuit in the close vicinity of the air gap, in the way shown in Fig. 8.11. In this figure, the conductors directed away from the reader are denoted by $\otimes$, while the conductors directed toward the reader are denoted by $\odot$. One pair of diametrically positioned conductors creates one turn of the rotor winding. These turns are connected in series and constitute a winding.

The rotor conductors are positioned along the rotor circumference in the manner that their line density varies as a sinusoidal function of the angular displacement θ. The function $N'_R(\theta)$ determines the number of conductors per unit length $R \cdot \Delta\theta$. The argument of the function is the angle θ, measured from the reference axis of the stator, denoted by (A) in Fig. 8.11, to the place on the rotor circumference where the conductor density $N'_R(\theta)$ is observed. The angle θ_m is also marked in the figure, and it defines the rotor displacement from the reference axis of the stator. When the rotor revolves at a constant speed Ω_m, the rotor position changes as $\theta_m = \theta_0 + \Omega_m t$, where θ_0 is the initial position. The reference axis of the rotor is denoted by (B). On the rotor reference axis, the angle θ is equal to θ_m. An arbitrary position (C) is shifted by $\theta - \theta_m$ with respect to the rotor reference axis. Since the highest line density of the rotor conductors $N'_{R\max}$ is at position $\theta = \theta_m$, the sinusoidal distribution of conductors can be described by function

$$N'_R(\theta) = N'_{R\,\max} \cdot \cos(\theta - \theta_m). \tag{8.19}$$

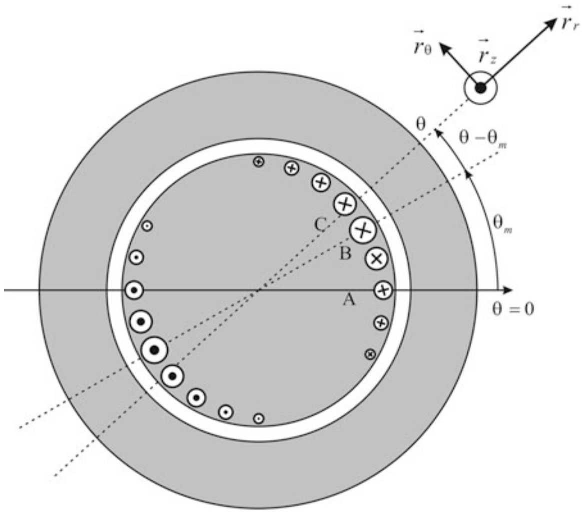

Fig. 8.11 Rotor current sheet is shifted with respect to the stator by θ_m. Maximum density of the rotor conductors is at position $\theta = \theta_m$

If the rotor winding carries electrical current i_2, the magnitude of sinusoidally distributed sheet of the rotor currents is $J_{R0} = N'_{R\max}\, i_2$.

In the case when a constant current $i_2 = I_2$ exists in the rotor conductors, the sheet of the rotor currents will create magnetic field in the air gap that would not move with respect to the rotor. The spatial orientation of such field is determined by the rotor position. By analogy with the stator field shown in Fig. 8.10, the north magnetic pole of the rotor is at position $\theta = \theta_m + \pi/2$, the radial component of the rotor field at $\theta = \theta_m$ is equal to zero, and the south magnetic pole of the rotor is at $\theta = \theta_m - \pi/2$. If the rotor does not move, position of the rotor magnetic poles does not change. When the rotor revolves, the field created by the DC current in the rotor conductors rotates with respect to the stator. The speed of the field rotation is equal to the rotor speed. In this case, position of the north magnetic pole of the rotor is $\theta = \theta_m + \pi/2 = \theta_0 + \Omega_m t + \pi/2$, where θ_0 is the rotor position at $t = 0$.

The line density of the rotor currents is given by function

$$J_R(\theta) = N'_R(\theta) \cdot i_2 = (N'_{R\,\max} \cdot i_2)\cos(\theta - \theta_m) = J_{R0}\cos(\theta - \theta_m) \qquad (8.20)$$

where $J_{R0} = N'_{R\max}\, i_2$ denotes the maximum line density of the rotor currents.

The components of the air gap magnetic field created by distributed stator winding have been analyzed in Sect. 8.3. In a like manner, it is necessary to determine the axial, tangential, and radial component of the magnetic field created in the air gap by distributed rotor winding. The air gap is cylindrical in shape; thus, it is convenient to adopt the unit vectors of the cylindrical coordinate system, the same system used in calculating the stator field. The axis (z) is oriented toward the reader ($\odot$), while the radial and tangential directions in position θ are shown in Fig. 8.11. On the basis of previously adopted notation rules, the axial, tangential, and radial components created by the rotor currents are denoted by H_z^R, H_θ^R, and H_r^R.

Question (8.10): Conductors of the stator and rotor are placed in close vicinity of the air gap. What are the negative effects of positioning the rotor conductors deeper in the rotor magnetic circuit, further away from the air gap?

Answer (8.10): The lines of the magnetic field of a single conductor placed deeper into the rotor magnetic circuit would close through the ferromagnetic material, where magnetic resistance is lower, instead of passing through the air gap and encircling the stator conductors. In cases where the rotor conductor is placed deep into the iron magnetic circuit, far away from the air gap, the rotor magnetic field and flux exist mainly in the rotor magnetic circuit and they do not extend neither to the air gap nor to the stator winding. For this reason, there is significant reduction of magnetic coupling between the rotor and stator windings. In such cases, most of the rotor flux is the leakage flux, the part of the rotor flux which does not encircle the stator windings. With the electromechanical conversion process being based on the magnetic coupling, an increased rotor leakage greatly reduces the electromagnetic torque and the conversion power. On the other hand, the rotor leakage is reduced by placing the rotor conductors in rotor slots, next to the air gap. Magnetic field of such conductors passes through the air gap and encircles conductors of the stator, contributing to the magnetic coupling between stator and rotor windings.

8.6.1 Axial Component of the Rotor Field

It is proved in Sect. 8.3 that the axial component of the magnetic field is equal to zero in cylindrical machines with axially placed conductors. Since electrical currents exist in the conductors placed along z axis of the cylindrical coordinate system, there are no currents in tangential direction. As a consequence, the axial component of the magnetic field in the air gap is equal to zero. The analysis of circulation of the field along the contour shown in Fig. 8.5 shows that the axial component of the field in the air gap is equal to zero, notwithstanding the stator and rotor currents.

8.6.2 Tangential Component of the Rotor Field

Tangential component of the rotor magnetic field $H_\theta{}^R$ is calculated in the air gap, next to the rotor magnetic circuit. The point of interest is in the air, and it resides on the boundary surface separating the rotor magnetic circuit and the air gap.

The line integral of the magnetic field along the contour shown in Fig. 8.6 helps calculating the magnetic field in the vicinity of the boundary surface between the ferromagnetic material and the air gap. The tangential component of the field is determined by the line density of the surface currents in the boundary plane. Conclusions drawn from Fig. 8.6 can be applied to determining the tangential

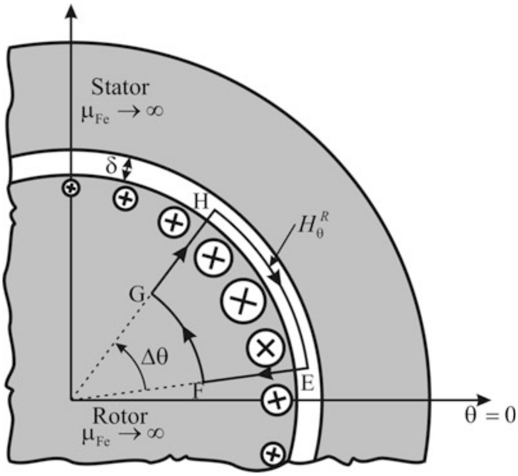

Fig. 8.12 Calculation of the tangential component of the magnetic field in the air gap due to the rotor currents, next to the rotor surface

field caused by the rotor currents. The field strength $H_\theta{}^R$ is determined by the line current density $J_R(\theta)$ of the current sheet representing the rotor currents. This statement will be proved by using the example presented in Fig. 8.12.

One should consider closed contour EFGH whose radial sides EF and GH are very short, while circular arcs FG and HE have approximately the same length $R\Delta\theta$, where R is diameter of the rotor. Circular arc FG passes through the iron part of the magnetic circuit, while circular arc HE passes through the air next to the rotor surface. Circulation of the vector of magnetic field along the closed contour EFGH is equal to the sum of all the currents passing through the surface leaning on the contour. In the considered case, there are rotor surface currents with line density $J_R(\theta)$. With $\Delta\theta \ll \pi$, it is justified to consider that the line current density does not change along the circular arc HE; thus, the line integral of the magnetic field along the closed contour is equal to the product of $J_R(\theta)$ and the length of the arc HE:

$$\oint_{EFGHE} \vec{H} \cdot \mathrm{d}\vec{l} = \int_{\theta_{EF}}^{\theta_{GH}} J_R(\theta) R \, \mathrm{d}\,\theta \approx J_R(\theta) \cdot R \cdot \Delta\theta. \tag{8.21}$$

Since the sides EF and GH are very short, while the magnetic field in iron, along the arc FG, is very low, the line integral along the closed contour is reduced to the integral of the component $H_\theta{}^R$ of the magnetic field in the air gap along the circular arc HE. With $\Delta\theta \ll \pi$, it is justified to assume that the field intensity $H_\theta{}^R$ does not change along the considered arc, and the integral is reduced to

$$\oint_{EFGHE} \vec{H} \cdot \mathrm{d}\vec{l} = \int_{H}^{E} H_\theta^R(\theta) \mathrm{d}l = \int_{\theta_{GH}}^{\theta_{FE}} H_\theta^R(\theta) R(-\mathrm{d}\theta) \approx -H_\theta^R(\theta) \cdot R \cdot \Delta\theta \tag{8.22}$$

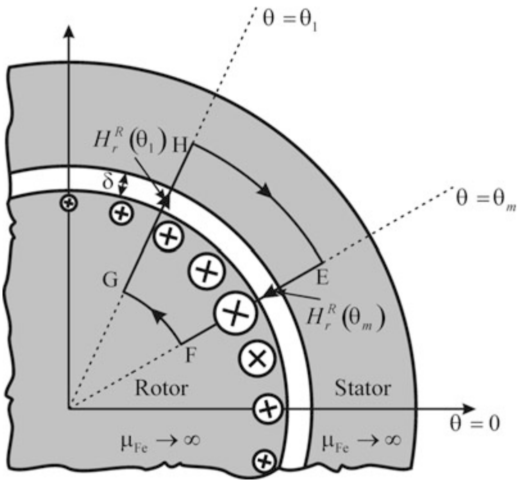

Fig. 8.13 Calculation of the radial component of the magnetic field caused by the rotor currents. Position θ_m corresponds to the rotor reference axis, while position θ_1 represents an arbitrary position where the radial component of the magnetic field is observed

Direction of the tangential component of magnetic field $H_\theta{}^R$ in the air gap, in close vicinity of the rotor surface, is opposite to the reference direction for tangential components in cylindrical coordinate system, and it is also opposite to the direction of the tangential component of the stator field. For this reason, there is a minus sign in (8.22).

On the basis of expressions (8.21) and (8.22), the component of magnetic field $H_\theta{}^R$ next to rotor surface is equal to the line density of the rotor currents:

$$H_\theta^R(\theta) = -J_R(\theta) = -J_{R0}\cos(\theta - \theta_m) \tag{8.23}$$

8.6.3 *Radial Component of the Rotor Field*

Radial component of the magnetic field in the air gap due to the rotor currents can be determined by calculating the line integral along the closed contour EFGH shown in Fig. 8.13. The side EF of the contour extends in radial direction, at position $\theta = \theta_m$, in the region with the maximum density of the rotor conductors directed toward the reader. Position θ_m represents the angular displacement of the rotor, and it is measured with respect to the stator reference axis. The side EF of the contour starts from the stator magnetic circuit, it passes through the air gap in direction opposite to the reference direction, and it ends up in the rotor magnetic circuit. The side GH is directed radially at position $\theta = \theta_1$. It starts from the magnetic circuit of the rotor, passes through the air gap in direction aligned with the reference radial direction, and it ends up in the stator magnetic circuit. The contour also comprises two circular arcs FG and HE of approximately the same length $R(\theta_1 - \theta_m)$, which pass through the magnetic circuits of the rotor and stator, respectively. Since the magnetic field strength H_{Fe} in iron is very small, it can be assumed that the magnetic field has nonzero values only along the sides EF and GH, which pass through the air gap. At the same time, the air gap length is much smaller than the machine radius ($\delta \ll R$). Therefore, it is justified to assume that the radial component of the magnetic field in the air gap does not exhibit significant changes along the sides EF and GH. With these assumptions, the circulation of the magnetic field along the contour becomes

$$\oint_C \vec{H} \cdot \mathrm{d}\vec{l} = +\delta \cdot H_r^R(\theta_1) - \delta \cdot H_r^R(\theta_m) \tag{8.24}$$

Circulation of the magnetic field along the closed contour is equal to the sum of all the currents passing through the surface encircled by the contour. In the case of the contour shown in Fig. 8.13, the sum of the currents passing through the contour is determined by calculating the integral of the line density $J_R(\theta)$ of surface currents from $\theta = \theta_m$ up to $\theta = \theta_1$:

$$\int_{\theta_m}^{\theta_1} J_R(\theta) R \mathrm{d}\theta = \int_{\theta_m}^{\theta_1} J_{R0} \cos(\theta - \theta_m) \cdot R \cdot \mathrm{d}\theta = R \cdot J_{R0} \cdot \sin(\theta_1 - \theta_m). \tag{8.25}$$

At position θ_1, the radial component of the magnetic field in the air gap caused by the rotor currents is

$$H_r^R(\theta_1) = H_r^R(\theta_m) + \frac{J_{R0} R}{\delta} \sin(\theta_1 - \theta_m) \tag{8.26}$$

For the purpose of deriving the radial component $H_r^R(\theta_1)$, it is necessary to determine the constant $H_r^R(\theta_m)$. In Sect. 8.3, where the calculation of the radial component of the stator magnetic field is carried out, it is shown that the average value of the radial component $H(\theta)$ in the air gap must be equal to zero. The proof was based on the fact that the field of the vector of magnetic induction B cannot have a nonzero flux through a closed surface, such as the cylinder enveloping the rotor. Namely, div $\boldsymbol{B} = 0$. Under circumstances, the same holds for the flux of the vector H through the cylindrical surface passing through the air gap and enveloping the rotor. Therefore, the constant $H_r^R(\theta_m)$ in (8.26) must be equal to zero. Since the position θ_1 can be arbitrarily chosen, the final expression for the radial component of the rotor magnetic field takes the form

$$H_r^R(\theta) = \frac{J_{R0} R}{\delta} \sin(\theta - \theta_m). \tag{8.27}$$

8.6.4 Survey of Components of the Rotor Magnetic Field

In the preceding section, the air gap magnetic field caused by the rotor currents is analyzed, assuming that the rotor winding has axially placed conductors, wherein the conductor density changes along the circumference as a sinusoidal function, reaching the highest density at position θ_m, also called the reference axis of the rotor. With the electrical current i_2 fed into the rotor conductors, the sheet of currents is formed on the rotor surface. The line density $J_R(\theta)$ of the surface currents exhibits the same sinusoidal change along the circumference as the density of the rotor conductors. The magnetic field is established in the air gap, while in iron, due to a very high-permeability μ_{Fe}, the magnetic field H_{Fe} is negligible. In the cylindrical coordinate system, the axial component of the field H is zero, while the tangential and radial components are determined by the expressions (8.29) and (8.30):

$$H_z^R(\theta) = 0 \tag{8.28}$$

$$H_\theta^R(\theta) = -J_R(\theta) = -J_{R0}\cos\,(\theta - \theta_m) \tag{8.29}$$

$$H_r^R(\theta) = \frac{J_{R0}R}{\delta}\,\sin(\theta - \theta_m) \tag{8.30}$$

The air gap δ is considerably smaller than radius R of the machine; thus, the radial component of the field is much higher than the tangential component.

Question (8.11): Consider a cylindrical machine of known dimensions, having the same number of conductors on the stator and the rotor. It is known that each conductor of the stator has electrical current in direction $\otimes$, while the rotor currents across the air gap have the current of the same strength but in the opposite direction ($\odot$). Determine the magnetic field in the air gap.

Answer (8.11): Since the air gap δ is very small ($\delta \ll R$), the opposite conductors of the stator and rotor are very close. Each stator conductor carrying the current in direction $\otimes$ has its counterpart across the air gap, the rotor conductor carrying the current in the opposite direction $\odot$. The distance between the two is rather small, $\delta \ll R$. For this reason, circulation of the magnetic field along the contour EFGH, shown in Fig. 8.13, gets equal to zero, as the sum of electrical currents passing through the integration contour gets zero. Therefore, the radial component of the magnetic field is equal to zero across the air gap. Regarding tangential component, it should be noted that the opposite directions of the currents in stator and rotor conductors contribute to tangential components of vector $\boldsymbol{H}$. This component is equal to the line density of the stator (or the rotor) sheet of surface currents.

8.7 Convention of Representing Magnetic Field by Vector

The subject of analysis in the preceding section was the magnetic field created by the rotor winding made out of series-connected conductors distributed sinusoidally along the rotor circumference. The left-hand part of Fig. 8.14 shows the lines of the radial field created by the rotor winding. The symbols $\otimes$ and $\odot$ indicate positions where the density of rotor conductors reaches its maximum. They also determine the reference axis of the rotor, which is perpendicular to the line $\otimes - \odot$ and which is determined by the angle θ_m. The symbols $\otimes$ and $\odot$ also indicate positions where the line density of the rotor current sheet has its maximum. The magnetic field lines shown in the figure correspond to sinusoidal change of the magnetic field H along the air gap circumference. The area of the rotor surface where the field lines exit the rotor and enter the air gap is denoted as the north (N) magnetic pole. In a like manner, the south (S) magnetic pole is defined and marked as the area where the field gets from the air gap into the rotor. In central parts of magnetic poles, the field strength H and the magnetic induction B assume their maximum values.

Magnetic field of the rotor winding can be represented in a concise way by introducing *the vector of the rotor flux*. Even though the flux is a directed scalar, it is possible to represent it as a vector by adding the course and direction to the scalar value.

A flux vector is determined by its course, direction, and amplitude. Vector $\boldsymbol{\Phi}_R$, shown in the right-hand side of Fig. 8.14, represents a sinusoidal distribution of the magnetic induction $\boldsymbol{B}$, the field lines of which are shown in the left-hand side of the figure. Direction of the flux vector is in accordance with direction of the field lines of $\boldsymbol{H}$ and $B = \mu H$. The amplitude of the flux vector $\boldsymbol{\Phi}_R$ is equal to the surface integral of the vector $\boldsymbol{B}$ over the surface leaning on one turn of the rotor winding. Therefore, the flux vector $\boldsymbol{\Phi}_R$ represents the flux in one turn. Alternatively, one can define the winding flux vector $\boldsymbol{\Psi}_R$ as the vector sum of all the fluxes in individual turns.

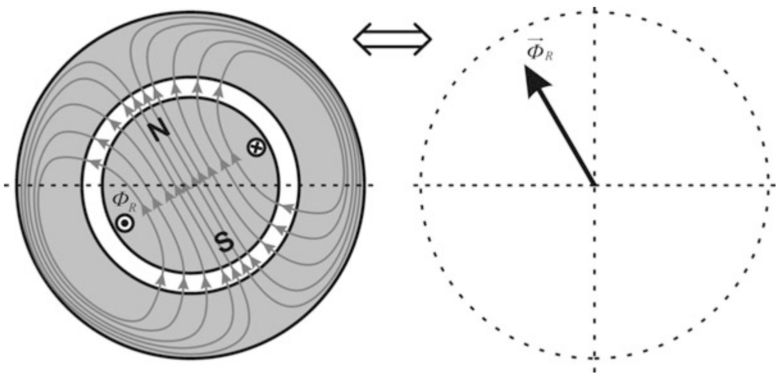

Fig. 8.14 Convention of vector representation of rotor magnetic field and flux

Question (8.12): Behold the left side of Fig. 8.14 and the two rotor conductors forming one rotor turn. Assume that these conductors are displaced several millimeters toward the stator and positioned across the air gap, on the inner surface of the stator magnetic circuit, while the electrical currents in these conductors remain the same. In the prescribed way, what used to be a rotor turn becomes a stator turn. Since the conductors denoted by $\otimes$ and $\odot$ are now on the surface of the stator magnetic circuit, the field created by the currents through these conductors becomes now the stator field. Sketch the field lines and compare them with the lines presented in the left-hand side of the figure. Denote positions of the north and south magnetic poles of the stator flux created by these conductors.

Answer (8.12): Radial component of the magnetic field in the air gap will not change by shifting the conductors. Direction of the tangential component of the field will change. Since radial component prevails over tangential component by an order of magnitude, it can be concluded that shifting the conductors will have no influence on the shape of the field lines. It is of interest to note that the north pole corresponds to the region where the magnetic field is directed from the magnetic circuit toward the air gap. In Fig. 8.14, the north pole of the stator is opposite to the south pole of the rotor.