Chapter 19
Mathematical Model of Synchronous Machine

This chapter introduces and explains mathematical model of synchronous machines. The model considers three-phase synchronous machines with excitation windings or permanent magnets on the rotor. This model does not include damper windings, which are introduced and explained in Chap. 21. The model represents transient and steady state behavior in electrical and mechanical subsystems of synchronous machines. Analysis and discussion introduce and explain Clarke and Park coordinate transforms. The model includes differential equations that express the voltage balance in stator and rotor windings, inductance matrix which relates flux linkages and currents, Newton differential equation of motion, expression for the air gap power, and expression for the electromagnetic torque. The model development process is very similar to that of the induction machine, which is detailed in Chap. 15. Therefore, some considerations are shortened or removed. The model obtained in this chapter is suitable for both isotropic (cylindrical) and anisotropic (salient pole) machines. This chapter closes with some basic considerations on the reluctant torque and synchronous reluctance machines.

19.1 Modeling Synchronous Machines

Dealing with synchronous generators and synchronous motors requires some basic knowledge on their electrical and mechanical properties. Electrical properties of machines involve steady state and transient relations between the machine voltages, currents, and flux linkages. Mechanical properties have to do with the rotor speed, electromagnetic torque, and motion resistances. For the purpose of designing the power supply and controls, it is required to know the equations and expressions that relate voltages and currents of synchronous machine. Steady state analysis relies on

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equivalent circuits that involve the machine parameters and phasors representing the relevant currents, voltages, and fluxes. Mechanical properties of synchronous machines are required to design and specify the interface to mechanical loads or driving turbines.

High-power synchronous machines are used as generators in electrical power plants. They are connected to three-phase network of constant frequency of $f = 50$ Hz or $f = 60$ Hz. Mechanical work is obtained via shaft from a steam or water turbine; it is converted into electrical energy and supplied to the network. Most synchronous generators have wound rotors. The excitation winding offers the possibility of changing the excitation flux by varying the excitation current. Changes in excitation flux affect the electromotive forces and stator voltages. This opens the possibility of controlling the voltage of the network and to change the reactive power. Synchronous machines of lower power are used in applications such as motion control, vehicle propulsion, industrial robots, and automated production machines. In these applications, synchronous machines act as motors. Their tasks include overcoming the motion resistances, driving the work machines, and controlling the speed and position of tools and work pieces. There is a tendency of designing and manufacturing medium- and low-power synchronous machines with constant excitation based on permanent magnets.

During transients, processes that take place in synchronous motors and generators, voltages, and currents are related by differential equations describing the voltage balance in the windings. The voltage balance equations describe the *electrical subsystem* of the machine. The *mechanical subsystem* is described by Newton differential equation of motion. The set of differential equations and expressions describing behavior of a synchronous machine is called *mathematical model* or *dynamic model*. In what follows, the mathematical model of synchronous machine is determined, with the aim of establishing the mechanical characteristic and the steady state equivalent circuit:

- Mechanical characteristic of synchronous machine is dependence of the electromagnetic torque on the rotor speed in steady state for given frequency and amplitude of the stator voltage.
- Steady state equivalent circuit is a network of resistances and inductances which serves for calculation of electrical currents and flux linkages of synchronous machine.

In the course of modeling, several approximations are taken, and certain phenomena of secondary importance are neglected. The iron losses are considered as negligible, and all ferromagnetic materials are assumed to be linear and free from saturation phenomena. Moreover, all the parasitic capacitances are neglected as distributed parameter effects. The model of synchronous machine will have:

- *N* differential equations expressing the voltage balance in windings
- Inductance matrix
- Expression for the torque
- Newton equation of motion

19.2 Magnetomotive Force

Most synchronous machines have the stator winding system with three-phase windings. There are also winding systems with 5-, 7-, 9-, 17-, or even more phase windings, but they are seldom used. The operation of synchronous machines with $N_{ph} > 3$ phases can be represented by an equivalent three-phase machine. In a three-phase synchronous machine with symmetrical power supply, the phase currents are

$$\begin{aligned} i_a &= I_m \cos \omega_e t, \\ i_b &= I_m \cos(\omega_e t - 2\pi/3), \\ i_c &= I_m \cos(\omega_e t - 4\pi/3). \end{aligned} \tag{19.1}$$

They have the same angular frequency and the same amplitude. Their initial phases are displaced by 2π/3. At the same time, magnetic axes of the phase windings are spatially displaced by 2π/(3*p*), where *p* is the number of pole pairs. Construction of three-phase stator windings is discussed in Chap. 14, while the aspects of machines with multiple-pole pairs are discussed in Chap. 17.

The stator phase currents create the stator magnetomotive force represented by the vector $\boldsymbol{F}_S = \boldsymbol{F}_a + \boldsymbol{F}_b + \boldsymbol{F}_c$. The phase currents given in (19.1) produce magnetomotive force which rotates at the speed $\Omega_e = \omega_e/p$, maintaining a constant amplitude of $F_S = 3/2\, NI_m$. In order to represent the vector $\boldsymbol{F}_S$, which resides in $\alpha\beta$ plane in Fig. 19.1, it is split in two components, corresponding to projections of the vector on the axes α- and β- of the orthogonal coordinate system. The $\alpha\beta$ coordinate

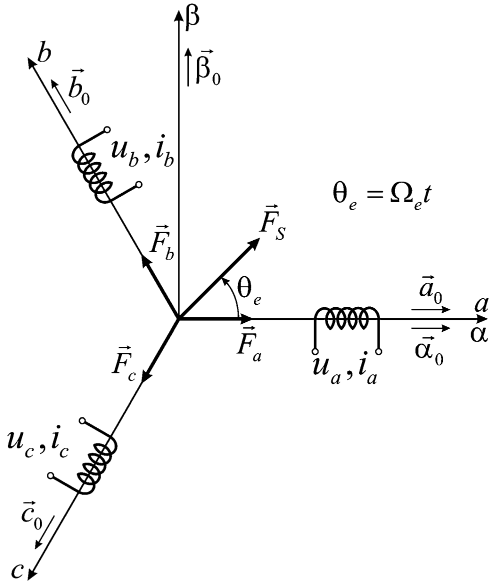

Fig. 19.1 Revolving vector of the stator magnetomotive force

system is positioned in such a way that the axis α coincides with the axis of the phase winding a. Directions of α– and β– axes are defined by unit vectors $\boldsymbol{\alpha}_0$ and $\boldsymbol{\beta}_0$. Direction of axes of the phase windings a, b, and c can be expressed in terms of these unit vectors, $\boldsymbol{a}_0 = \boldsymbol{\alpha}_0$, $\boldsymbol{b}_0 = -½\,\boldsymbol{\alpha}_0 + \boldsymbol{\beta}_0$ sqrt(3)/2, and $\boldsymbol{c}_0 = -½\,\boldsymbol{\alpha}_0 - \boldsymbol{\beta}_0$ sqrt (3)/2. Magnetomotive forces of individual phases are

$$\begin{aligned} \vec{F}_a &= Ni_a\vec{\alpha}_0, \\ \vec{F}_b &= Ni_b\left(-\frac{1}{2}\vec{\alpha}_0 + \frac{\sqrt{3}}{2}\vec{\beta}_0\right), \\ \vec{F}_c &= Ni_c\left(-\frac{1}{2}\vec{\alpha}_0 - \frac{\sqrt{3}}{2}\vec{\beta}_0\right), \end{aligned} \tag{19.2}$$

and the resultant magnetomotive force is equal to

$$\vec{F}_s = \vec{F}_a + \vec{F}_b + \vec{F}_c = \frac{3}{2}NI_m\left[\vec{\alpha}_0 \cos\omega_e t + \vec{\beta}_0 \sin\omega_e t\right] \tag{19.3}$$

The phase currents in a three-phase winding are not independent variables, and this makes the modeling more difficult. For star-connected phase windings, the phase currents are related by $i_a + i_b + i_c = 0$. Similar difficulty exist in delta-connected phase windings, where $u_a + u_b + u_c = 0$. With $i_c = -i_a - i_b$, there are only two independent currents in voltage balance equations. Therefore, it is necessary to replace i_c by $(-i_a - i_b)$ in voltage balance equations for phase c. In addition, the angle between magnetic axes of the phase windings is 2π/3. For windings with spatial displacement of π/2, their mutual inductance is equal to zero. With displacement of 2π/3, there is a nonzero mutual inductances between all the phases, and this contributes to more complex voltage balance equations. The above shortcomings can be avoided by introducing the two-phase equivalent of the three-phase machine. The two-phase equivalent machine can be made with one-phase winding oriented in direction of unit vector $\boldsymbol{\alpha}_0$ and the other-phase winding oriented in direction of unit vector $\boldsymbol{\beta}_0$. With two independent phase currents i_α and i_β and with the mutual inductance between the two orthogonal windings equal to zero, the two-phase model is readily understood.

19.3 Two-Phase Equivalent

The stator current vector $\boldsymbol{i}_S$ is determined by dividing the vector $\boldsymbol{F}_S$ by the number of turns of the stator phase winding. The vector $\boldsymbol{i}_S$ can be represented as the sum of two components which are projections of the vector on the axes of $\alpha\beta$ orthogonal coordinate frame. These components are i_α and i_β, and they can be considered as

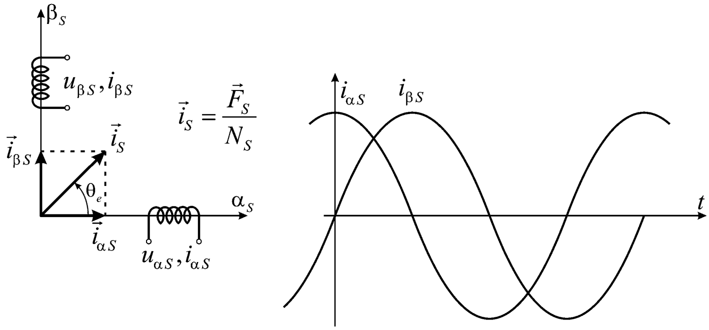

Fig. 19.2 Two-phase representation of the stator winding

the phase currents of the equivalent two-phase winding, with the phase windings aligned with $\alpha-$ and β–axes. Introduction of the two-phase equivalent makes the mathematical model more concise and intuitive. The number of phase currents is reduced to two, and the mutual inductance between the phase windings is equal to zero. The two-phase winding system with phase currents i_α and i_β produces the magnetomotive force with projections F_α and F_β. These projections are proportional to corresponding currents, namely, $F_\alpha = Ni_\alpha$ and $F_\beta = Ni_\beta$. At the same time, the flux $\Psi_{S\alpha}$ of the phase winding α is projection of the stator flux vector $\boldsymbol{\Psi}_S$ on the axis α. The absence of mutual inductance and the circumstance that the currents and flux linkages in two phases correspond to projections of relevant vectors facilitate understanding and using the two-phase mathematical model.

While introducing the two-phase equivalent, it is of interest to maintain the same flux, the same magnetic field energy, and the same torque as in original, three-phase machine. This is achieved if the amplitude and spatial orientation of the magnetomotive force $\boldsymbol{F}_S$ remain unaltered. With the same magnetomotive force, the same flux and the same energy of magnetic field are obtained. There is no unique way to maintain the same vector $\boldsymbol{F}_S$ with the two-phase equivalent. It can be obtained with a set of windings with lower currents and with more turns per phase or with another set of windings with larger currents and lower number of turns per phase.

Removal of the three-phase winding and its replacement with the two-phase winding can be carried out in such way that the number of turns remains unchanged, $N_{abc} = N_{\alpha\beta}$. Then, vector of the stator magnetomotive force $\boldsymbol{F}_S$ remains the same provided that $i_\alpha(t) = i_a(t) - i_b(t)/2 - i_c(t)/2 = 3/2\, I_m \cos\omega_e t$ and $i_\beta(t) = (\text{sqrt}(3)/2) \cdot (i_b(t) - i_c(t)) = 3/2\, I_m \sin\omega_e t$. It should be noted that in the considered case, the peak and rms values of the phase currents in the two-phase equivalent are 50% larger than corresponding currents in the three-phase winding (Fig. 19.2).

Starting from previous relations for the phase currents i_α and i_β, transformation of the three-phase variables to their two-phase equivalent variables can be written in the following matrix form:

$$\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = K_I \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} \tag{19.4}$$

Three-phase to two-phase transform (3Φ/2Φ transform) was discussed in Chap. 15. The following paragraphs are a brief reinstatement of previous considerations.

19.4 Clarke 3Φ/2Φ Transform

Three-phase to two-phase transform is known as Clarke transform, named after the author who proposed the first formulation of the transform. The matrix given in (19.4) is called *transformation matrix*. Presence of the adjustable coefficient K_I allows for a certain degree of freedom in defining the two-phase equivalent for the original three-phase machine. The number of turns of the equivalent machine does not have to be the same to that of the original machine, provided that the magnetomotive force remains unaltered. Namely, the condition of invariability of the magnetomotive force F_S can be accomplished by choosing a two-phase equivalent with $N_{\alpha\beta} = mN_{abc}$ turns. Then, the currents in the phase windings $i_\alpha(t)$ and $i_\beta(t)$ should be m times lower in order to maintain the same value of F_S. For that reason, the coefficient K_I in (19.4) has to be $K_I = 1/m$. The ratio of the peak currents of the two-phase equivalent and the three-phase original is equal to (2/3)/m.

In order to obtain the model of the two-phase machine, it is also necessary to transform the voltages and fluxes of the original machine to α-β coordinate system. Clarke transform for the voltages and fluxes is given by the following expressions:

$$\begin{bmatrix} u_\alpha \\ u_\beta \end{bmatrix} = K_U \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} \tag{19.5}$$

$$\begin{bmatrix} \Psi_\alpha \\ \Psi_\beta \end{bmatrix} = K_\Psi \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \Psi_a \\ \Psi_b \\ \Psi_c \end{bmatrix} \tag{19.6}$$

There is possibility to assign different values to coefficients K_U, K_I, and K_Ψ. For practical reasons, 3Φ/2Φ transform is adopted where all the three coefficients are equal, $K_U = K_I = K_\Psi = K$. In this way, the two-phase equivalent maintains the same ratio of currents, voltages, and fluxes as the original machine, and the 3Φ/2Φ transform is invariant in terms of impedances and inductances.[1]

Question (19.1): Is it possible to actually make a two-phase machine with $N_{\alpha\beta} = mN_{abc}$ turns which produces the same magnetomotive force $\boldsymbol{F}_S$ of the stator winding as the original three-phase synchronous machine and which has the voltages and currents (u_α, u_β, i_α, i_β) that are equal to those obtained by applying the Clarke transform to the voltages and currents of the original machine? It is assumed that transformation matrices used for the flux, voltage, and current are equal, namely, that they have the same leading coefficient ($K_U = K_I = K_\Psi$).

Answer (19.1): The same magnetomotive force F_S results in the same flux $\Phi_S = F_S/R_\mu$. At the same time, the angular speed of the revolving magnetic field has to be the same in both 2-phase and 3-phase machines. For that reason, both machines have the same electromotive force induced in one turn. With that in mind, the phase voltages of the two-phase equivalent are equal to $u_{\alpha\beta} = m\ u_{abc}$. Maintaining the amplitude of F_S requires that the phase currents of the two-phase equivalent are $i_{\alpha\beta} = (3/2)\cdot(i_{abc}/m)$. Finally, maintaining the ratio of voltages and currents requires that the ratio u_{abc}/i_{abc} is equal to the ratio $u_{\alpha\beta}\ /i_{\alpha\beta}$. Summarizing the above considerations,

$$\frac{u_{\alpha\beta}}{i_{\alpha\beta}} = \frac{m\,u_{abc}}{(3/2)(i_{abc}/m)} = \frac{2m^2}{3}\frac{u_{abc}}{i_{abc}} = \frac{u_{abc}}{i_{abc}}$$

$$\Rightarrow \quad m = \sqrt{\frac{3}{2}} \ \Rightarrow \ K_U = K_I = K_\Psi = \sqrt{\frac{2}{3}}.$$

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Solution to Question 19.1 proves the possibility to actually make the two-phase equivalent machine with the same flux, torque, and magnetomotive force and with the same impedances and inductances as the original machine. The voltages and currents of the actual two-phase machine are obtained by applying the Clarke transform to the original variables. Considered two-phase equivalent must have sqrt(3/2) times more turns than the original machine. Voltages, currents, and fluxes of the two-phase equivalent are obtained by Clarke transform of the original variables, that is, voltages, currents, and fluxes of the three-phase winding. The 3Φ/2Φ transform to be applied must have the coefficients $K_U = K_I = K_\Psi =$ sqrt(3/2). The actual two-phase machine has the same torque and power as the original machine. Therefore, besides being impedance-invariant, considered Clarke transform is also power-invariant.[2]

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[1] Invariability of impedances and inductances is discussed in Chap. 15. Impedance-invariant transform results in an equivalent machine where all the impedances are the same as relevant impedances of the original machine.

[2] Coordinate transforms do not have to be power-invariant. Generally, coordinate transforms such as Clarke transform provide another mathematical formulation of the considered dynamic system. The new model does not have to correspond to any real system. An example to that is the model of a resistor, $u_1 = Ri_1$, obtained by applying the "transform" $u_1 = 2u$, $i_1 = 2i$ to the original resistor, described by $u = Ri$. The model $u_1 = Ri_1$ is impedance-invariant, but it is not power-invariant.

If the three-phase winding is removed from the magnetic circuit of the original machine and replaced by the two-phase winding with sqrt(3/2) times more turns, one obtains a two-phase synchronous machine which replaces the three-phase machine. The voltages and currents obtained by using 3Φ/2Φ transform with K = sqrt(2/3) are equal to voltages and currents that can be measured on the actual two-phase machine windings. Hence, considered 3Φ/2Φ transform with K = sqrt (2/3) results in the two-phase equivalent machine that can be actually made. Notice at this point that the possibility of actually making the machine that comes out of the 3Φ/2Φ transform is not of particular importance.

Clarke transform with K = sqrt(2/3) is invariant in terms of the magnetomotive force, power, impedance, and inductance. In addition, it can be practically applied by making a machine with voltages and currents equal to those obtained by the transform. In spite of that, transform with coefficient K = sqrt(2/3) is seldom used. The presence of irrational number such as sqrt(2/3) in calculations is the reason for this particular transform to be rarely used.

In selecting the coordinate transform, there is freedom to choose the coefficient K so as to facilitate the use of the model. In practice, transforms with $K_U = K_I = K_\Psi = 2/3$ and with $K_U = K_I = K_\Psi = 1$ are often used. With 3Φ/2Φ transform of currents which uses $K_I = 1$, one obtains the currents i_α and i_β with peak values and rms values 50% larger than those of the original variables. Transform which results in currents i_α and i_β with peak and rms values equal to those of the phase current i_a, i_b, and i_c, it is required to apply the coefficient $K = 2/3$. This choice has two shortcomings.

Clarke transform that uses $K = 2/3$ is not power-invariant. Namely, the power calculated in $\alpha\beta$ domain is not the same as the power of the original machine. Notice at this point that the rms values of voltages and currents are the same for $\alpha\beta$ and *abc* variables. The original machine has three-phase windings, while the equivalent machine in $\alpha\beta$ domain has only two-phase windings. Therefore, calculated power is not the same since $P_{\alpha\beta} = (2/3)P_{abc}$. As a consequence, whenever using the 3Φ/2Φ transform with $K = 2/3$, the actual power of the original machine has to be calculated on the basis of the expression $P_{abc} = (3/2)\ P_{\alpha\beta}$.

It is not possible to make an actual two-phase machine that would have the same voltages and currents as those calculated by the transform. In other words, it is not possible to rewind the existing three-phase machine and make an equivalent two-phase machine with voltages and currents that correspond to the values obtained by the considered 3Φ/2Φ transform. On the other hand, the choice $K = 2/3$ results in transformed voltages and currents in $\alpha\beta$ domain that have the same amplitude and the same rms value as those of voltages and currents of the original three-phase machine. With $K = 2/3$, the virtual two-phase equivalent must have a larger number of turns. In order to maintain the amplitude of the vector $\boldsymbol{F}_S$, it is necessary to have $N_{\alpha\beta} = (3/2)N_{abc}$. With the same flux per turn $\Phi = F_S/R_\mu$, the turn electromotive forces are the same in both the original and the equivalent machine. Hence, in an attempt to make an actual two-phase equivalent, the two-phase machine has to be made with the same electromotive force per turn and with 50% more turns than the original machine. With $N_{\alpha\beta} = (3/2)N_{abc}$, the phase voltages of such a machine are $u_{\alpha\beta} = (3/2)\, u_{abc}$, 50% larger than the original phase voltage. At the same time, 3Φ/2Φ transform with $K = 2/3$ results in $u_{\alpha\beta} = u_{abc}$. This shows that the transform that uses $K = 2/3$ results in a two-phase machine that does not have a real, actual counterpart.

Favorable properties of the 3Φ/2Φ transform with coefficient $K = 2/3$ are invariability in terms of impedance and inductance, as well as the circumstance that the peak and rms values of the original machine are equal to those of the two-phase equivalent. The absence of invariability in terms of power is corrected by applying the formula $P_{abc} = (3/2)P_{\alpha\beta}$.

In further analysis, it will be assumed that the three-phase stator winding is replaced by the two-phase equivalent by applying Clarke 3Φ/2Φ transform which uses the coefficient $K = 2/3$. Unless otherwise stated, the analysis starts with an assumption that considered synchronous machine has one pair of magnetic poles ($p = 1$) and that the electrical frequency ω corresponds to mechanical speed Ω.

19.5 Inductance Matrix and Voltage Balance Equations

A synchronous machine with two-phase stator winding and wound rotor is shown in Fig. 19.3. The voltage balance equations of the considered windings are

$$u_{\alpha s} = R_s i_{\alpha s} + \frac{\mathrm{d}}{\mathrm{d}t}\Psi_{\alpha s}, \quad u_{\beta s} = R_s i_{\beta s} + \frac{\mathrm{d}}{\mathrm{d}t}\Psi_{\beta s}, \quad u_R = R_R i_R + \frac{\mathrm{d}}{\mathrm{d}t}\Psi_R. \tag{19.7}$$

Variables $u_{\alpha s}$ and $u_{\beta s}$ represent the phase voltages supplied to the two-phase stator winding, while u_p represents the voltage supplied to the excitation winding. The fluxes $\Psi_{\alpha s}$, $\Psi_{\beta s}$, and Ψ_R represent total flux linkages of the windings α_s and

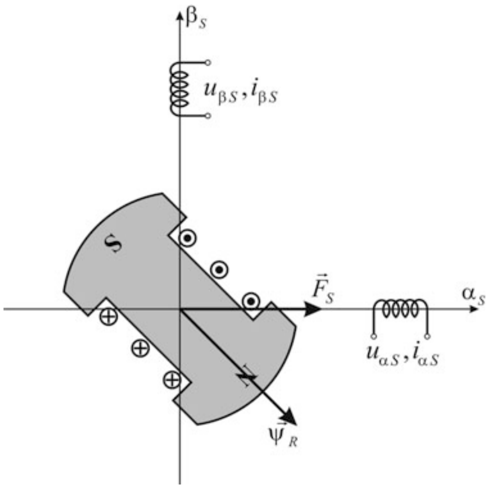

Fig. 19.3 Synchronous machine with the two-phase stator winding and the excitation winding on the rotor

β_s and of the excitation winding. Relation between currents $i_{\alpha s}$, $i_{\beta s}$, and i_R and flux linkages is given by the inductance matrix. Since the rotor is in motion, the angular displacement of the excitation winding magnetic axis with respect to α_s axis of the stator is $\theta_m = \Omega_m t$, assuming that the rotor speed Ω_m does not change. With variable θ_m, mutual inductances of the inductance matrix are variable, and the inductance matrix is a nonstationary matrix:

$$\begin{bmatrix} \Psi_{\alpha s} \\ \Psi_{\beta s} \\ \Psi_R \end{bmatrix} = \begin{bmatrix} L_s & 0 & L_m \cos\theta_m \\ 0 & L_s & L_m \sin\theta_m \\ L_m \cos\theta_m & L_m \sin\theta_m & L_R \end{bmatrix} \cdot \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \\ i_R \end{bmatrix} \tag{19.8}$$

The flux in phase α of the stator winding is equal to $\Psi_{\alpha s} = L_S i_{\alpha s} + L_m \cos(\theta_m)\, i_R$. Variable mutual inductances result in the voltage balance equations that have trigonometric functions such as $\cos(\theta_m)$, where the state variable θ_m appears as the function argument. This causes difficulties in deriving a legible steady state equivalent scheme:

$$\begin{aligned} u_{\alpha s} &= R_s i_{\alpha s} + \frac{\mathrm{d}}{\mathrm{d}t}\left(L_S i_{\alpha s} + L_m \cos(\theta_m) i_R\right) \\ &= R_s i_{\alpha s} + L_S \frac{\mathrm{d}}{\mathrm{d}t} i_{\alpha s} + L_m \cos(\theta_m) \frac{\mathrm{d}}{\mathrm{d}t} i_R - \omega_m L_m i_R \sin(\theta_m). \end{aligned}$$

In addition, considered model has the state variables such as the phase currents $i_{\alpha s}$ and $i_{\beta s}$ which have sinusoidal change even in the steady state. Therefore, their time derivatives assume nonzero values in steady state conditions, and this creates difficulties in the steady state analysis and hinders formulation of controls of the machine. The above mentioned shortcomings are removed by applying Park coordinate transform. This transform represents the machine vectors in revolving *dq* coordinate frame. Its application to synchronous machines is very much the same as the application of Park transform to induction machines, discussed in Chap. 15.

19.6 Park Transform

The model of synchronous machine in stationary α_S–β_S coordinate frame has two significant drawbacks which make its use more difficult and hinder the analysis, conclusion making, and formulation of the control law. These drawbacks are the following:

- Differential equations expressing the voltage balance comprise trigonometric functions of state variables. This affects usability of the model and results in impractical steady state equivalent circuits. Variable coefficients in voltage balance equations appear due to changes in mutual inductances, which make elements of the inductance matrix. These changes take place due to variation in relative position between the rotor, which carries the excitation winding, and the stator, which carries the stator phase windings.

- First-time derivatives of state variables have nonzero values even in steady state conditions. State variables of the machine model which is formulated in stationary α_S–β_S coordinate frame are projections of relevant vectors on α_S- and β_S-axes. At steady state, the rotor speed, the flux amplitude, the electromagnetic torque, and the conversion power are all constant. Yet, the stator phase currents, flux linkages of individual phases, their magnetomotive forces, and electromotive forces are all AC quantities. They exhibit sinusoidal change even in steady state, and their first-time derivatives are not equal to zero. This circumstance makes the analysis of steady state operation more difficult.

The model deficiencies mentioned above are removed by applying transformation of all the state variables to rotating coordinate system with orthogonal *d*- and *q*-axes. This *dq* coordinate frame revolves at the rotor speed, and it is called *synchronous coordinate frame*. In most cases, *d*-axis is made collinear with magnetic axis of the excitation winding. In permanent magnet machines, *d*-axis is made collinear with the rotor flux vector which is produced by permanent magnets. Transformation of currents, voltages, flux linkages, and magnetomotive forces into *dq* coordinate frame is called *Park transform*, and it has been explained in detail in Chap. 15.

Park transform can be conceived as a replacement of the existing stator phases α_S and β_S by imaginary, *virtual windings* residing in synchronously rotating *dq* coordinate frame, positioned in such way that the *d*-axis coincides with magnetic axis of the excitation winding. This new coordinate system rotates at the same speed as the rotor, therefore the name *synchronous coordinate frame*. The voltages, currents, and flux linkages of the excitation winding do not need to be transformed, as they reside already within the target *dq* coordinate frame. Transformation procedure and notation used hereafter are the same as the ones used in Chap. 15. The subsequent paragraphs are reduced to a brief reinstatement of the Park transform, already detailed in Chap. 15.

With $\theta_{dq} = \theta_m$, the new *dq* coordinate system revolves at the same speed as the rotor flux $\boldsymbol{\Psi}_R$. In steady state conditions, relative position between vectors $\boldsymbol{\Psi}_R$ and $\boldsymbol{F}_S$ does not change. Hence, *dq* frame revolves in synchronism with the vector of the stator magnetomotive force $\boldsymbol{F}_S$ as well. Therefore, in steady state conditions, projections of both vectors on *d*- and *q*-axes are constant. The same conclusion applies to voltage, current, and stator flux vectors. Hence, Park transform provides a set of variables in synchronously rotating *dq* frame that all have constant values in steady state conditions.

Advantage of placing the *d*-axis along magnetic axis of the excitation winding is the circumstance that all of the excitation flux extends along the *d*-axis. Namely, *q* component of the excitation flux Ψ_R is equal to zero. For the setup in Fig. 19.4, which includes virtual windings *d* and *q* and the excitation winding, mutual inductance between the virtual phase *q* and the excitation winding is equal to zero, which simplifies further considerations. In synchronous machines with

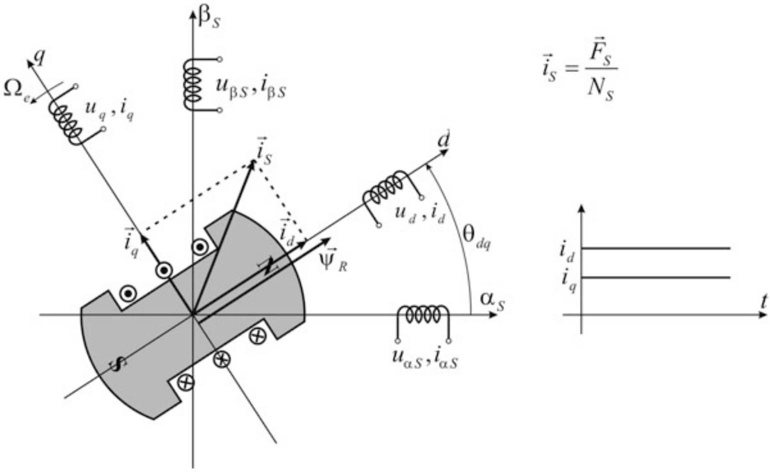

Fig. 19.4 Transformation of stator variables to a synchronously rotating coordinate system. The angle θ_{dq} is equal to the rotor angle θ_m

permanent magnets, the *d*-axis is positioned so as to coincide with direction of the rotor flux Ψ_R, the flux which is caused by permanent magnets.

By adopting the synchronously rotating coordinate system with *d*-axis aligned with the rotor magnetic axis, one obtains the system of windings shown in Fig. 19.4, with constant mutual inductances between virtual stator phases and the excitation winding and with steady state variables that assume constant values in steady state conditions. As an example, projections i_d and i_q of the stator current vector $\boldsymbol{i}_S = \boldsymbol{F}_S/N_S$ on axes of the *dq* frame are constant in the steady state, when the vector $\boldsymbol{i}_S$ maintains both the amplitude and relative position with respect to *dq*-axes. The same can be proved for all relevant state variables. By transforming stator variables from stationary $\alpha_S - \beta_S$ frame to synchronously rotating *dq* frame, one obtains the model with all the relevant state variables constant in steady state conditions. This facilitates analysis of steady state operating regimes.

The virtual stator windings *d* and *q* cannot be actually made. Instead, they serve as graphical means of grasping the effects of Park transform. One can make a thought experiment of *removing* the $\alpha_S - \beta_S$ stator windings and *mounting* new, virtual windings, with their magnetic axes collinear with the axes of *dq* coordinate frame (Fig. 19.4).

19.7 Inductance Matrix in *dq* Frame

Virtual *dq* windings do not change relative positions with respect to the excitation winding. Virtual stator phase winding of *d*-axis coincides with magnetic axis of the excitation winding. The mutual inductance $L_{d\text{-}R}$ between the two windings is equal to L_m. At the same time, mutual inductance between virtual stator phase winding of q-axis and the excitation winding is equal to zero. In addition, mutual inductance between mutually orthogonal windings d and q is also zero. Hence, the inductance matrix with $3 \times 3 = 9$ elements has only five nonzero elements. Their values L_S, L_R, and L_m are constant:

$$\begin{bmatrix} \Psi_d \\ \Psi_q \\ \Psi_R \end{bmatrix} = \begin{bmatrix} L_s & 0 & L_m \\ 0 & L_s & 0 \\ L_m & 0 & L_R \end{bmatrix} \cdot \begin{bmatrix} i_d \\ i_q \\ i_R \end{bmatrix}. \tag{19.9}$$

Currents i_d and i_q in these new, virtual windings must produce the same vector of the stator magnetomotive force $\boldsymbol{F}_S$ that was created previously with phase windings α_S and β_S. To meet this condition, currents i_d and i_q in virtual windings must assume the values $i_d = i_{\alpha s}\cos(\theta_m) + i_{\beta s}\sin(\theta_m)$ and $i_q = -i_{\alpha s}\sin(\theta_m) + i_{\beta s}\cos(\theta_m)$. Matrix form of these relations is given in (19.11). All the remaining variables,[3] such as the voltages and flux linkages, have to be transformed from $\alpha_S - \beta_S$ frame into synchronously rotating dq frame. Applying Park transform to all the relevant variables, one obtains voltages, currents, flux linkages, and magnetomotive forces in synchronously rotating dq frame. Each variable such as i_d and i_q is equal to projection of relevant vector on the axes of the dq coordinate frame. At this point, it is necessary to derive the voltage balance equations in dq frame.

The angle $\theta_{dq} = \theta_m$ between the d-axis and the α_S-axis is

$$\theta_{dq} = \theta_{dq}(0) + \int_0^t \omega_m \mathrm{d}\tau = \theta_{dq}(0) + \int_0^t p\Omega_m \mathrm{d}\tau \tag{19.10}$$

where p is the number of magnetic pole pairs. Park transform is given in expression

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = [T] \cdot \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \end{bmatrix} = \begin{bmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{bmatrix} \cdot \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \end{bmatrix} \tag{19.11}$$

For a two-pole machine, electrical representation of the rotor speed $\omega_m = p\Omega_m$ is equal to the mechanical speed Ω_m of the rotor. With multiple-pole pair machines, where $p > 1$, spatial displacement between the north and south magnetic poles becomes π/p, while the actual distance between orthogonal axes becomes $\pi/(2p)$. Therefore, Figs. 19.4 and 19.5 do not represent anymore an adequate spatial

[3] Winding currents can be treated as the state variables. In this case, flux linkages cannot be the state variables, as they are calculated from currents in (19.9). On the other hand, one can promote the flux linkages into state variables. In the latter case, winding currents are calculated from the flux linkages and therefore do not represent the state variables. The voltages across the windings are external driving forces and do not represent the state variables. In mechanical subsystem, the state variables are the speed and position of the rotor.

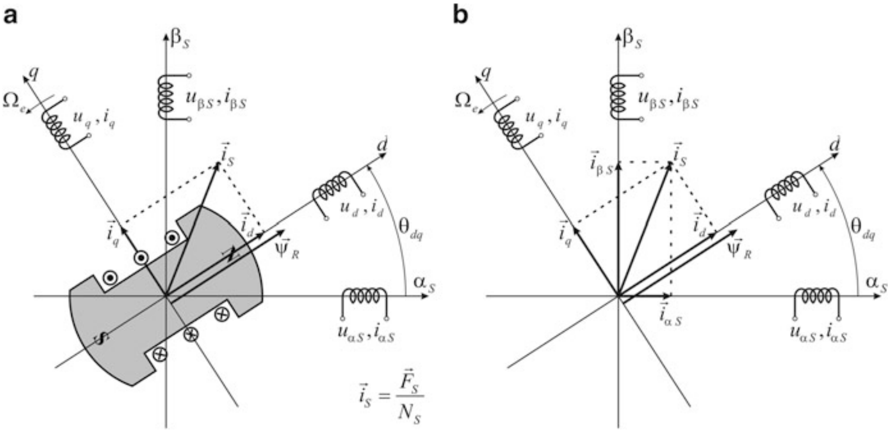

Fig. 19.5 Transformation of stator variables to a synchronously rotating coordinate system. The angle θ_{dq} is equal to the rotor angle θ_m

disposition of relevant axes and windings. In both figures, an angular displacement of 2π corresponds to the actual spatial displacement of $2\pi/p$ in synchronous machines with $2p$ magnetic poles. Equations 19.10 and 19.11 can be used to determine the angle θ_{dq} and to perform Park rotational transform in both two-pole and multipole synchronous machines.

19.8 Vectors as Complex Numbers

Vectors of the current, voltage, and flux can be represented by using complex notation, whereby projections of each vector on orthogonal axes $\alpha\beta$ or dq are represented by real and imaginary parts of a complex number, such as $\underline{i}_{dq} = i_d + \mathrm{j}i_q$, $\underline{i}_{\alpha\beta s} = i_{\alpha s} + \mathrm{j}i_{\beta s}$. Taking into account that $\mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$, Park transform of stator currents from $\alpha\beta$ to dq coordinate frame can be represented by expression $\underline{i}_{dq} = \mathrm{e}^{-\mathrm{j}\theta}\, \underline{i}_{\alpha\beta s}$. Using complex numbers to represent vectors simplifies the process of deriving the voltage balance equations in dq coordinate frame. By introducing complex notation in representing the stator voltage $\underline{u}_{\alpha\beta s} = u_{\alpha s} + \mathrm{j}u_{\beta s}$, the voltage balance equations for the stator winding can be represented by a single equations (19.12). This equation employs complex numbers $\underline{u}_{\alpha\beta s}$, $\underline{i}_{\alpha\beta s}$, and $\underline{\Psi}_{\alpha\beta s}$:

$$u_{\alpha s} = R_s i_{\alpha s} + \frac{\mathrm{d}}{\mathrm{d}t}\Psi_{\alpha s}, \quad u_{\beta s} = R_s i_{\beta s} + \frac{\mathrm{d}}{\mathrm{d}t}\Psi_{\beta s},$$
$$\Rightarrow \quad \underline{u}_{\alpha\beta s} = R_s \underline{i}_{\alpha\beta s} + \frac{\mathrm{d}}{\mathrm{d}t}\underline{\Psi}_{\alpha\beta s}. \tag{19.12}$$

Voltages, flux linkages, and currents in synchronous, *dq* coordinate frame can be represented by complex numbers. Real part of a complex number corresponds to projection of relevant vector on *d*-axis, while imaginary part corresponds to projection of relevant vector on *q*-axis. By doing so, *dq* plane is interpreted as a complex plane with real *d*-axis and imaginary *q*-axis:

$$\underline{i}_{dq} = i_d + \mathrm{j}i_q$$

It is also possible to define another complex plane, with real α-axis and imaginary β-axis. The voltages, flux linkages, and currents in stationary $\alpha\beta$ coordinate frame can be represented by complex numbers defined in $\alpha\beta$ plane. Real and imaginary parts of a complex number correspond to projection of relevant vector on α− and β−axes:

$$\underline{i}_{\alpha\beta} = i_{\alpha s} + \mathrm{j}i_{\beta s}$$

Park transform of stator currents from $\alpha\beta$ frame into *dq* frame is written as

$$\underline{i}_{dq} = i_d + \mathrm{j}i_q = e^{-j\theta_m}(i_{\alpha s} + \mathrm{j}i_{\beta s}) \tag{19.13}$$

Relations between complex representations of voltages and flux linkages in *dq* and $\alpha\beta$ coordinate frames are

$$\begin{aligned} \underline{u}_{dq} &= e^{-\mathrm{j}\theta_m} \cdot \underline{u}_{\alpha\beta s}, \\ \underline{\Psi}_{dq} &= e^{-\mathrm{j}\theta_m} \cdot \underline{\Psi}_{\alpha\beta s}. \end{aligned} \tag{19.14}$$

19.9 Voltage Balance Equations

Voltages u_d and u_q across the virtual stator phases *d* and *q* in synchronously rotating coordinate frame can be obtained by applying Park transform to $\alpha\beta$ voltages:

$$\underline{u}_{dq} = u_d + \mathrm{j}u_q = \underline{u}_{\alpha\beta S} e^{-\mathrm{j}\theta_m} = \left(R_S \underline{i}_{\alpha\beta S} + \mathrm{d}\underline{\Psi}_{\alpha\beta S}/\mathrm{d}t\right) e^{-\mathrm{j}\theta_m}. \tag{19.15}$$

Variables $\underline{i}_{\alpha\beta S}$ и $\underline{\Psi}_{\alpha\beta S}$ of the stationary coordinate frame can be expressed in terms of *dq* variables by applying the inverse Park transform, $\underline{i}_{\alpha\beta S} = \underline{i}_{dq}\exp(-\mathrm{j}\theta_m)$:

$$\begin{aligned} \underline{u}_{dq} &= \left(R_S \underline{i}_{dq} e^{+\mathrm{j}\theta_m} + \mathrm{d}\left(\underline{\Psi}_{dq} e^{+\mathrm{j}\theta_m}\right)/\mathrm{d}t\right) e^{-\mathrm{j}\theta_m} \\ &= R_S \underline{i}_{dq} + \mathrm{d}\underline{\Psi}_{dq}/\mathrm{d}t + \mathrm{j}\omega_m \underline{\Psi}_{dq} \end{aligned} \tag{19.16}$$

where $\omega_m = p\Omega_m$. Therefore, the voltage balance equations in dq frame do not have the usual form of $u = Ri + \mathrm{d}\Psi/\mathrm{d}t$. They comprise an additional factor which appears as a consequence of applying rotational transform. Angle θ_m denotes the position of magnetic axis of the rotor with respect to the stator phase a. The angular frequency $p\Omega_m = \omega_m$ represents the rotor speed. In cases where synchronous machine has more than one pair of magnetic poles ($p > 1$), it is necessary to take into account the circumstance that $p\Omega_m = \omega_m$. The angle θ_m defines the transformation matrix of Park rotational transform. With $p > 1$, this angle is p times larger than the mechanical displacement of the rotor. Starting from the voltage balance equations in stationary $\alpha\beta$ coordinate frame

$$\underline{u}_{\alpha\beta s} = R_s \underline{i}_{\alpha\beta s} + \frac{\mathrm{d}}{\mathrm{d}t}\underline{\Psi}_{\alpha\beta s},$$

one obtains

$$\underline{u}_{dq} = e^{-\mathrm{j}\theta_m}\left(R_s \underline{i}_{\alpha\beta s} + \frac{\mathrm{d}}{\mathrm{d}t}\underline{\Psi}_{\alpha\beta s}\right) = e^{-\mathrm{j}\theta_m}\left(R_s e^{\mathrm{j}\theta_m}\underline{i}_{dq} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{e^{\mathrm{j}\theta_m}\Psi_{dq}}\right)\right)$$

Equation which expresses the voltage balance of the stator windings in synchronously rotating dq coordinate system is

$$\underline{u}_{dq} = R_s \underline{i}_{dq} + \frac{\mathrm{d}}{\mathrm{d}t}\underline{\Psi}_{dq} + \mathrm{j}\omega_m \underline{\Psi}_{dq} \tag{19.17}$$

Equation 19.17 can be split into real and imaginary parts. Each of them represents a scalar equation

$$\mathrm{Re}\{\underline{u}_{dq}\} \to u_d = R_s i_d + \frac{\mathrm{d}}{\mathrm{d}t}\Psi_d - \omega_m \Psi_q,$$
$$\mathrm{Im}\{\underline{u}_{dq}\} \to u_q = R_s i_q + \frac{\mathrm{d}}{\mathrm{d}t}\Psi_q + \omega_m \Psi_d.$$

Therefore, the voltage balance equation with complex variables can be split in two scalar equations:

$$\begin{aligned} u_d &= R_s i_d + \frac{\mathrm{d}}{\mathrm{d}t}\Psi_d - \omega_m \Psi_q, \\ u_q &= R_s i_q + \frac{\mathrm{d}}{\mathrm{d}t}\Psi_q + \omega_m \Psi_d. \end{aligned} \tag{19.18}$$

The voltage balance equations of the stator windings get affected by Park transform, and they obtain additional factors such as $\omega_m \Psi_d$. At the same time, the voltage balance equation in excitation winding remains unchanged. This equation does not get affected by Park transform. Actually, the excitation current, voltage, and flux linkage do not get transformed by Park transform. Namely, the excitation winding is fastened to the rotor, and its magnetic axis coincides with d-axis of synchronously rotating coordinate frame. Thus, transient phenomena of the excitation winding get modeled by the following equation:

$$u_R = R_R i_R + \frac{\mathrm{d}}{\mathrm{d}t}\Psi_R \tag{19.19}$$

19.10 Electrical Subsystem of Isotropic Machines

Mathematical model of synchronous machine describes *electrical* and *mechanical* subsystem. The former is described by the voltage balance equations and the latter by Newton equation of motion. With two equivalent phase windings on the stator and one excitation winding on the rotor, synchronous machine has three windings and, hence, three differential equations expressing the voltage balance in these windings. Besides, the model includes the inductance matrix, which provides relations between flux linkages and currents, as well as the expression for the electromagnetic torque.

Rotor magnetic circuit of a synchronous machine may be of cylindrical form, and in this case, magnetic resistance R_μ is the same in all directions. Cylindrical rotor with which maintains the same magnetic resistance in all directions is called *isotropic*. Since the self-inductance of each winding depends on the ratio N^2/R_μ, a constant magnetic resistance results in constant self-inductances. The stator phase windings have the same number of turns, and therefore, $L_d = L_q = L_S$. In cases where magnetic resistances in d- and q-axes are different, the rotor is called *anisotropic*. In this case, self-inductances are not the same, $L_d \neq L_q$.

Voltage balance equations in (19.8) are written for the windings shown in Fig. 19.6, and they are applicable to both isotropic and anisotropic synchronous machines. The difference between the former and the latter appears in the inductance matrix.

Voltage balance equations in virtual stator phase windings in dq coordinate frame and in the excitation winding are given by

$$\begin{aligned}
u_d &= R_s i_d + \frac{\mathrm{d}\Psi_d}{dt} - \omega_m \Psi_q\ , \\
u_q &= R_s i_q + \frac{\mathrm{d}\Psi_q}{dt} + \omega_m \Psi_d\ , \\
u_R &= R_R i_R + \frac{\mathrm{d}\Psi_R}{dt},
\end{aligned} \tag{19.20}$$

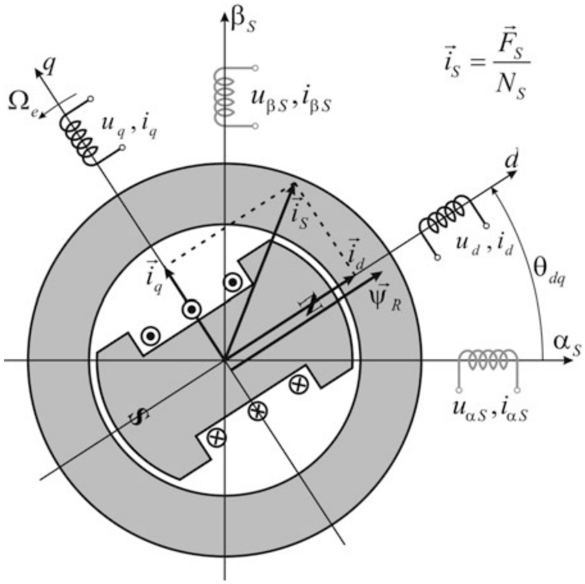

Fig. 19.6 Model of a synchronous machine in the *dq* coordinate system

while the relations between fluxes and currents are determined by the inductance matrix, given in (19.9). This matrix equation can be split into three scalar expressions:

$$\Psi_d = L_s^* i_d + L_m i_R, \tag{19.21}$$

$$\Psi_q = L_s^* i_q, \tag{19.22}$$

$$\Psi_R = L_R i_R + L_m i_d \tag{19.23}$$

In isotropic machines, $L_d = L_q = L_S^*$. With anisotropic machines where $L_d \neq L_q$, parameter L_S^* in (19.21) is replaced by L_d and parameter L_S^* in (19.22) is replaced by L_q. Flux of the excitation winding is equal to $\Psi_R = L_R i_R + L_m i_d$. Coefficient L_R represents self-inductance of the excitation winding, while L_m is mutual inductance of the excitation winding and virtual stator phase winding in *d*-axis. Mutual inductance L_m can be determined from measurement of the largest mutual inductance between the excitation winding and one of the stator phase windings, *a*, *b*, and *c*. The mutual inductance assumes the largest value when the rotor position and the position of magnetic axis of the excitation winding coincide with magnetic axis of the stator phase winding. For synchronous machines with isotropic cylindrical rotor, where the magnetic circuit has the same magnetic resistance in all directions, the flux linkage of the virtual stator phase winding in *d*-axis is equal to $\Psi_d = L_s i_d + L_m i_R$, while the flux of the virtual stator phase winding in *q*-axis is equal to $\Psi_q = L_s i_q$.

19.11 Torque in Isotropic Machines

Electromagnetic torque of an isotropic synchronous machine can be derived by analyzing the balance of power. It is necessary to consider the electrical power, delivered to synchronous machine from a three-phase voltage source, mechanical power on the rotor shaft, and the power of losses in iron, copper, and mechanical subsystem. Starting from expression for electrical power delivered by the source, $p_e = (3/2)\,(u_d i_d + u_q i_q)$, and by using the voltage balance equations for the stator windings, one obtains the following relation:

$$\begin{aligned} p_e &= \frac{3}{2}\left(u_d i_d + u_q i_q\right) \\ &= \frac{3}{2} R_s \left(i_d^2 + i_q^2\right) + \frac{3}{2}\left(i_d \frac{\mathrm{d}\Psi_\mathrm{d}}{dt} + i_q \frac{\mathrm{d}\Psi_q}{\mathrm{d}t}\right) + \frac{3}{2}\omega_m\left(\Psi_d i_q - \Psi_q i_d\right) \\ &= p_{Cu1} + \frac{\mathrm{d}W_m}{\mathrm{d}t} + p_{em}. \end{aligned} \tag{19.24}$$

In the above expression, the first factor on the right represents the power of losses in the stator windings, also called copper losses. There are also iron losses in stator magnetic circuit, where magnetic inductance B changes with an angular frequency of ω_m. One of the four approximations taken in modeling electrical machines is that the iron losses are rather small and therefore negligible (see Chap. 6, Sect. 6.2). Therefore, considered balance of power does not include the iron losses. The second factor in the above equation is $\mathrm{d}W_m/\mathrm{d}t$, and it defines the rate of change of the energy accumulated in magnetic coupling field. If the machine operates with a constant flux, the energy of the magnetic field is constant as well, and the factor $\mathrm{d}W_m/\mathrm{d}t$ is equal to zero. The third and the last factor of the above equation is $p_{em} = (3/2)\omega_m(\Psi_d i_q - \Psi_q i_d)$. It is obtained by subtracting the losses from the input electrical power. Therefore, it represents the rate of change of the electrical energy into mechanical work. It is transferred through the air gap to the rotor by means of electromagnetic interaction of the stator and rotor fields in the air gap. The power p_{em} is called *power of electromechanical conversion.*

Synchronous machines have the stator magnetic field revolving in synchronism with the rotor. In steady state, the angular frequency of the source ω_e is equal to the angular frequency of rotation $p\Omega_m = \omega_m$. The question arises whether $\omega_e = \omega_m$ during transients. Namely, during transient processes, where the electromagnetic torque T_{em} changes, the angle between the stator field $\boldsymbol{F}_S$ and the rotor flux $\boldsymbol{\Psi}_R$ may change as well. The vector $\boldsymbol{F}_S$ revolves at the speed ω_e/p, while the rotor flux $\boldsymbol{\Psi}_R$ revolves at the rotor speed Ω_m. When the angle between the two vectors changes, there is a temporary difference between ω_e and $p\Omega_m = \omega_m$. When the machine enters a new steady state, the vectors assume and maintain a new relative position, and the equation $\omega_e = p\Omega_m = \omega_m$ is reinstated.

The change between ω_e and ω_m allows the angle between vectors $\boldsymbol{F}_S$ and $\boldsymbol{\Psi}_R$ to change during transients. Yet, they do remain in synchronism, namely, they revolve at the same speed in steady state conditions. If the *dq* coordinate system is introduced by aligning the axes *d* with the rotor magnetic axis, then the speed of rotation of this *dq* coordinate frame remains equal to ω_m in both steady state and transient conditions. Therefore, the voltage balance equation (19.20) comprises the angular frequency ω_m in both steady state and transient conditions.

It is of interest to calculate the amount of p_{em} that is passed to the output shaft as mechanical power. With rotor excitation based on permanent magnets, there is no excitation winding and no losses in the rotor. The power p_{em} passed to the rotor through the air gap is converted into mechanical power. This mechanical power may not be equal to the output mechanical power of the machine, due to losses in mechanical subsystem such as friction and air resistance to rotor motion. Therefore, it is called *internal mechanical power*, and it is denoted by $p_{mR} = p_{em}$. When the rotor has an excitation winding, this winding has copper losses $R_R {i_R}^2$. These losses are supplied from an external power source that provides the voltage u_R and the current i_R to the excitation winding. In steady state, this power source supplies the power $u_R i_R = R_R {i_R}^2$ to the excitation winding. Therefore, equation $p_{mR} = p_{em}$ applies as well to synchronous machines with electromagnetic excitation.

Internal mechanical power is equal to the product of the internal electromagnetic torque T_{em} and the rotor speed $\Omega_m = \omega_m/p$. This torque is a mechanical interaction between the stator and the rotor, caused by electromagnetic forces generated by the coupling field. The electromagnetic torque is calculated from expression

$$T_{em} = \frac{p_{mR}}{\Omega_m} = p\frac{p_{em}}{\omega_m} = \frac{3p}{2}\left(\Psi_d i_q - \Psi_q i_d\right) \tag{19.25}$$

The above expression is further simplified for isotropic machines, where the magnetic circuit has the same magnetic resistance in all directions and where equation $L_d = L_q = L_s$ applies. By introducing relations $\Psi_d = L_s i_d + L_m i_R = L_s i_d + \Psi_{Rm}$ and $\Psi_q = L_s i_q$ in the above torque expression, one obtains

$$T_{em} = \frac{3p}{2}\left(\Psi_d i_q - \Psi_q i_d\right) = \frac{3p}{2} L_m i_R i_q = \frac{3p}{2}(L_m i_R) i_q = \frac{3p}{2}\Psi_{Rm} i_q \tag{19.26}$$

The flux component $\Psi_{Rm} = L_m i_R$ represents the part of the excitation flux which encircles the stator winding. It is slightly smaller than the flux of the excitation winding due to a finite amount of magnetic leakage flux. In machines with permanent magnet excitation, the flux Ψ_{Rm} represents the part of the rotor flux, caused by permanent magnets, that encircles the stator windings. A small amount of the flux of permanent magnets does not reach the stator core and does not contribute to the process of electromechanical conversion.

19.12 Anisotropic Rotor

The rotor magnetic circuit may have a noncylindrical form which introduces the difference in magnetic resistances along *d*- and *q*-axes. This results in different self-inductances L_d and L_q of virtual stator phase windings in *dq* frame. Cylindrical structures, where magnetic resistance is not dependent on direction of the field, are called *isotropic*, which means the ones that are having the same properties in all directions. In isotropic machines, inductances L_d and L_q are equal. When magnetic resistance changes with direction of the field, then, the machine is *anisotropic*, and inductances L_d and L_q are different. Salient features of anisotropic machines will be presented in the following section, along with the corresponding expression for the torque.

The flux of the virtual stator phase winding in *d*-axis is equal to $\Psi_d = L_d i_d + L_m i_R$, while the flux in the virtual stator phase winding *q* is equal to $\Psi_q = L_q i_q$. Excitation flux does not contribute to the stator flux in *q*-axis. With cylindrical rotor, inductances L_d and L_q have the same value, $L_d = L_q = L_s$. Construction of an anisotropic rotor is shown in Fig. 19.7, with different magnetic resistances in *d*- and *q*-axes. In the left side of the figure, there is a cross section of magnetic circuit of the rotor with electromagnetic excitation. This magnetic circuit is shaped to achieve a low magnetic resistance in *d* and to facilitate establishing the excitation flux. Conductors of the excitation winding are placed on the sides of the magnetic circuit, directed along *q*-axis. For this reason, magnetic resistance to the flux directed along *q*-axis is relatively high because the path of the *q*-axis flux includes relatively large air-filled segments. A higher magnetic resistance results in a smaller inductance $L \sim N^2/R_\mu$; thus, the circumstance that $R_{\mu d} < R_{\mu q}$ results in $L_d > L_q$.

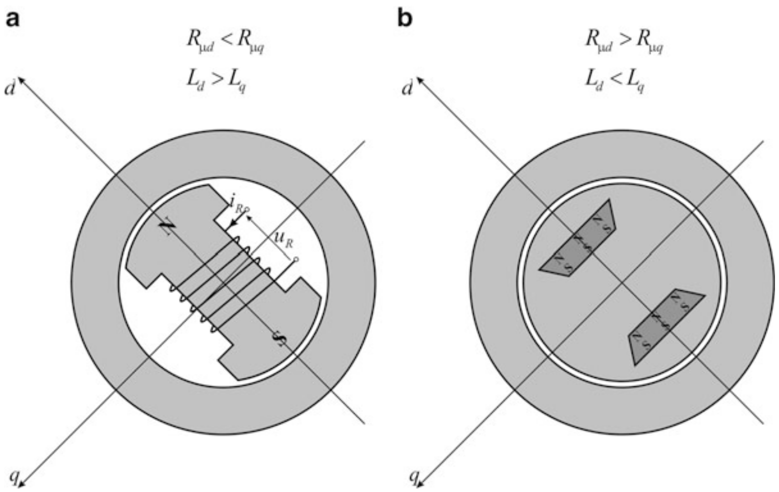

Fig. 19.7 (**a**) Anisotropic rotor with excitation winding and with different magnetic resistances along *d*- and *q*-axes. (**b**) Anisotropic rotor with permanent magnets

The second example in Fig. 19.7 shows a rotor magnetic circuit with permanent magnets built in the interior of the rotor. The magnets are inserted along d-axis. Differential permeability of permanent magnets is close to μ_0, and their presence on the flux path in d-axis increases magnetic resistance $R_{\mu d}$. With $R_{\mu d} > R_{\mu q}$, the phase winding inductances are $L_d < L_q$.

Flux linkages of an anisotropic machine are

$$\begin{aligned} \Psi_d &= L_d i_d + L_m i_R, \\ \Psi_q &= L_q i_q, \\ \Psi_R &= L_R i_R + L_m i_d. \end{aligned} \tag{19.27}$$

19.13 Reluctant Torque

Differences between self-inductances L_d and L_q of virtual stator phases d and q affect the expression for the electromagnetic torque:

$$\begin{aligned} T_{em} &= \frac{3p}{2}\left(\Psi_d i_q - \Psi_q i_d\right) \\ &= \frac{3p}{2}\Psi_{Rm} i_q + \frac{3p}{2}\left(L_d - L_q\right) i_d i_q. \end{aligned} \tag{19.28}$$

The torque expression contains an additional component, proportional to the difference of self-inductances in d- and q-axes. This torque component is called *reluctant* torque because it appears as a consequence of differences in magnetic resistances, also called *reluctances*. Reluctant torque may exist even in synchronous machines with no excitation winding and no permanent magnets. It is sufficient to have different magnetic resistances in d- and q-axes and different inductances L_d and L_q. In the absence of the rotor flux, only the stator field exists in the air gap. This torque tends to bring the rotor in position where it pays the minimum magnetic resistance to the stator flux. Hence, it acts toward aligning the rotor axis of minimum magnetic resistance with the stator flux. There are synchronous machines that are made with no permanent magnets and no excitation windings. Instead, their rotors have considerable differences of the magnetic resistances between the direct d-axis and quadrature q-axis. This increases the difference L_d–L_q and, hence, the reluctant torque. These machines are called *synchronous reluctance machines*.

Question (19.2): A synchronous machine has an anisotropic rotor with no permanent magnets and no excitation winding. The amplitude of the stator current is limited to I_m. Determine the largest possible value of the reluctant torque. All parameters affecting the reluctant torque are known.

Answer (19.2): Reluctant torque is proportional to the product of currents i_d and i_q. When the amplitude of the stator current is limited, the current components can be expressed as $i_d = I_m\cos(\xi)$ and $i_q = I_m\sin(\xi)$, where ξ is the angle between the stator current vector and d-axis. The reluctant torque is then

$$T_{em} = \frac{3p}{2}\left(L_d - L_q\right)I_m^2 \cos\xi \ \sin\xi = \frac{3p}{4}\left(L_d - L_q\right)I_m^2 \ \sin 2\xi.$$

The highest reluctant torque is obtained for $\xi = \pi/4$, when $i_d = i_q = I_m/\mathrm{sqrt}(2)$, and it is equal to $(3p/4)(L_d - L_q){I_m}^2$.

19.14 Reluctance Motor

Synchronous reluctance machines are used in applications where the size and weight have no particular importance and where the prevailing goal is to have a construction which is robust, simple, and low cost. Reluctance machines have no active parts on the rotor. Rotor has only magnetic circuit made to have different magnetic resistances in direct and quadrature axes. The rotor magnetic circuit can be obtained by stacking the iron sheets in the way shown in Fig. 19.8. By stacking the iron sheets, one obtains a small magnetic resistance along the sheets and a large magnetic resistance in direction perpendicular to the sheets. The flux that passes in direction perpendicular to the sheets passes a number of times through the insulation gaps between the sheets, which increase the equivalent magnetic resistance. In the prescribed way, one obtains an anisotropic rotor, the rotor with different magnetic resistances in d- and q-axes. At the same time, the external appearance of the rotor is cylindrical, with no salient poles and with a low air drag. Therefore, the rotor may reach high speeds without significant motion resistances and without jeopardizing mechanical integrity of the machine.

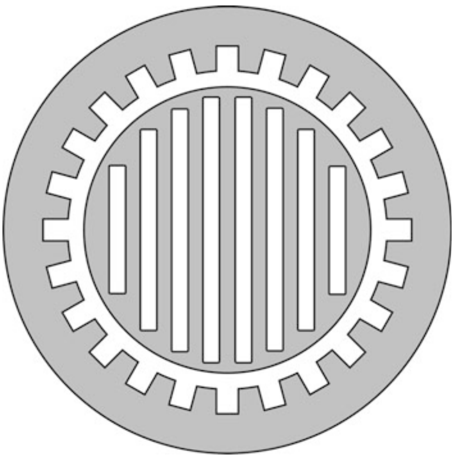

Fig. 19.8 Rotor of a reluctant synchronous machine

Advantage of synchronous reluctance machines, compared to wound rotor machines, is that the former have considerably simpler construction and they do not need to have external supply to the excitation winding through the slip rings. Compared to permanent magnet machines, reluctance machines are easier and cheaper to manufacture. One advantage of reluctance machines is the fact that the air gap flux depends on the stator current only, which opens the possibility to adjust the flux to different operating conditions, such as the field weakening.

With stator currents of reluctance machine equal to zero, the flux is equal to zero as well. At the same time, the electromotive force $\omega_m \Psi$ is equal to zero as well. In cases where the stator winding of synchronous reluctance machine is disconnected from the source and opened, the voltages across the terminals are equal to zero, and they do not pose electrical hazard. At the same time, the fact that there are no induced voltages between the open winding terminals of a reluctance motor means that short-circuiting the terminals would not result in any stator currents. This means that reluctance motor with short-circuited stator winding and with rotor in motion does not have any stator current and does not generate any torque. Synchronous machines with permanent magnet excitation do not have the same behavior. The rotor flux of a permanent magnet machine is present even in cases where the machine is disconnected from the source. The voltage across the terminals of the opened stator winding is equal to the electromotive force $E_0 = \omega_m \Psi_{Rm}$. When the stator terminals are short-circuited, a relatively high short-circuit current is established, approximately equal to $I_{ks} \approx E_0/X_S = \Psi_{Rm}/L_S$. With low stator inductances L_S of synchronous machines with surface mounted magnets, the short-circuit currents are very large, and they result in the stator magnetomotive forces that may demagnetize and destroy the magnets. Besides, the short-circuit current I_{ks} produces braking torque.[4] Reluctance machines do not have the short-circuit current, they have no braking torque in short-circuit conditions, and their voltage between the stator terminals is equal to zero in cases where the machine is disconnected from the source:

$$T_{em} = \frac{3p}{2}\left(L_d - L_q\right) i_d i_q \,. \tag{19.29}$$

One shortcoming of reluctance machines is their lower specific power and lower specific torque. Due to the absence of the excitation flux, the torque expression does not have component which is proportional to the product $\Psi_{Rm} i_q$. Instead, it has only the reluctant torque which is proportional to the product of currents i_d and i_q.

[4] With the stator winding in short circuit and with the short-circuit current I_{KS}, machine does not receive any electrical power from the source. At the same time, there are copper losses in the stator winding, proportional to $R_S {I_{KS}}^2$. The power of copper losses is supplied from the only access to the machine which remains available, this access being the shaft, where the machine receives or delivers the power $T_{em}\Omega_m$. In short-circuit conditions, mechanical power is absorbed through the shaft. There is a braking torque component proportional to $R_S {I_{KS}}^2/\Omega_m$ which accounts for the copper losses in short circuit.

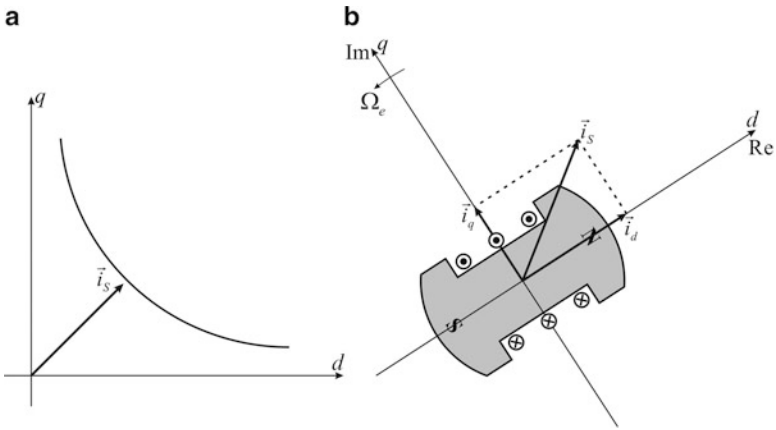

Fig. 19.9 (**a**) Constant torque hyperbola in the $i_d - i_q$ diagram. (**b**) Positions of the rotor, *dq* coordinate system, and complex plane

A thought experiment of inserting permanent magnets into existing reluctance machine would add the component $\Psi_{Rm}i_q$ to the torque, hence increasing the specific torque and power of the machine. It can be concluded that the ratio T_{em}/I_S between the electromagnetic torque and the stator current is considerably lower in reluctance machines than in synchronous machines with permanent magnets or with excitation winding. For developing the same torque, the stator current of reluctance machine will be significantly larger compared to the required stator current of an excited synchronous machine. For this reason, the power and efficiency of reluctance machines are lower than those of excited synchronous machines.

In order to mitigate the unfavorable ratio T_{em}/I_S of reluctance machines, their control is conceived to maximize the torque-per-amp ratio. It is of interest to obtain the maximum possible torque for the given amplitude of the stator current. The left side of Fig. 19.9 shows a *dq* coordinate frame with currents i_d and i_q on the abscissa and ordinate. The root locus denoting constant torque $T_{em} \sim i_d i_q$ is represented by a hyperbola. Various pairs of i_d and i_q provide the same product $i_d i_q$. Thus, there is a certain degree of freedom in controlling the machine, and it is to be used to minimize the losses and maximize the torque-per-amp ratio. In Fig. 19.9, the amplitude of the stator current is proportional to the radius vector that starts from the origin and ends at the selected operating point on hyperbola $T_{em} \sim i_d i_q$. The smallest amplitude is achieved with radius vector at an angle of $\pi/4$ with respect to *d*- and *q*-axes, that is, with $i_d = i_q$. Wherever possible, reluctance machines are controlled with $i_d = i_q$.