# Chapter 12
# Modeling and Supplying DC Machines

In this chapter, mathematical model is developed for DC machines with excitation windings and DC machines with permanent magnet excitation. The block diagram of the model is used to provide a brief introduction to the torque control. Steady-state equivalent circuits are derived and explained for armature and excitation windings. These circuits are used to introduce and analyze mechanical characteristic of separately excited DC machine and determine the steady-state speed. The chapter provides basic elements for the control of the rotor speed. Steady-state operation of DC generators is explained along with basic output characteristics. Typical applications of DC machines are classified on the basis of the speed and torque changes within the four quadrants of $T_{em}$–$\Omega_m$ plane. On that ground, the basic requirements are specified for the power supply of the armature windings. The operation of switching power converter with H-bridge is briefly explained, along with the basic notions on pulse-width modulation (PWM). The impact of pulsed power supply on the machine operation is considered by studying the ripple of the armature current. The chapter closes with an overview of most common power converter topologies used in supplying DC machines.

Analysis of electrical and mechanical characteristics of a DC machine is based on mathematical model. The model contains differential equations and algebraic relations describing transient processes in the machine. In DC electrical machines, the excitation flux is established along *direct* axis, while the rotor flux (*armature reaction*) appears along quadrature axis. The two axes are orthogonal, and the mutual inductance between the excitation winding and the armature windings is

S.N. Vukosavic, *Electrical Machines*, Power Electronics and Power Systems,
DOI 10.1007/978-1-4614-0400-2_12, 

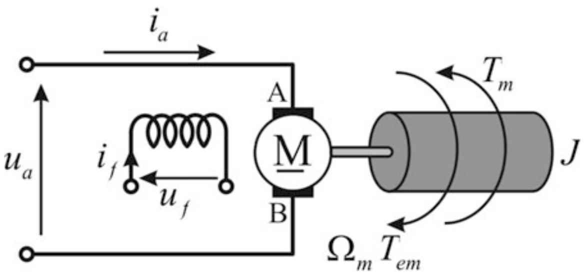



**Fig. 12.1** Connections of a DC machine to the power source and to mechanical load

equal to zero.[1] Namely, changes in excitation current do not have an immediate impact on the rotor flux. At the same time, changes of the rotor current do not affect the excitation flux. The absence of interaction between direct and quadrature axes makes the transient phenomena of these axes decoupled. Transients in armature winding do not affect[2] the excitation winding. Hence, differential equation describing the changes of the excitation flux and current does not have factors proportional to the rotor flux and the armature current. For this reason, mathematical model of DC machine is relatively simple and clear. The flux control loop is decoupled from the torque control loop, and their design and application are quite straightforward.

The subject of modeling is a DC machine connected to the source $u_a$, feeding the armature winding, and to the source $u_f$, feeding the excitation winding. Connections of DC machine to electrical sources and mechanical load are shown in Fig. 12.1. Notations $u_a$ and $u_f$ are used in the figure since the winding voltages can be variable and change in time. In steady state, when the supply voltages are constant, these quantities are denoted by $U_a$ and $U_f$. The shaft of the machine rotates at the angular speed of $\Omega_m$. Revolving masses of inertia $J$ are accelerated or decelerated by electromagnetic torque $T_{em}$ and load torque $T_m$. Reference directions of the two torques are opposite, and they are shown in the figure.

The analysis of transient phenomena in a DC machine presented here results in differential equations that make up the mathematical model. According to conclusions of the preceding section, the mathematical model includes:

[1] *Note*: Mutual inductance of orthogonal windings is equal to zero if magnetic circuit is linear, that is, in cases where magnetic saturation does not occur. Otherwise, flux in one of the two orthogonal axes changes the operating point ($B$,$H$) on nonlinear magnetizing curve of ferromagnetic material, which affects magnetic resistance and flux in the other axis. Namely, the lines of the excitation flux and the lines of the rotor flux pass through the same magnetic circuit. Orthogonal fluxes share the same ferromagnetic material on both stator and rotor. Variation of one of these fluxes changes degree of saturation of ferromagnetic material (iron), acting indirectly upon the other flux. Nonlinearity of magnetic circuit leads to coupling of the orthogonal axes in all cases where their flux linkages share the same magnetic circuit.

[2] Due to nonlinear $B(H)$ characteristic of iron, magnetic circuit may saturate. In cases where the saturation level is altered by the armature current, there is a change in magnetic resistance on the path of the excitation flux. Through this secondary effect, called *cross saturation*, the armature current may affect the excitation flux.

- Differential equations of voltage balance in the windings
- Differential equation describing changes of angular speed (Newton equation)
- Algebraic relations between fluxes and currents (inductance matrix)
- Expression for electromagnetic torque

Unless otherwise stated, the process of modeling electrical machines throughout this book includes the four approximations discussed in the preceding sections:

- Parasitic capacitances are neglected (as well as the energy of electrical field).
- Spatial distribution of the energy of magnetic field is neglected. It is assumed that the energy is concentrated in discrete elements such as inductances. Thus, the equivalent circuits are represented as lumped parameter networks.
- Losses in magnetic circuit are neglected (i.e., losses in iron).
- Nonlinearity of magnetic circuit is neglected, that is, there is no magnetic saturation.

## 12.1 Voltage Balance Equation for Excitation Winding

The magnetomotive force along quadrature axis is created by the rotor conductors, and it has no influence on the excitation flux. Therefore, instantaneous value of the flux $\psi_f$ in the excitation winding is $\psi_f = N_f \Phi_f = L_f i_f$. In further considerations, instantaneous values of currents, flux linkages, and voltages are dealt with. Therefore, notation $i_f$ is used, denoting the variables that change in time, such as the excitation current $i_f(t)$. For brevity, further expressions are written by using representation such as $i_f$, without an explicit specification such as $i_f(t)$, showing that the considered variable is a time-varying function.

Coefficient of self-induction of the excitation winding is given by (11.6). The excitation winding has a finite electrical resistance $R_f$, and the voltage balance in this winding is expressed by the equation

$$u_f = R_f i_f + \frac{\mathrm{d}\psi_f}{\mathrm{d}t} = R_f i_f + L_f \frac{\mathrm{d}i_f}{\mathrm{d}t}. \tag{12.1}$$

Excitation flux $\Phi_f$ passes through the main poles. At the same time, $\Phi_f$ is the flux in a single turn of the excitation winding. This flux is proportional to the excitation current. On the basis of (11.7), the excitation flux is

$$\Phi_f = L'_f i_f = \left(\frac{L_f}{N_f}\right) i_f. \tag{12.2}$$

Therefore, the excitation winding can be represented by an $R$-$L$ circuit, as shown in the left-hand side of Fig. 12.2. In the case when a DC voltage $U_f$ is fed to the terminals of the excitation winding, the excitation current increases exponentially toward the final value $i_f(\infty) = I_f = U_f/R_f$, which is reached in the steady state. It is

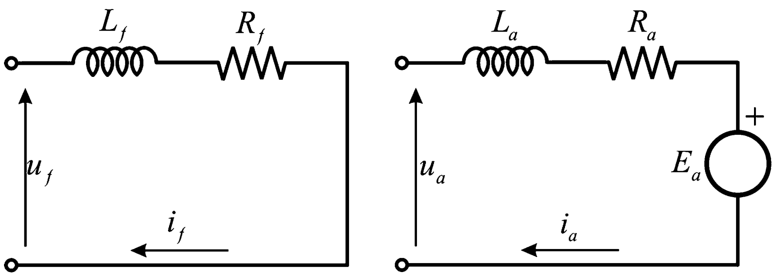



**Fig. 12.2** Voltage balance in the excitation winding (*left*) and in the armature winding (*right*)

of interest to determine the change in the excitation current in the case where the initial value is $i_f(0)$ and the excitation voltage is $u_f(t) = U_f$ for $t > 0$. During transient, instantaneous value of the excitation current is

$$i_f = i_f(0)e^{-\frac{t}{\tau}} + i_f(\infty)\left(1 - e^{-\frac{t}{\tau}}\right) = \frac{U_f}{R_f}\left(1 - e^{-\frac{t}{\tau}}\right), \tag{12.3}$$

where $\tau = L_f/R_f$ is the electrical time constant of the excitation winding. In steady state, relation between the excitation flux and voltage across the excitation winding is

$$\Phi_f = L'_f I_f = L'_f \frac{U_f}{R_f}. \tag{12.4}$$

**Question (12.1):** Determine time constant of the excitation winding of a DC machine with the main poles cross section $S = 0.01\ \text{m}^2$, with the air gap $\delta = 1$ mm, $N_f = 4{,}000$, and with resistance of the excitation winding of $R_f = 400\ \Omega$.

**Answer (12.1):** Magnetic resistance for the excitation flux is equal to

$$R_\mu = \frac{2 \cdot \delta}{\mu_0 \cdot S} = 159155\,\text{H}^{-1}.$$

Inductance of the excitation winding is equal to

$$L_f = \frac{N_f^2}{R_\mu} = 100,5\,\text{H}.$$

Time constant of the excitation winding is equal to

$$\tau = \frac{L_f}{R_f} = 0,251\,\text{s}.$$

## 12.2 Voltage Balance Equation in Armature Winding

Rotor winding has two parallel branches connected between the brushes A and B. The equivalent internal resistance of armature winding $R_a = N_R R_1/4 + \Delta R$ can be measured between the brushes at standstill. The number of rotor conductors is $N_R$, while $R_1$ is resistance of a single conductor. The equivalent resistance of the commutator, including the brushes and collector, is denoted by $\Delta R$. In addition to resistance, rotor (armature) winding has self-inductance $L_a$. Namely, the presence of electrical currents in rotor conductors creates magnetomotive force called armature reaction. Consequently, the rotor flux is created, inversely proportional to the magnetic resistance $R_\mu$. The magnetic resistance along the path of the rotor flux is relatively high, since the rotor flux passes through the neutral zone between the main poles, where the lines of magnetic field face a very large air gap. Inductance of the armature winding $L_a$ is proportional to the square of the number of turns and inversely proportional to the magnetic resistance of the magnetic circuit containing the rotor flux. This magnetic resistance is significantly higher than the one encountered by the excitation flux. This is due to the fact that the excitation flux passes through a very small air gap $\delta$, while the armature reaction faces a very large air gap under the auxiliary poles. Smaller DC machines are made with no auxiliary poles at all, and they have even larger magnetic resistance in quadrature axis. In most DC machines, the number of turns in excitation winding is much larger than the number of turns in armature winding. As a consequence, armature inductance $L_a$ is two or three orders of magnitude smaller compared to the inductance $L_f$ of the excitation winding. In every coil, electrical current changes at the rate $\mathrm{d}i/\mathrm{d}t \sim u/L$, proportional to the applied voltage and inversely proportional to the coil inductance. Therefore, the armature current in DC machines changes at a rate which is two or three orders of magnitude higher than the rate of change of the excitation current.

**Question (12.2):** Determine the equivalent internal resistance $R_a$ of armature winding having a total of $N_R = 40$ conductors. Resistance of each conductor is 0.1 Ω, while the equivalent resistance of the mechanical commutator with two brushes is equal to $\Delta R = 0.2\ \Omega$. Determine the self-inductance of the rotor winding $L_a$. The equivalent cross section of the magnetic circuit comprising the rotor flux is $S = 0.1\ \mathrm{m}^2$. Distance between the rotor and stator in the neutral zone is $d = 20$ mm. Determine time constant of the armature winding circuit.

**Answer (12.2):** Resistance of the armature winding is equal to $R_a = 0.1\ \Omega \cdot 40/4 + 0.2\ \Omega = 1.2\ \Omega$.

Magnetic resistance along the rotor flux path is

$$R_\mu = \frac{2 \cdot d}{\mu_0 \cdot S} = 318\ 310\,\mathrm{H}^{-1}.$$

The armature winding has 40 conductors and 20 turns. Inductance of the armature winding is equal to

$$L_a = \frac{20^2}{R_\mu} = 1.256\,\mathrm{mH}.$$

Time constant of the excitation winding is equal to

$$\tau_a = \frac{L_a}{R_a} = 1.047\,\mathrm{ms}.$$

***

In addition to the voltage drop due to resistance $R_a$ and inductance $L_a$, the rotor circuit has induced electromotive force $E_a$, proportional to the rotor speed $\Omega_m$ and to the excitation flux $\Phi_f$. The voltage balance equation for the armature winding takes the following form:

$$u_a = R_a i_a + L_a \frac{\mathrm{d}i_a}{\mathrm{d}t} + E_a = R_a i_a + L_a \frac{\mathrm{d}i_a}{\mathrm{d}t} + k_e \Phi_f \Omega_m. \tag{12.5}$$

Equivalent circuit of the armature winding is shown in the right-hand side of Fig. 12.2. There are no changes of the electrical current in steady-state conditions, and the first derivative of the current is equal to zero. In steady-state conditions, (12.5) assumes the form

$$U_a = R_a I_a + E_a = R_a I_a + k_e \Phi_f \Omega_m. \tag{12.6}$$

Model of the machine includes expressions for the induced electromotive force and electromagnetic torque derived earlier:

$$E_a = k_e \Phi_f \Omega_m, \quad T_{em} = k_m \Phi_f i_a. \tag{12.7}$$

$$k_m = \frac{N_R}{2\pi}, \quad k_e = \frac{N_R}{2\pi}.$$

## 12.3 Changes in Rotor Speed

In addition to modeling transients in the windings, which represent the electrical subsystem, it is necessary to model the mechanical subsystem of the machine and to derive differential equation describing changes in the rotor angular speed. The rotor is coupled to a work machine or a driving machine by means of its shaft. Equivalent inertia of all rotating parts is denoted by $J$. It comprises inertia of the rotor, shaft, work machine, coupling elements, transmission elements, and of all the parts moving along with the rotor at speed $\Omega_m$. The rotor speed is affected by:

- Electromagnetic torque $T_{em}$
- Friction torque $k_F\Omega_m$
- Load torque $T_m$
- Inertial torque $J\mathrm{d}\Omega_m/\mathrm{d}t$

According to notation presented in Fig. 12.1, reference direction for electromagnetic torque is positive, meaning that positive value of this torque acts in direction of increasing algebraic value of the rotor speed. Load torque $T_m$ represents mechanical load of the work machine which resists to motion and affects the rotor speed. Reference direction of this torque is negative, meaning that positive value of this torque acts in the direction of reducing the algebraic value of the speed. Friction torque resists the motion in either direction; thus, it acts in the direction of reducing the speed absolute value. Inertial torque $J\mathrm{d}\Omega_m/\mathrm{d}t$ represents the torque required to change the speed and provide the acceleration $\mathrm{d}\Omega_m/\mathrm{d}t$. Equation (12.8) expresses the balance of all the torque components mentioned above. As a matter of fact, (12.8) is Newton's second law of motion applied to rotation. Since the friction torque can be two orders of magnitude lower than $T_{em}$ and $T_m$, it is often neglected:

$$J\frac{\mathrm{d}\Omega_m}{\mathrm{d}t} = T_{em} - T_m - k_F\Omega_m. \tag{12.8}$$

## 12.4 Mathematical Model

Equations derived so far represent the mathematical model of DC machine. The model can be used for analysis of transient processes and steady states, and it is also called *dynamic model*. A concise review of these equations is presented here.

Voltage balance in the excitation winding:

$$u_f = R_f i_f + \frac{\mathrm{d}\psi_f}{\mathrm{d}t} = R_f i_f + L_f\frac{\mathrm{d}i_f}{\mathrm{d}t}.$$

Voltage balance in the armature winding:

$$u_a = R_a i_a + L_a\frac{\mathrm{d}i_a}{\mathrm{d}t} + E_a.$$

Expressions for the electromotive force and torque:

$$E_a = k_e\Phi_f\Omega_m,$$
$$T_{em} = k_e\Phi_f i_a.$$

Relation of excitation current to excitation flux:

$$\Phi_f = L_f' i_f.$$

Newton equation:

$$J\frac{d\Omega_m}{dt} = T_{em} - T_m - k_F\Omega_m.$$

## 12.5 DC Machine with Permanent Magnets

Figure 11.1 shows cross sections of DC machine with excitation winding on the stator and DC machine with permanent magnets on the stator. Excitation flux can be obtained by using a DC current $I_f$ in excitation winding, which creates the magnetomotive force and flux along direct axis of the machine. The machine can also be made without an excitation winding. Instead, permanent magnets can be inserted instead of main poles to provide the excitation flux. The advantage of having permanent magnets is the absence of the excitation winding. There is no need to have a separate power supply for the excitation winding. At the same time, the overall efficiency of the machine is increased due to absence of copper losses in the excitation winding. A disadvantage of DC machines with permanent magnet excitation is that the flux cannot be changed. The flux is defined by $B(H)$ characteristics of the magnets and by the magnetic resistance of the magnetic circuit. Specifically, the flux is closely related to the remanent magnetic induction of permanent magnets. Hence, the permanent magnet excitation is not suitable for applications requiring variable flux. In machines with excitation winding, excitation flux can be varied by changing the excitation voltage and current.

Mathematical model of DC machines with permanent magnets is obtained by removing one differential equation from the model derived in the preceding section. The flux $\Phi_f$ is constant and determined by characteristics of the magnet. Since a DC machine with permanent magnets does not have an excitation winding, the differential equation describing the voltage balance in this winding is omitted.

## 12.6 Block Diagram of the Model

The mathematical model can be presented in the form of a diagram, shown in Fig. 12.3. Individual blocks in this diagram contain transfer functions obtained by applying Laplace transform to differential equations of the model. As an example, voltage balance differential equation of the excitation winding has time domain form of

$$u_f = R_f i_f + L_f\frac{di_f}{dt}.$$

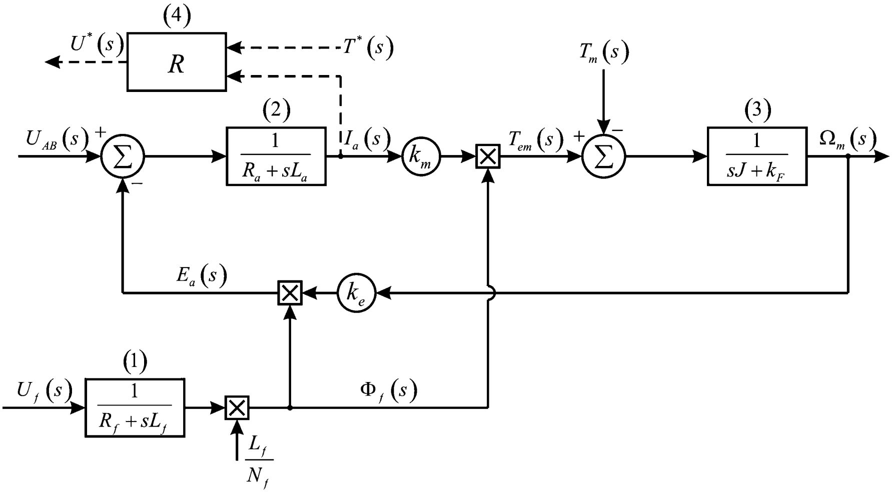



**Fig. 12.3** Model of a DC machine presented as a block diagram

Application of Laplace transform to this differential equation results in an algebraic equation with complex images of the excitation current and excitation voltage. In equation

$$U_f(s) = R_f I_f(s) + sL_f I_f(s) - i_f(0),$$

$s$ denotes *Laplace operator*, that is, *differentiation operator*, while $U_f(s)$ and $I_f(s)$ are complex images of the originals $u_f(t)$ and $i_f(t)$. Assuming that initial value of the excitation current is $i_f(0) = 0$, the equation takes the form

$$U_f(s) = \left(R_f + sL_f\right) I_f(s). \tag{12.9}$$

Excitation winding is a subsystem with excitation voltage at the input. The voltage is used as the control variable which determines the change of excitation current. The excitation current comes as a consequence or reaction to the excitation voltage. Therefore, the voltage is a control input, while the current is the output of the considered system. On the basis of the previous equation, the transfer function of block (1) in Fig. 12.3 is $I_f(s)/U_f(s) = 1/(R_f + sL_f)$. Block (2) represents the transfer function of the armature winding, while block (3) is Newton differential equation.

The torque and flux of DC machine depend on the excitation and armature voltages, and this is shown on the left-hand side of the diagram. The excitation current and flux are dependent on the excitation voltage, and they vary according to transfer function $I_f(s)/U_f(s) = 1/(R_f + sL_f)$. Electrical time constant $\tau_f = L_f/R_f$ of the excitation winding ranges between 200 ms and 10 s. Variation of the armature current depends on the voltage difference between the external voltage and induced electromotive force ($U_a - E_a$). Variation of the current d$i_a(t)$/d$t$ is positive when $U_a - E_a - R_a\ i_a(t) > 0$; otherwise, variation of the current is negative. Time constant of the armature winding $\tau_a = L_a/R_a$ ranges from 1 up to 100 ms.

The excitation voltage and armature voltage are *inputs* to the system. They are *control* variables that affect the state of DC machine. The armature current, excitation current, as well as the rotor speed are *state variables* of the considered dynamic system. The state variables are the reaction of the system to the external control variables. Variables such as the armature current, excitation flux, electromagnetic torque, and the rotor speed are the system *outputs*.

Electromagnetic torque $T_{em}$ is equal to the product of the excitation flux, armature current, and constant $k_m$. Torque $T_{em}$ represents input variable of the mechanical subsystem, that is, control force which determines variation of the rotor speed. Speed of rotation increases when $T_{em}$ exceeds the sum of all torques resisting the movement. When the electromagnetic torque equals the sum of resisting torques, $T_m + k_F\Omega_m$, the rotor speed remains constant. If $T_{em} < T_m + k_F\Omega_m$, the rotor speed decreases. It should be noted that block (3) of the diagram corresponds to the friction torque $k_F\Omega_m$. The friction torque is usually smaller than the rated torque by two orders of magnitude. Therefore, friction is often neglected, and the transfer function is represented by $1/(J\cdot s)$.

## 12.7 Torque Control

Block (4) in Fig. 12.3 is denoted by R, and it does not belong to the mathematical model of DC machine. Connections of this block are made by dotted lines. This block illustrates the possibility of controlling the torque of the machine, which is discussed here. DC motors are often used in motion control applications, where they provide the means for controlling the speed and position of tools and workpieces in automated production lines. They are also used for running elevators, conveyors, and similar devices. In motion control tasks, electrical motors are used to provide a variable torque $T_{em}$ which should be equal to the torque reference $T^*$, calculated within the motion controller which is not shown in the figure. The torque reference $T^*$is determined so as to overcome the motion resistances and ensure desired speed and/or position. Its change depends on desired speed changes and on forces and torques resisting the motion. The torque $T_{em}$ should be as close to the reference $T^*$ as possible. *Torque control* implies a set of actions and measures conceived to maintain the electromagnetic torque $T_{em}$ at the desired reference value $T^*$. In cases where the reference changes, controlled variable $T_{em}$ should track these changes. The torque $T_{em}$ is proportional to the product of the armature current and the flux. At the first glimpse, the torque control can be done either by changing the flux or by changing the armature current. Yet, only the later approach is used in practice. This is due to the fact that the flux changes are rather slow. Moreover, flux control is not available with DC machines having permanent magnet excitation. With DC machines having an excitation winding, the time constant of the armature winding $\tau_a$ is considerably smaller than the time constant of the excitation winding. While only slow variations of the flux are possible, the current $i_a$ can be changed quickly. Thus, regulation of the torque implies regulation of the armature current. The speed of the torque response is defined by the speed of response of the armature current. Starting from the voltage balance equation of the armature winding, variation of the current is determined by equation

$$\frac{\mathrm{d}i_a}{\mathrm{d}t} = \frac{1}{L_a}(u_a - R_a i_a - E_a). \tag{12.10}$$

Therefore, variations of the armature current can be accomplished by varying the armature voltage $u_a$. For this reason, DC motors are supplied from static power converters, power electronic devices that provide variable armature voltage. For the purpose of current control, it is necessary to measure the armature current and compare it to the reference in order to establish the error $\Delta i_a = i^* - i_a$. If the armature current is below the reference ($\Delta i_a > 0$), armature voltage should be increased in order to obtain $\mathrm{d}i_a/\mathrm{d}t > 0$. From (12.10), it can be concluded that increasing armature voltage leads to increasing armature current; thus, error $\Delta i_a$ is reduced. In a like manner, if the current is too high ($\Delta i_a < 0$), the voltage should be reduced. The algorithm that calculates the control variable $u^*$ from the error $\Delta i_a$ is called *control algorithm*. Device or block diagram which implements such algorithm is called *regulator* or *controller*. Regulator can often be described by transfer function. The error $\Delta i_a$ is an input to the regulator, while the control variable $u^*$ is the output. Control algorithm affects the speed and character of the system dynamic response. Block (4) in Fig. 12.3 indicates the method of connecting such regulator. The regulator output $u^*$ represents the desired armature voltage. This voltage reference is fed to the static power converter which supplies the armature winding. A more detailed analysis of the regulation problem is beyond the scope of this book. Hence, design of the regulator structure and setting of its parameters are left out of discussion.

Design of regulators and controllers requires some basic knowledge on transient processes in electrical machines and their mathematical models. These models and processes are studied and exercised in this book.

## 12.8 Steady-State Equivalent Circuit

It is of interest to analyze the steady-state operation of DC machines. In steady state, there are no changes in the rotor speed nor in electrical currents in the windings. During transients, instantaneous value of electrical current is denoted by $i(t)$, while in steady state it is denoted by $I$. Steady state in excitation winding is defined by equation $U_f = R_f I_f$. This relation is represented by the equivalent circuit

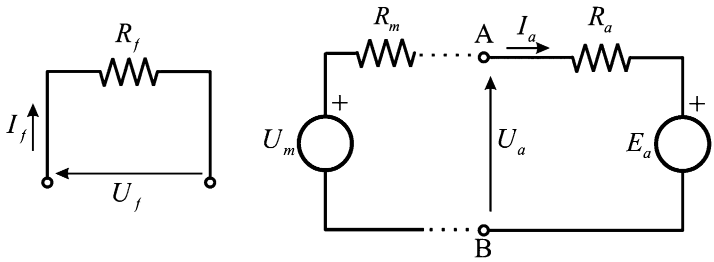



**Fig. 12.4** Steady-state equivalent circuits for excitation and armature winding

given in the left-hand part of Fig. 12.4. The voltage balance equation of the armature winding is given by expression

$$U_a = R_a \cdot I_a + k_e \cdot \Phi_f \cdot \Omega_m.$$

Therefore, the steady-state value of the armature current is $I_a = (U_a - E_a)/R_a$, where $U_a$ is the voltage fed to the brushes. This equation can be represented by the steady-state equivalent circuit given in the right-hand part of Fig. 12.4. The circuit can be used for determination of the current, torque, and power of a DC machine. Given the rotor speed and the excitation flux, one can calculate the electromotive force, find the difference $U_a - E_a$, determine the armature current $I_a$, and find the torque and power. In cases where the voltage and current are known, the equivalent circuit can be used to calculate the electromotive force and determine the rotor speed according to expression $\Omega_m = E_a/(k_e\Phi_f)$.

The voltage $U_a$ between the brushes A and B in Fig. 12.4 is equal to $U_a = R_aI_a + E_a$. When DC machine is used as a motor, it is supplied from an external source of DC voltage. This source is shown in Fig. 12.4. It has internal resistance $R_m$ and no load voltage $U_m$. With $R_m \approx 0$, the armature voltage $U_a$ is approximately equal to $U_m$. Further on, whenever the armature voltage is supplied from an external source $U_m$, it is assumed that $U_a = U_m$.

**Question (12.3):** For a DC machine, it is known that $k_e\Phi_f = k_m\Phi_f = N_R\Phi_f/(2\pi) =$ 1 Wb. Rotor shaft is coupled to a work machine which resists the motion and provides the load torque $T_m$. Machine runs in steady state, where the electromagnetic torque $T_{em}$ is equal to the load torque $T_m$. The rotor speed is constant and equal to $\Omega_m = 100$ rad/s. The armature winding is fed from a voltage source $U_a = 110$ V. Equivalent resistance of the armature winding is $R_a = 1\ \Omega$. (1) Determine the electromagnetic torque, power delivered by the source, and power of electromechanical conversion. (2) Assuming that the rotor shaft is decoupled from the work machine, and a new steady state is reached, determine the rotor speed. (3) Assume that the rotor shaft is coupled to the work machine which maintains the rotor speed at $\Omega_m = 100$ rad/s, notwithstanding changes in electromagnetic torque $T_{em}$. If the source voltage is reduced to 90 V, determine the electromagnetic torque and power of electromechanical conversion in new steady-state conditions.

**Answer (12.3):**

(1) $I_a = 10$ A, $T_{em} = 10$ Nm, $P_{source} = 1{,}100$ W, $P_{em} = 1{,}000$ W.
(2) $T_{em} = 0 \rightarrow I_a = 0 \rightarrow U_a = E_a \rightarrow \omega_m = 110$ rad/s.
(3) $I_a = -10$ A, $T_{em} = -10$ Nm, $P_{em} = -1{,}000$ W; machine is operating in the generator mode.

## 12.9 Mechanical Characteristic

Mechanical characteristic of a DC machine is a curve in $T_{em}$–$\Omega_m$ plane which relates the torque and speed in steady-state operation, where the load torque $T_m$, the armature voltage $U_a$, and the excitation voltage $U_f$ remain constant and do not change. It can be expressed either as $T_{em}(\Omega_m)$ or as $\Omega_m(T_{em})$.

The following considerations assume that a DC machine runs in the steady state. It has constant excitation flux and constant voltage $U_a$ applied to the armature winding. In Fig. 12.5, it is shown that an external voltage source $U_m$ is connected to the brushes, providing the required armature voltage. The voltage balance equation is $U_m = U_a = R_a I_a + k_e \Phi_f \Omega_m$. In conditions where the armature current is equal to zero, the electromotive force is equal to the supply voltage. Therefore, with $I_a = 0$, the rotor angular speed is $\Omega_0 = U_m/(k_e \Phi_f)$. In this condition, the electromagnetic torque $T_{em}$ is equal to zero as well. Therefore, the speed $\Omega_0$ is called *no load speed*. With constant supply voltage $U_m$ and with $E_a = k_e \Phi_f \Omega_m = U_m - R_a I_a$, any increase in armature current reduces the electromotive force. With constant flux, the electromotive force is proportional to the rotor speed. Therefore, an increase in armature current decreases the rotor speed. On the other hand, the electromagnetic torque $T_{em}$ is proportional to the armature current. In steady-state conditions, $T_{em} = T_m = k_m \Phi_f I_a$. Hence, any increase in the load torque decreases the rotor speed.

The following considerations assume that DC machine runs in the steady state. Therefore, the rotor speed is constant; hence, $J \cdot d\Omega_m/dt = 0$. With friction torque being neglected, the load torque $T_m$ is equal to the electromagnetic torque $T_{em}$. Therefore, the Newton equation reduces to $T_{em} = T_m + k_F . \Omega_m \approx T_m$. Hence, in the steady state and with no friction, the electromagnetic torque matches the load torque. Therefore, the armature current $I_a = T_{em}/(k_m \Phi_f)$ is proportional to the

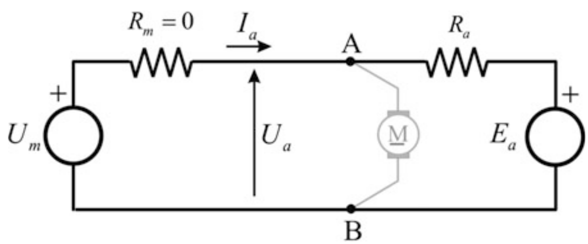



**Fig. 12.5** Supplying armature winding from a constant voltage source

load torque $T_m$. For that reason, an increase in the load torque $T_m$ results in an increase of the armature current, hence decreasing the rotor speed.

Previous considerations are drawn for steady-state operation of DC machine with constant supply voltages and constant load torque. They show that the rotor speed decreases with the load torque and, hence, with the electromagnetic torque. Steady-state relation of the speed and torque is called *mechanical characteristic*, and it can be represented by a curve in $T_{em}$–$\Omega_m$ or $\Omega_m$–$T_{em}$ plane. While the steady-state equivalent circuit relates the voltages and currents at electrical terminals of a DC machine, the mechanical characteristic relates the rotor speed and torque at the rotor shaft, wherein the shaft represents the mechanical access to the machine. The mechanical characteristic depends on the supply voltages and DC machine parameters. Variations in armature voltage $U_a$ affect the no load speed $\Omega_0$ and alter the mechanical characteristic $T_{em}(\Omega_m)$. Variation in excitation voltage changes the excitation current and flux, changing in such way the mechanical characteristic.

Mechanical characteristic can be determined by starting from voltage balance equation for the armature winding, given in (12.6). Calculation of function $T_{em}(\Omega_m)$ starts with $T_{em} = k_m \Phi_f I_a$. It is required to express the armature current in terms of the rotor speed. From voltage balance equation $U_a = R_a I_a + k_e \Phi_f \Omega_m$, depicting the voltage balance in armature winding in the steady state, the armature current is found to be

$$I_a = \frac{U_m - E_a}{R_a} = \frac{U_m - k_e\,\Phi_f \Omega_m}{R_a},$$

and the electromagnetic torque is calculated according to expression

$$\begin{aligned} T_{em} &= k_m\,\Phi_f \frac{U_m - k_e\,\Phi_f \Omega_m}{R_a} = k_m\,\Phi_f \frac{U_m}{R_a} - \frac{k_m k_e\,\Phi_f^2}{R_a}\Omega_m \\ &= T_0 - S \cdot \Omega_m. \end{aligned} \tag{12.11}$$

Torque $T_0 = k_m \Phi_f U_m / R_a$ in (12.11) is *start-up torque*, the electromagnetic torque developed by DC machine when the rotor is at standstill. If the rotor speed is zero, the induced electromotive force is zero as well, and the armature winding has *start-up* current $I_0 = U_m / R_a$. It will be shown later that the start-up current has very large values which could damage the machine or power supply feeding the machine. In order to restrain the armature current to acceptable values, at low speeds, it is necessary to reduce the armature voltage. This can be accomplished by adjusting the power supply voltage $U_m$ according to the rotor speed. In cases where the power supply is not adjustable, it is necessary to connect a series resistor in order to increase the total resistance $\Sigma R$ of the armature circuit and hence limit the current. Parameter $S$ in expression (12.11) is the slope or the *stiffness* of mechanical characteristic. Slope $S$ of mechanical characteristic $T_{em} = T_0 - S\Omega_m$ determines the ratio between the torque change $\Delta T_{em}$ and the speed change $\Delta \Omega_m$. Large stiffness means that small variations of the rotor speed would result in large variations of the torque.

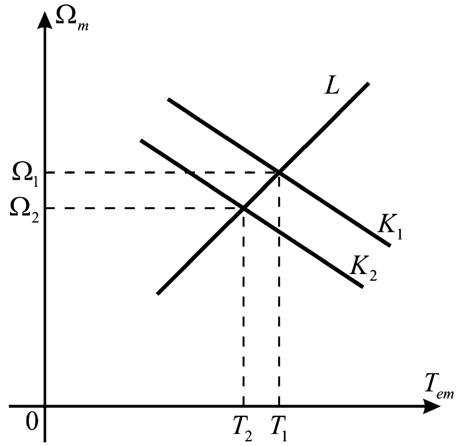



**Fig. 12.6** Steady state at the intersection of the machine mechanical characteristics and the load mechanical characteristics

Similarly to electrical machines which have their own mechanical characteristic $T_{em}(\Omega_m)$, work machines connected to the rotor shaft have mechanical characteristics $T_m(\Omega_m)$ of their own, determining the change of the load torque with the rotor speed. Characteristic $T_m(\Omega_m)$ is called *load characteristic*. Since the steady state is reached with $T_m = T_{em}$, the steady-state operating point in $T$–$\Omega_m$ plane is at the intersection of the two mechanical characteristics, the one of the electrical machine and the one of the load. In Fig. 12.6, the load characteristic is represented by a straight line $L$. There are two different mechanical characteristics of a DC machine shown in the figure, $K_1$ and $K_2$. The characteristics $K_1$ and $K_2$ are obtained for different supply voltages of the armature winding. If DC machine has characteristic $K_1$, steady state is reached at point ($T_1$, $\Omega_1$). With characteristic $K_2$, steady state is reached at point ($T_2$, $\Omega_2$). Therefore, it is possible to change mechanical characteristic of the machine and change the steady-state speed and torque by varying the armature supply voltage $U_m = U_a$.

## *12.9.1 Stable Equilibrium*

The equilibrium reached at the intersection of the two mechanical characteristics can be stable or unstable. When the operating point is displaced from the stable equilibrium by action of external disturbances, it returns to the same point after certain transient phenomena. The unstable equilibrium is retained only in the absence of disturbances. When the operating point is displaced from the unstable equilibrium, it does not return to the same point. An example of an unstable equilibrium is a ball positioned precisely at the peak of the hill. Left alone, it remains at the peak. Any disturbance would move the ball OFF the peak and make it roll all the way down the slope.

Stability of the steady-state operating point depends on the stiffness of the two mechanical characteristics. From the mechanical characteristic of DC machine $T_{em} = T_0 - S_{em}\Omega_m$ and load characteristic $T_m = T_{0m} - S_m\Omega_m$, it can be concluded that the speed change $\Delta\Omega_m$ results in the electromagnetic torque change of $\Delta T_{em} = -S_{em}\Delta\Omega_m$ and the load torque change of $\Delta T_m = -S_m\Delta\Omega_m$. The influence of parameters $S_{em}$ and $S_m$ on dynamic behavior of the system comprising one DC machine and one work machine can be studied by starting from the steady-state operating point where the DC machine runs at angular speed $\Omega_1$ and develops electromagnetic torque $T_1$, while the load machine resists to motion by the same torque, $T_m = T_1$. In steady state, the speed does not change, and Newton equation reads

$$J\frac{\mathrm{d}\Omega_m}{\mathrm{d}t} = T_{em} - T_m = T_1 - T_1 = 0.$$

The system is susceptible to external disturbances that may produce small changes of the torque and speed. If a small variation of the rotor speed $\Delta\Omega_m$ occurs for any reason, the rotor speed becomes $\Omega_1 + \Delta\Omega_m$, while the electromagnetic torque changes to $T_1 - S_{em}\Delta\Omega_m$. At the same time, the load torque becomes $T_1 - S_m\Delta\Omega_m$. Since $\Omega_1$ is a constant, Newton equation becomes

$$\begin{aligned} J\frac{\mathrm{d}(\Omega_1 + \Delta\Omega_m)}{\mathrm{d}t} &= J\frac{\mathrm{d}\Delta\Omega_m}{\mathrm{d}t} = T_{em} - T_m \\ &= T_1 - S_{em}\Delta\Omega_m - T_1 + S_m\Delta\Omega_m \\ &= (S_m - S_{em})\Delta\Omega_m. \end{aligned} \tag{12.12}$$

With $S_m - S_{em} > 0$, a positive value of $\Delta\Omega_m$ gives a positive value of the first derivative $\mathrm{d}(\Delta\Omega_m)/\mathrm{d}t$. Therefore, disturbance $\Delta\Omega_m$ will progressively increase. A negative disturbance $\Delta\Omega_m$ gives a negative value of the first derivative $\mathrm{d}(\Delta\Omega_m)/\mathrm{d}t$. In this case, disturbance will progressively advance toward negative values of ever larger magnitude. Hence, the steady-state operating point with $S_m - S_{em} > 0$ is unstable. Namely, any disturbance, whatever the size and however small, puts the system into instability.

With $S_m - S_{em} < 0$, a positive value of $\Delta\Omega_m$ gives a negative value of the first derivative $\mathrm{d}(\Delta\Omega_m)/\mathrm{d}t$. Therefore, disturbance $\Delta\Omega_m$ will decrease and gradually converge toward zero, bringing the system to the original steady-state operating point. This dynamic behavior is called *stable* since the system returns to the initial steady state after being disturbed and moved from the equilibrium. On the other hand, systems that progressively move away from the initial state and do not return are called *unstable*.

**Question (12.4):** Starting from (12.12) and assuming that stiffness of the characteristic and inertial torque are known, and that $\Delta\Omega_m(0) = A$, determine the change $\Delta\Omega_m(t)$.

**Answer (12.4):** Solution of differential equation $dy/dx = ay$ is $y(x) = y(0) \cdot e^{ax}$. In (12.12), $y = \Delta\Omega_m$, $y(0) = A$, $x = t$, and $a = (S_m - S_{em})/J$. Therefore, the change of $\Delta\Omega_m$ is determined by expression

$$\Delta\Omega_m(t) = \Delta\Omega_m(0) \cdot e^{\frac{S_m - S_{em}}{J}t}.$$

## 12.10 Properties of Mechanical Characteristic

Mechanical characteristic of a DC machine is shown in Fig. 12.7, including the intersection with the abscissa and ordinate. The intersection with the ordinate represents no load speed $\Omega_0$. This speed is achieved when the electromagnetic torque is equal to zero. With zero torque, the armature current is equal to zero as well. In the absence of the voltage drop $R_a I_a$, the electromotive force $k_e \Phi_f \Omega_m$ is equal to the armature voltage. On the basis of (12.11), no load speed is

$$\Omega_0 = \frac{T_0}{S} = \frac{U_m}{k_e \Phi_f}. \tag{12.13}$$

The intersection with the abscissa represents the initial torque $T_0$ which is developed when the rotor is at standstill. The initial torque is equal to

$$T_0 = k_m \Phi_f \frac{U_m}{R_a}. \tag{12.14}$$

The slope of the mechanical characteristic determines the ratio of $\Delta T_{em}$ and $\Delta\Omega_m$, as shown in Fig. 12.7. Mechanical characteristic is often represented by the function $T_{em}(\Omega_m) = T_0 - S\Omega_m$. In other words, the stiffness $S$ is considered positive if the

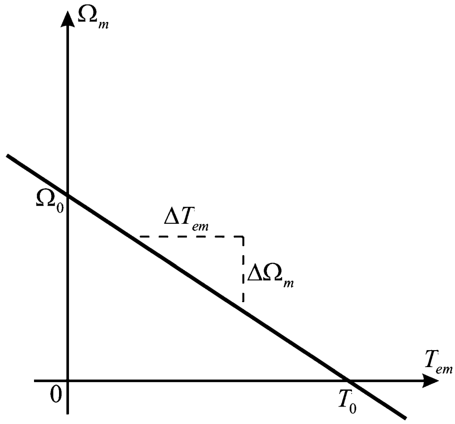



**Fig. 12.7** No load speed and nitial torque

torque drops as the speed increases. Therefore, the slope is determined according to expression

$$S = -\frac{\Delta T_{em}}{\Delta \Omega_m}. \tag{12.15}$$

Slope of the mechanical characteristic of a DC machine with the armature power supply as shown in Fig. 12.5 is equal to

$$S = \frac{k_m k_e \Phi_f^2}{R_a}. \tag{12.16}$$

Mechanical characteristic can be also represented by function $\Omega_m = f(T_{em})$. Equation (12.11) can be presented in the form

$$\Omega_m = \Omega_0 - \frac{1}{S} T_{em}. \tag{12.17}$$

Power supply of the armature winding has a finite internal resistance, as well as the conductors connecting the supply to the brushes. In addition, the armature circuit may have a resistor inserted in series with the purpose of reducing the initial current and initial torque. At the same time, series resistor may be used to alter the mechanical characteristic and change the rotor speed. In practice, total resistance in the armature circuit is higher than the equivalent resistance $R_a$ of the armature winding and mechanical collector. For this reason, expressions (12.11), (12.14), and (12.16) should use $\Sigma R$ instead of $R_a$. Notation $\Sigma R$ represents the sum of all resistances in the armature circuit, namely, the sum of internal resistance of the power source, resistance of wiring and connections, inserted series resistances, and the equivalent resistance of the armature winding, collector, and brushes.

## 12.11 Speed Regulation

In cases without inserted series resistances, total resistance of the armature circuit $\Sigma R$ is very small. In practice, resistance $R_a$ of DC machines in conjunction with usually encountered armature currents $I_a$ results in a voltage drop $R_a I_a$ of only $U_a$/1,000 .. $U_a$/100, where $U_a$ is the armature voltage in most common operating conditions. Hence, the value of $R_a$ ranges from $(U_a/I_a)$/1,000 to $(U_a/I_a)$/100. Therefore, the slope $S$ of the mechanical characteristic is relatively high. This means that, during variations of the torque, variations of the speed will be very small. From (12.17), a high value of the slope $S$ of the mechanical characteristic ensures that the rotor speed has only slight changes and remains close to the no load speed. According to (12.13), no load speed is determined by the armature voltage $U_a = U_m$. Therefore, the speed can be changed by varying the armature voltage.

In cases where the rotor speed is to be varied while the power source voltage $U_m$ is constant, the armature voltage and the speed can be changed by inserting a variable series resistance $R_{ext}$ in the armature circuit. In this way, the armature voltage is reduced to $U_a = U_m - R_{ext}I_a$, and this reduces the rotor speed. Insertion of a variable series resistance is a simple but inefficient way of controlling the rotor speed. The power losses due to Joule effect in series resistance are proportional to the square of the armature current. More efficient way of controlling the speed is the use of power source that provides variable voltage $U_m$. Continuous and lossless change of the armature voltage is feasible with static power converters that employ semiconductor power switches.

In conditions where $R_{ext} = 0$, variation of the supply voltage $U_m$ changes the no load speed and maintains the slope of the mechanical characteristic. Equation (12.16) proves that changes in $U_m$ do not affect the slope $S$ of the mechanical characteristic. On the other hand, no load speed $\Omega_0$ is proportional to the supply voltage (12.13). By changing the supply voltage $U_m$, a family of mechanical characteristics is obtained, all of them having the same slope. Supply voltage $U_m$ determines the intersection of each of these characteristics with the $\Omega_m$ axis of the $T_{em}$–$\Omega_m$ plane, as shown in Fig. 12.8. As a matter of fact, changes in the armature supply voltage result in translation of the mechanical characteristic in direction of $\Omega_m$ axis. Translation of the mechanical characteristic can be used to change the intersection with the load characteristic and, hence, change the running speed for the given load. In other words, the rotor speed can be changed by altering the armature supply voltage. In Fig. 12.8, mechanical characteristics $K_1$, $K_2$, $K_3$, and $K_4$ are given, each one obtained with different armature voltage. Characteristic $K_4$ is obtained for the case when the armature supply voltage is equal to zero. This characteristic passes through the origin.

Diagram in Fig. 12.8 is divided in four quadrants. In quadrant I, the rotor speed and electromagnetic torque both have positive values. Their product represents the power of electromechanical conversion, and it has positive value in the first quadrant, where the electrical machine operates as a motor. In the second and fourth quadrants, direction of the electromagnetic torque is opposite to direction of the rotor speed. In these quadrants, torque and speed have opposite signs, and their product assumes a negative value. In these quadrants, the power of electromechanical conversion is negative, and the machine operates as a generator. In generator mode, electrical machine creates electromagnetic torque which resists the motion, namely, it *brakes* and acts toward decreasing the rotor speed. To keep the rotor running, generator requires water turbines, steam turbines, or other similar devices that provide the driving torque that runs the rotor and maintains the rotor speed. In the third quadrant, the machine operates in the motor mode, quite like in the first quadrant. The difference is that both torque and speed in the third quadrant are negative.

The need for DC machines to operate in one or more quadrants depends upon the mechanical load or work machine used in actual application. A DC machine can be used to run a fan in a blower application. Direction of the air flow does not change. For that reason, DC motor runs in the same direction, without a need to change direction of the rotor speed. The air resistance produces the load torque which is

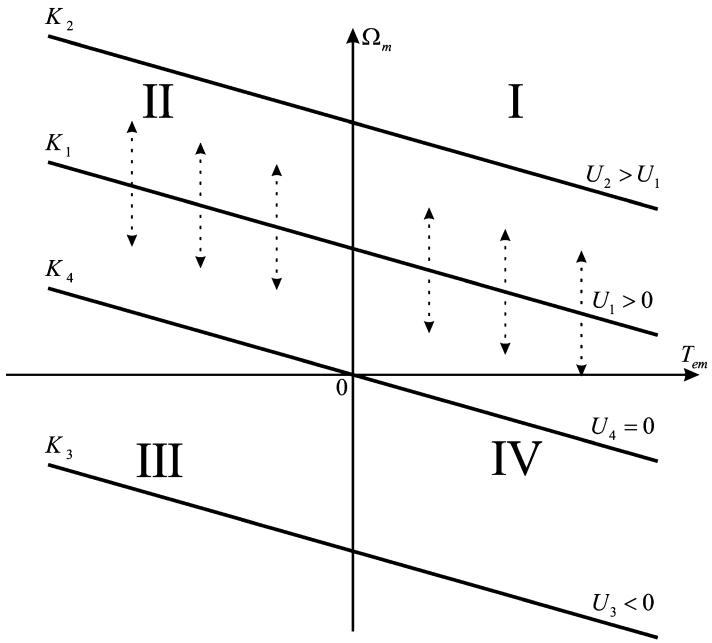



**Fig. 12.8** The impact of armature voltage on mechanical characteristic

opposite to the rotor speed. Hence, direction of the torque $T_m = T_{em}$ does not change either. Hence, a DC machine used in typical blower application operates in the first quadrant.

A number of DC machines are used to control the motion of various parts, tools, objects, or vehicles. The motion is usually made in both directions. The speed has positive sign while moving in one direction and negative sign while moving in the other direction. Moreover, each motion cycle starts with an acceleration phase, where the speed increases, and ends with a braking phase, when the speed is decreased and brought back to zero (Fig. 12.10). In acceleration, the torque has the same direction as the speed, while in braking phase, the torque changes direction and acts against the speed. Hence, motion from one position to another involves the torque of both directions and the speed in only one direction. Coming back to the original position (Fig. 12.10) involves the speed of the opposite direction. Therefore, a forth-and-back motion requires the speed and torque changes with all the four possible combinations of signs, ($T_{em} > 0$, $\Omega_m > 0$), ($T_{em} < 0$, $\Omega_m > 0$), ($T_{em} > 0$, $\Omega_m < 0$), and ($T_{em} < 0$, $\Omega_m < 0$). In other words, it is required to accomplish the *four-quadrant operation*.

**Question (12.5):** A work machine resists the motion by developing the torque $T_m = 0{,}001\ \Omega^2$. For a DC motor with independent excitation and with constant excitation flux, the following parameters are known: $R_a = 0.1\ \Omega$, $k_m \Phi_f = 1$ Wb, and $U_a = 100$ V. Determine speed of rotation in steady state.

**Answer (12.5):** Steady-state values of the electromagnetic torque $T_{em}$ and the load torque $T_m$ are equal. The steady-state speed corresponds to the intersection of the mechanical characteristic of the motor and the load characteristic. On the basis of (12.11), the electromagnetic torque is $T_{em} = T_0 - S\Omega_m$. For the given parameters, $T_0 = 1 \cdot 100/0.1$ Nm $= 1{,}000$ Nm, while $S = 1 \cdot 1/0.1$ Nm·s/rad. Equation $T_{em}(\Omega_m) = T_m(\Omega_m)$ results in a quadratic equation in terms of $\Omega_m$. Positive solution of this quadratic equation is 99.02 rad/s, which is the speed of the considered system at steady state.

## 12.12 DC Generator

DC machines can operate in the generator mode. If the rotor is put to motion by means of a steam or hydroturbine, the machine receives mechanical work which is converted to electrical energy. Mechanical power supplied to the shaft is the product of the rotor speed and the turbine torque $T_T$ which keeps the rotor in motion and maintains the speed. With rotor in motion, the electromotive force $E_a = k_e\Phi_f\Omega_m$ is induced in the armature winding, and it is available between the brushes. The voltage $U_a$ can be used to supply DC electrical loads such as the light bulbs, heaters, and similar. The armature voltage of DC generator is often denoted by $U_G$. With a resistive load connected between the brushes, the load current is established in direction which is opposite to the adopted reference direction of the armature current $I_a$. Respecting the reference direction of the armature current, electrical current in generator mode has negative sign. For this reason, analysis of DC generators is often made by assuming a new reference direction of the current, opposite to the one used in motoring mode. Equivalent circuit in Fig. 12.9 includes electrical current $I_G = -I_a$ which circulates from brush B to brush A. Brush A represents positive pole of the voltage supplied to electrical consumers. Starting from equation

$$U_G = U_a = R_a I_a + k_e\Phi_f\Omega_m,$$

and introducing substitution $I_G = -I_a$, one obtains

$$U_G = k_e\Phi_f\Omega_m - R_a I_G = E_a - R_a I_G, \tag{12.18}$$

which determines variation of the generator voltage as function of consumer current $I_G$. Starting from (12.18), the current–voltage characteristic is obtained, given in Fig. 12.9. No load voltage is equal to $E_a$. Slope $\Delta U/\Delta I$ determines the voltage drop experienced by electrical consumers. The voltage drop $\Delta U$ is proportional to the consumer current. The slope $\Delta U/\Delta I$ is equal to the armature resistance $R_a$. In cases when the electrical load is connected over long lines with considerable resistance, the slope $\Delta U/\Delta I$ is equal to the sum of the armature resistance and the line resistances.

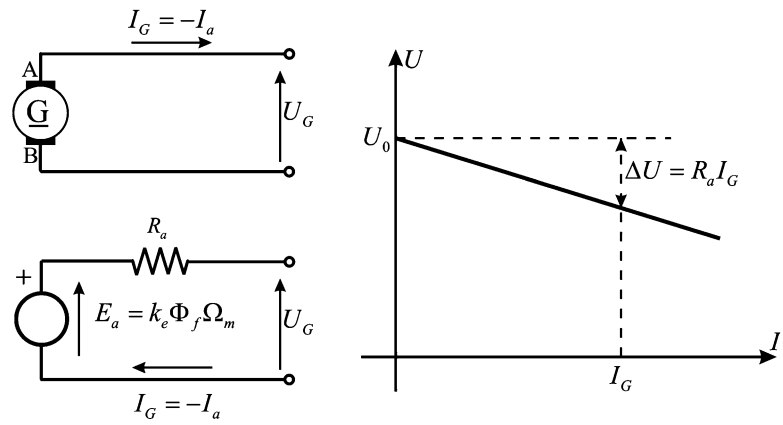



**Fig. 12.9** Voltage–current characteristic of a DC generator

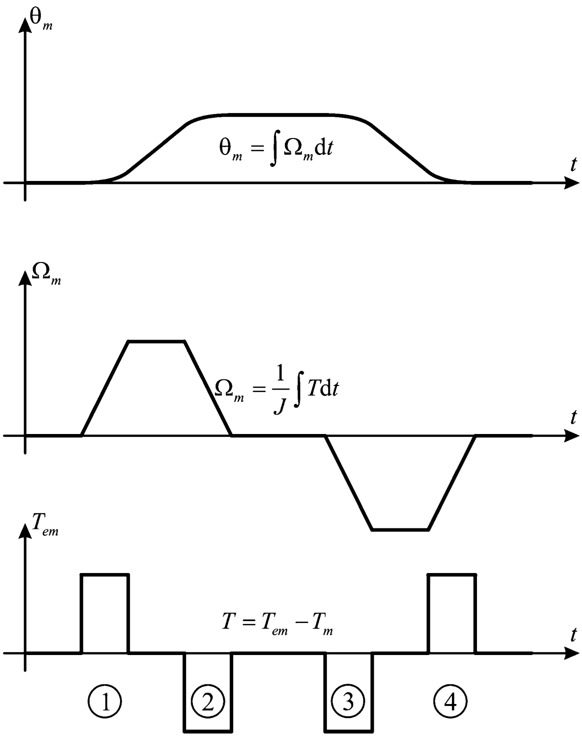



**Fig. 12.10** Variations of the position, speed, and torque within one cycle

When supplying electrical consumers and loads operating with DC currents, it is of interest to keep the supply voltage constant. Due to the voltage drop $\Delta U$ caused by a finite resistance of the armature winding, the voltage across consumers depends on the load current. In order to keep the voltage $U_G$ at the desired value notwithstanding the changes in electrical current, it is required to increase the electromotive force and maintain the voltage $E_a - R_a I_G$. In practical applications, it is necessary to measure the voltage $U_G$, to compare the measurement with the desired value, and then to adjust the excitation voltage and current in order to obtain the excitation flux which results in desired electromotive force. The voltage across the load $U_G = E_a - R_a I_G$ remains constant if the changes in the voltage drop $R_a I_G$ are matched and compensated for by contemporary changes of the electromotive force $E_a$. In the prescribed way, it is possible to achieve the voltage regulation of a DC generator.

When a DC generator supplies resistive load which absorbs the current $I_G > 0$, the armature current is then $I_a = -I_G < 0$. Since the armature current $I_a$ is negative, the electromagnetic torque $T_{em}$ is negative as well. With $T_{em} < 0$, the electromagnetic torque resists the motion and acts against the rotor speed. In other words, DC machine provides a *braking* torque. Therefore, the power of electromechanical conversion $P_{em} = T_{em}\Omega_m$ is negative. This means that the mechanical work is being converted to electrical energy. The generator receives mechanical work from a driving turbine by means of the shaft. The product of the driving torque $T_T$ of the turbine and the rotor speed represents mechanical power which is delivered to the machine. The torque $T_T$ acts in the direction opposite to the previously adopted reference direction for the load torque $T_m$. With reference directions shown in Fig. 12.1, the generator mode implies $T_{em} < 0$ and $T_m < 0$.

**Question (12.6):** A hydroturbine drives DC generator at angular speed of $\Omega_m = 100$ rad/s. The parameters $R_a = 1\ \Omega$ and $k_m\Phi_f = 1$ Wb are known, while the resistance of the load connected between the brushes is $R_L = 4\ \Omega$. Determine the voltage across the load, the torque $T_T = -T_{em}$ delivered to the rotor by the turbine, the turbine power $P_T = T_T\omega_m$, and the power $P_G = U_a I_G$ delivered to the load. Why is $P_T > P_G$?

**Answer (12.6):** Electromotive force of the generator is $E_a = 100$ V. Generator current is $I_G = -I_a = 100$ V/(1 Ω + 4 Ω) = 20 A. Voltage across the consumer is 80 V. The electromagnetic torque is $-1{\cdot}20$ Nm = $-20$ Nm. The turbine torque is $T_T = -T_{em} = 20$ Nm. The turbine power is $P_T = 2{,}000$ W. The power delivered to the consumer is $P_G = 1{,}600$ W. The difference $R_a I_a^2 = 400$ W is converted to heat in the rotor windings.

## 12.13 Topologies of DC Machine Power Supplies

Whether used as electrical motors or generators, DC machines are often connected to static power converters. Variable speed applications require continuous voltage change of the power supply connected to the armature winding. On the other hand, DC voltages obtained from DC networks or batteries are mostly constant. Cases where a variable voltage DC load such as DC machine has to be connected to a constant voltage source are frequently encountered. In such cases, it is necessary to use a DC/DC static power converter which conditions the armature voltage according to needs. Moreover, DC machines are often supplied from AC mains. In these cases, it is necessary to use static power converter which converts constant AC voltages in adjustable DC voltage. Most common topologies of static power converters used in conjunction with DC machines are discussed in the following section.

### *12.13.1 Armature Power Supply Requirements*

A DC motor takes electrical energy from power source, performs electromechanical conversion, and delivers mechanical work to the output shaft. Electrical motors are usually cylindrical rotating machines which deliver the driving torque to work machines by means of the rotor shaft and subsequent mechanical couplings. Mechanical power delivered to work machine is determined by the product of the driving torque and the speed of rotation.

It is of interest to specify the required characteristics of the power source supplying the armature winding. A DC motor is mainly used for controlling the motion of tools, workpieces, semifinished articles, finished articles, packaging machines, manipulators, vehicles, and other objects. A typical motion cycle includes start from initial position, motion toward the targeted position, reaching the target and resting at the target position, and then turning back to the initial position. Representative motion cycle is depicted in Fig. 12.10 by typical changes in the position $\theta_m$, speed $\Omega_m$, and torque $T_{em}$ in the course of moving from the start position to the target and coming back. In order to get a closer specification for the armature power supply, it is of interest to observe the torque and speed changes during this motion.

Characteristic phases of the motion cycle are denoted by numbers 1 to 4. In phase 1, the torque has positive value, and it accelerates the motor, increasing the speed and initiating the motion toward target position. It is observed in Fig. 12.10 that the desired speed is reached soon and then the torque reduces while the speed remains constant. In constant speed interval between the phases 1 and 2, the torque is very low. With constant speed and with no need to provide the acceleration torque $J\mathrm{d}\Omega_m/\mathrm{d}t$, the torque reduces to a very small friction, and it is considered as equal to zero. In phase 2, position $\theta_m$ gets close to the target position. For this reason, it is necessary to brake and reduce the speed. Negative torque is developed in order to reduce the speed to zero and eventually stop at the target position. In the course of coming back to the initial position, the speed and torque required in phases 3 and 4 are of the opposite direction compared to the speed and torque required in phases 1 and 2. It can be concluded that the motion cycle given in Fig. 12.10 comprises the following four combinations of the speed and torque directions:

- $T_{em} > 0$, $\Omega_m > 0$ (phase 1)
- $T_{em} < 0$, $\Omega_m > 0$ (phase 2)

- $T_{em} < 0$, $\Omega_m < 0$ (phase 3)
- $T_{em} > 0$, $\Omega_m < 0$ (phase 4)

In the first and third phases, the electrical machine operates in motor mode, while in the second and fourth phases, it operates in generator mode. Hence, throughout the motion cycle depicted in Fig. 12.10, the operating point ($T_{em}$–$\Omega_m$) has to pass through all the four quadrants of the torque-speed plane. This can be used to specify the armature power supply and define the required voltages and currents. The torque $T_{em} = k_m \Phi_f I_a$ is determined by the armature current. Direction of the electromagnetic torque is determined by the sign of the armature current $I_a$. At the same time, the voltage drop $R_a I_a$ is often neglected, and the armature voltage $U_a$ is assumed to be close to the induced electromotive force $E_a = k_e \Phi_f \Omega_m$. Therefore, the sign of the voltage $U_a$ is determined by direction of the rotor speed. With $U_a \approx k_e \Phi_f \Omega_m$ and $I_a = T_{em}/(k_m \Phi_f)$, the change of the operating point in $T_{em}$–$\Omega_m$ plane can be used to envisage the required voltages and currents in ($I_a - U_a$) plane. In this way, it is possible to specify the characteristics of the power source intended for supplying the armature winding. From the previous conclusions and from relations $T_{em} = k_m \Phi_f I_a$ and $U_a \approx E_a = k_e \Phi_f \Omega_m$, it can be concluded that, in the course of motion cycle depicted in Fig. 12.10, the voltage and current of the armature winding change signs in the following way:

- $I_a > 0$, $U_a > 0$ (phase 1)
- $I_a < 0$, $U_a > 0$ (phase 2)
- $I_a < 0$, $U_a < 0$ (phase 3)
- $I_a > 0$, $U_a < 0$ (phase 4)

Therefore, the power source for supplying the armature winding should provide voltages and currents of both directions and in all the four combinations. These requirements are crucial for the topology of the static power converter intended for supplying the DC machine.

There are applications of DC motors where the speed and torque do not change the sign. In these cases, the power source supplying the armature winding is more simple. In earlier mentioned example of a fan driver, the machine operates in the first quadrant, and armature winding can be supplied by a static power converter with strictly positive voltages and currents, $I_a > 0$ and $U_a > 0$.

### 12.13.2 Four Quadrants in *T*–*Ω* and *U*–*I* Diagrams

If a DC machine is used to effectuate the motion shown in Fig. 12.10, in different phases of this motion, it passes through all the four quadrants of the $T$–$\Omega$ plane. If direction of the excitation flux does not change, direction of the torque is determined by direction of the armature current, while direction of the electromotive force is determined by direction of the angular rotor speed. Applying the

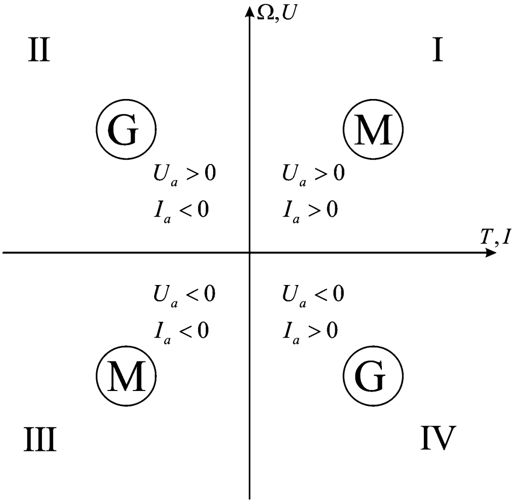



**Fig. 12.11** Four quadrants of the $T$–$\Omega$ and $U$–$I$ diagrams

torque-current relation ($T_{em} = k_m \Phi_f I_a$) and the voltage-speed relation ($U_a \approx E_a = k_e \Phi_f \Omega_m$), the quadrants of the $T$–$\Omega$ plane and the quadrants of the $U$–$I$ plane can be shown by the common Fig. 12.11.

#### 12.13.2.1 I Quadrant

The machine develops a positive torque and rotates in reference direction; thus, $T_{em} > 0$ and $\Omega_m > 0$. The armature voltage and current are positive due to $U_a \approx E_a$ and $E_a \sim \Omega_m$ and due to $I_a \sim T_{em} > 0$. Power taken from the source is positive, $P_i = U_a I_a > 0$. Power of electromechanical conversion is also positive, $P_{em} = T_{em}\Omega_m = E_a I_a > 0$. The machine operates in motor mode.

#### 12.13.2.2 II Quadrant

The machine develops a negative torque, while the rotor speed is positive considering the reference direction; thus, $T_{em} < 0$ and $\Omega_m > 0$. The armature voltage is positive, but the current is negative. Power taken from the source is negative because the voltage and current do not have the same sign. Power of the electromechanical conversion is also negative because the direction of the torque and speed does not have the same sign. The machine operates in the generator mode; therefore, it resists the motion and brakes.

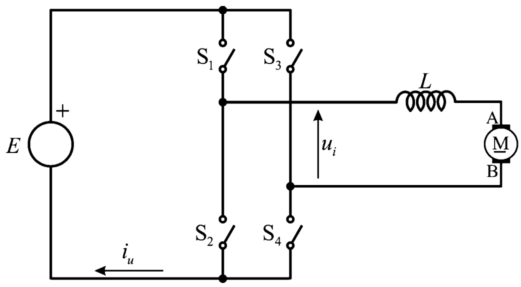



**Fig. 12.12** Topology of the converter intended for supplying the armature winding

#### 12.13.2.3 III Quadrant

The machine develops a negative torque and rotates opposite to the reference direction; thus, $T_{em} < 0$ and $\Omega_m < 0$. The armature voltage and current are negative because $U_a \approx E_a \sim \Omega_m < 0$, while $I_a \sim T_{em} < 0$. Power taken from the source is positive, $P_i = U_a I_a > 0$. Power of the electromechanical conversion is also positive, $P_{em} = T_{em}\Omega_m = E_a I_a > 0$. The machine operates in the motor mode.

#### 12.13.2.4 IV Quadrant

The machine develops a positive torque and rotates opposite to the reference direction; thus, $T_{em} > 0$ and $\Omega_m < 0$. The armature current is positive, but the voltage is negative. Power taken from the source is negative because the voltage and current do not have the same signs. Power of the electromechanical conversion is also negative because the torque acts in direction opposite to the speed. The machine operates in the generator mode. It resists the motion and brakes.

### *12.13.3 The Four-Quadrant Power Converter*

Topology of the static power converter which is used for supplying armature winding of a DC machine supporting the motion cycle shown in Fig. 12.10 is presented in this section. Electrical circuit of the power converter is shown in Fig. 12.12. The basic requirements are described in previous section. The converter should supply the armature winding by variable voltages and variable currents in all four possible combinations of their polarities.

In the left-hand part of the figure, $E$ denotes a DC supply with constant voltage which feeds the static power converter. The voltage $E$ is obtained either from a battery or a *rectifier*. The rectifier is a static power converter comprising diodes or other semiconductor power switches, and it converts electrical energy of AC voltages and currents to electrical energy of DC voltages and currents. It is supplied either from a single-phase or from a three-phase network. At the input of a rectifier, there are AC voltages and currents. Mains-supplied rectifier has the AC quantities at the line frequency (50 or 60 Hz). The rectifier performs AC/DC conversion and feeds the electrical energy in the form of DC voltages and currents. A diode rectifier provides the output DC voltage $E$ which is proportional to the rms value of the AC voltages across the input terminals. Therefore, mains-supplied diode rectifiers cannot provide variable DC voltage $E$.

Receiving the energy from the DC supply $E$, it is necessary to perform the conversion and provide variable armature voltage of both polarities, positive and negative. Solution which satisfies the needs is the bridge comprising four switches, $S_1$, $S_2$, $S_3$, and $S_4$. Within these preliminary considerations, the switches are considered to be ideal. This means that they do not carry any electrical current when turned OFF (open). At the same time, the voltage drop across the switch which is turned ON (closed) is considered negligible and equal to zero. This means that any switch in the state of conduction (closed) does not have any *conduction losses*. Moreover, it is also assumed that the processes of closing and opening the switch do not involve any losses. Hence, there are no *commutation losses*. The transients of changing the switch state are called *commutation*.

#### 12.13.3.1 Power Switches

Real mechanical switches as well as semiconductor power switches carry a small amount of *leakage current* even when switched OFF. Besides, in their state of conduction (ON, closed), they have a small voltage drop across the switch. Hence, real switches do have a certain amount of conduction losses. Each process of turning ON or OFF a semiconductor power switch and each process of closing or opening a mechanical switch involve energy losses. The energy loss incurred in each commutation is multiplied by the number of commutations per second to obtain the commutation losses.

Mechanical switches have contacts which close (get in touch) or open (get detached) in order to operate the switch. Turn-OFF commutation losses in mechanical switches arise due to an intermittent electrical arc which appears during separation of contacts. Even though the contacts are being detached, the current continues for a short while through an electric arc which breaks up in the space between contacts. The contacts are disengaged quickly, and the arc is very brief. Yet, it contributes to energy losses. Turn-ON commutation losses of mechanical switches arise due to electrical current being established prior to proper closing of the contacts, which contributes to the commutation losses.

Contemporary static power converters feeding the armature winding do not use mechanical switches. Instead, semiconductor power switches are used, such as BJT (bipolar junction transistors), MOSFET (metal oxide field effect transistors), and IGBT (insulated gate bipolar transistors). In semiconductor power switches, commutation losses arise due to phenomena of a different nature. Due to transient processes within semiconductor power switches, the change from the OFF state, characterized by $u = E$ and $i \approx 0$, into the ON state, characterized by $u \approx 0$ and $i = I_a$, a brief *commutation* interval $\Delta t_c$ exists where considerable voltage and considerable current exist at the same time. Commutation time $\Delta t_c$ is different for BJT, MOSFET, and IGBT transistors, and it ranges from 100 ns up to 1 μs. The time integral of the *ui* product during the commutation interval represents the energy loss incurred during one commutation event. In both cases of mechanical and semiconductor switches, power of commutation losses is dependent on the energy loss of single commutation and on the number of commutations per second.

By closing switches $S_1$ and $S_4$, the voltage $+E$ is established between brushes A and B. By closing switches $S_2$ and $S_3$, the voltage $-E$ is established between brushes A and B. Therefore, the switching bridge shown in Fig. 12.12 provides voltages of both polarities. Current through the closed switches is equal to the armature current. As the armature current has both directions, the switches should be capable of conducting the current in both directions. According to previous considerations, the voltage and current direction depend on the quadrant where the operating point of the machine resides.

#### 12.13.3.2 Switching States

Each of the four switches is closed (ON) or opened (OFF). Assuming that all the switches can be controlled independently, the number of switching states for the four switches is $2^4 = 16$. When the switches are connected to the switching bridge, shown in Fig. 12.12, the number of available switching states is reduced.

Considering switches $S_1$ and $S_2$, the switching state $S_1 = S_2 =$ ON would bring the power source $E$ into short circuit. At the same time, the switching state $S_1 = S_2 =$ OFF would leave no path for the armature current and cannot be used either. Hence, the branch (arm) $S_1$–$S_2$ has only two available switching states, and these are ($S_1 =$ ON, $S_2 =$ OFF) and ($S_1 =$ OFF, $S_2 =$ ON). The same holds for branch $S_3$–$S_4$. With two branches (arms) and with two possible switching states in each branch, the number of distinct switching states for the entire switching bridge is four.

The same conclusion regarding the number of possible switching states can be obtained by reasoning whether the number of switches being turned ON at any given instant should be 0, 1, 2, 3, or 4. Considering the switching bridge in Fig. 12.12, it has to be noted that the number of switches is turned ON 2 at each instant. First of all, it can be neither 4 nor 3. If the number of turned-ON switches is 3, then either the branch $S_1$–$S_2$ or the branch $S_3$–$S_4$ would bring the source $E$ into short circuit. The source $E$ is either a battery or a diode rectifier, and it has a very small internal resistance. Therefore, turning ON of one entire branch would lead to very high current through the source and through the switches, leading very quickly to their permanent damage. On the other hand, the number of turned-ON switches cannot be less than 2. This is due to the fact that the switching bridge must provide the path for the armature current at any instant, due to the fact that the armature current cannot be interrupted. The armature current gets from the branch $S_1$–$S_2$ to the brush A and then from the brush B to the branch $S_3$–$S_4$. For the purpose of

**Table 12.1** Switching states

| $S_1$ | $S_2$ | $S_3$ | $S_4$ | $u_i$ |
|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | $-E$ |
| 1 | 0 | 0 | 1 | $+E$ |
| 1 | 0 | 1 | 0 | 0 |

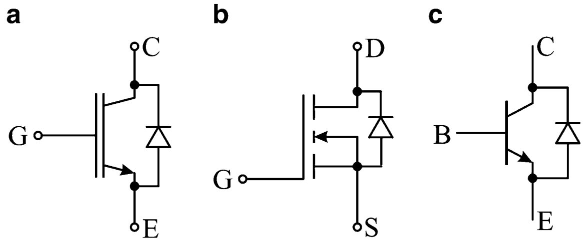



**Fig. 12.13** Notation for semiconductor power switches. IGBT transistor switch (**a**), MOSFET transistor switch (**b**), and BJT (bipolar) transistor switch (**c**)

providing the path for the armature current, it is necessary that one of the switches in branch $S_1$–$S_2$ and one of the switches in branch $S_3$–$S_4$ gets turned ON. The last statement confirms the hypothesis that the number of switches turned ON at each instant is 1 in each branch (arm) and 2 in the switching bridge as a whole.

Available switching states for the bridge in Fig. 12.12 are given in Table 12.1. For each of the four switches, notation 0 represents the OFF state, while notation 1 represents the ON state. The column on the right-hand side shows voltage $u_i$ obtained at the output of the switching bridge in conditions where the given switching state is applied.

The switching bridge shown in Fig. 12.12 makes use of semiconductor power switches. They are mostly power transistors applying BJT, MOSFET, or IGBT technology. The type of transistor to be used in static converter depends upon the operating voltage, operating current, commutation frequency, cooling conditions, required reliability, price, and also upon other factors. Each of transistor technologies has its advantages, disadvantages, and characteristic application area. The most frequently used notation for the mentioned transistors is given in Fig. 12.13.

#### 12.13.3.3 MOSFET, BJT, and IGBT Transistors

The outline of most salient features of contemporary semiconductor power switches is included so that the reader may have an overview of practical voltage drops, commutation characteristics, and switch control requirements. Further study of power electronics is out of the scope of this book.

The state of power transistors is controlled by the third, control electrode. In BJT transistor, control electrode is called base. Positive base current brings the BJT power transistor in ON state, while ceasing the base current and exposing the base to negative voltage turns the transistor OFF. Switching of IGBT and MOSFET transistors is accomplished by varying the voltage of the control electrode called gates. The gate voltage of +15 V is larger than the threshold $V_T \in$ [+4 V .. +6 V], and it brings the transistor into conduction state (ON). Turning OFF is achieved by applying $-15$ V to the gate. Supplying the base current to large BJT transistors may involve considerable amount of power, while the gate control of IGBT and MOSFET transistors is virtually lossless.

All the three families of power transistors have very small currents while in OFF state. Their ON behavior is different. The voltage drop across power transistor in the state of conduction (ON) is rather small. Roughly, it varies between 100 mV and 3 V. Bipolar junction transistor (BJT) is turned ON by feeding the base which is sufficiently high current to bring the transistor to the state of *saturation*, when the voltage $V_{CE} = V_{BE} - V_{BC}$ across collector and emitter terminals is very small. Small BJT transistors in the state of saturation may have $V_{CE}$ as low as 200 mV. Power BJT transistors have their internal voltage drops and may have the values of $V_{CE}$ anywhere between 500 mV and 1 V. On the other hand, large-current BJT transistors have relatively low current gain $\beta = I_C/I_B$ and require very large base current. For that reason, most semiconductor power switches in BJT technology have two transistors connected in *Darlington* configuration where the voltage drop in ON state is $V_{CE} = 2V_{BE} - V_{BC}$, ranging between 1.5 and 3 V.

The MOSFET and IGBT transistors are turned ON by applying +15 V to the gate. MOSFET transistors in ON state behave as a resistor and have voltage drop of $R_{ON}I_{DS}$, where $R_{ON}$ is the "ON" of the MOSFET channel. Transistors made for operating voltages below 100 V may have $R_{ON}$ of only 1 mΩ, resulting in very low voltage drops. Therefore, these transistors are preferred choice for all applications with low operating voltages. Due to specific properties of power MOSFET switches, their resistance $R_{ON}$ increases with the maximum sustainable voltages. Due to $R_{ON} \sim U^{2.5}$, transistor made to sustain twice the voltage would have 5.6 time larger resistance in ON state. For that reason, high-voltage MOSFET transistors are rarely used due to their large voltage drop. IGBT power transistors are developed as a hybrid of BJT and MOSFET technologies, combining positive characteristics of the both. Therefore, they are widely used and made available for voltages up to several kilovolts and currents above 1 kA.

#### 12.13.3.4 Freewheeling Diodes

Electrical current in armature winding changes direction to provide both motoring and braking torques. Therefore, each of the switches has to be ready to conduct electrical currents in both directions. Power switches in Fig. 12.13 are mostly made with power transistors. Placing one power transistor in place of the switches $S_1$, $S_2$, $S_3$, and $S_4$ is not sufficient since power transistors operate with only one direction of current. When turned ON, bipolar transistor conducts the current that enters collector and goes to emitter. An attempt to establish emitter current of opposite direction is of little use. Power transistors are suited for bidirectional currents. Any inverse current may result in significant losses and eventually damage semiconductor device. Therefore, the use of transistors with inverse current is not of interest in static power converters. Transistors such as BJT, IGBT, and MOSFET are used to conduct electrical current only in one direction. For this reason, each of the switches $S_1$.. $S_4$ has one power transistor and one semiconductor power diode. Element denoted by (C) in Fig. 12.13 is a parallel connection of one bipolar transistor conducting in CE direction and one power diode conducting in EC direction. All the four switches shown in Fig. 12.12 are constructed in the prescribed way. Therefore, each of $S_1$.. $S_4$ switches should be considered as a parallel connection of one power transistor and one power diode.

#### 12.13.3.5 Available Output Voltages

According to Table 12.1, a positive voltage across the armature winding is obtained by turning ON the switches $S_1$ and $S_4$. In this switching state, positive armature current circulates through power transistors within switches $S_1$ and $S_4$. Otherwise, with negative armature current, the current is established through power diodes within switches $S_1$ and $S_4$, connected in parallel with power transistors. Negative voltage across the armature winding is obtained by turning ON the switches $S_2$ and $S_3$. The same way as the previous, this switching state can be used for armature currents in both directions. There are also the switching states $S_1 = S_3 =$ ON and $S_2 = S_4 =$ ON which provide the armature voltage $u_i = 0$.

The switching structure in Fig. 12.12 with four available switching states allows feeding the armature winding by voltages and currents of both polarities. Therefore, it is compatible with the need to operate DC machine in all the four quadrants in $T_{em}$–$\Omega_m$ plane. Prescribed method cannot provide continuous change in armature voltage. Namely, there are only four available switching states, and they provide the output voltages of $+E$, $-E$, or 0. Hence, the armature winding can be supplied by the voltage that assumes one of the three discrete values. Instantaneous value of the armature voltage cannot have a continuous change. On the other hand, the armature voltage can be supplied by the train of pulses. The change (modulation) of the pulse-width changes the average value of the armature voltage.

**Question (12.7):** What are the switching states that provide armature voltage equal to zero?

**Answer (12.7):** By turning ON switches $S_2$ and $S_4$, armature voltage is made equal to zero. Armature winding is short circuited also when switches $S_1$ and $S_3$ are switched ON.

### *12.13.4 Pulse-Width Modulation*

According to analysis summarized in Table 12.1, the output voltage $u_i$ may have one of the three available values, $+E$, $-E$, or 0. It has been shown that application of DC electrical motors requires continuous variation of the supply voltage.

The switching structure in Fig. 12.12 cannot produce continuous change of the output voltage. On the other hand, it is possible to devise a sequence of switching states that would repeat in relatively short periods called *switching periods*. Each switching state may have adjustable duration and provide the armature voltages +$E$, $-E$, or 0. Sequential repetition of discrete voltages +$E$, $-E$, and/or 0 would result in an average voltage $+E > U_{av} > -E$. The voltage $u_i$ would assume the form of a train of pulses of variable width. The width of the pulses would affect the average voltage $U_{av}$ within each switching period. It is obvious, though, that variation of pulse width cannot result in continuous change of the instantaneous voltage. It is possible to change only the average value of the armature voltage within each switching period. Variation of pulse width is called *pulse-width modulation*.

#### 12.13.4.1 Armature Voltage Requirements

In the areas of industrial robots, electrical vehicles, and in majority of applications involving motion control, there is a need for continuous variation of the speed of electrical motors. The steady-state armature voltage is equal to $U_a = R_a I_a + k_e \Phi_f \Omega_m$. The voltage drop $R_a I_a$ is usually much lower than the electromotive force. Windings of electrical machines are made to have small resistance, so as to reduce losses due to Joule effect and increase the efficiency. Therefore, it is justified to assume that $U_a \approx k_e \Phi_f \Omega_m$. With constant flux, the armature voltage is proportional to the rotor speed. As the speed changes continuously, the voltage must have continuous changes as well. The available voltage sources are usually batteries or diode rectifiers with constant voltage $E$. There is a possibility to use a series resistance $\Delta R$ in the armature circuit and to reduce the voltage by $\Delta R I_a$. The *rheostat* approach to the voltage regulation allows the voltage $U_a = E - \Delta R I_a$ to be changes varying resistor $\Delta R$. This regulation is not convenient as it has poor energy efficiency. It is accompanied by the losses due to Joule effect in the resistor. In cases where $U_a = ½E$ is required, one half of the input power is converted into heat in series resistor $\Delta R$, while the other half is transferred to DC machine.

#### 12.13.4.2 Pulse-Width Modulation

Electrical machines are supplied by variable voltage from switching power supplies. An example of power supply based on switching bridge is shown in Fig. 12.12. It does not contain series resistors or similar elements which would bring in power losses. Neglecting rather small conduction and commutation losses incurred in power switches, the switching bridge in Fig. 12.12 is virtually lossless. According to Table 12.1, the switching state ($S_1 = S_4 =$ ON and $S_2 = S_3 =$ OFF) provides the armature voltage $U_a = +E$, where $E$ is DC voltage fed to the input terminals of the switching bridge. The source $E$ is often called *primary source*. If the switching

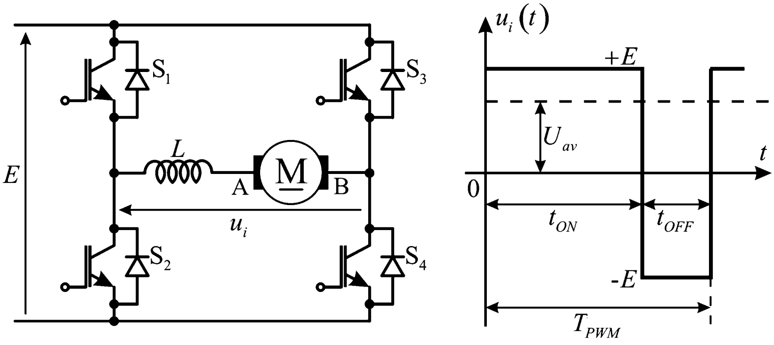



**Fig. 12.14** Pulse-width modulation

state is changed and the other *diagonal* is activated where ($S_1 = S_4 =$ OFF and $S_2 = S_3 =$ ON), the armature voltage becomes $U_a = -E$. None of the two[3] considered states allows continuous voltage variations. However, by fast, periodic change of switching states, the output voltage resembles a train of pulses. The width of such voltage pulses can be altered by changing the dwell time of corresponding switching states. The average voltage of the pulse train depends on the amplitude and width of individual pulses. By a continuous variation of the pulse width, it is possible to accomplish a continuous variation of the average voltage.

#### 12.13.4.3 Average Voltage

If the armature voltage obtained at the output of the switching bridge changes periodically, intervals with $S_1 = S_4 =$ ON are replaced by intervals when $S_2 = S_3$ = ON. Within one switching period $T$, the switching bridge assumes the first switching state and then changes to the second switching state. The switching period is usually close to 100 μs. During one period, the switching state with diagonal $S_1$–$S_4$ turned ON is retained over the interval $t_{ON}$, where $0 < t_{ON} < T$. During the remaining part of the period, diagonal $S_2$–$S_3$ is turned ON. The form of the output voltage obtained across the armature winding is shown in Fig. 12.14. The average voltage within the period $T$ is proportional to the pulse width $t_{ON}$:

$$U_{av} = \frac{1}{T}\int_0^T u_i(t) \cdot \mathrm{d}t = \frac{2t_{ON} - T}{T}E. \tag{12.19}$$

[3] There are two more switching states that provide $U_a = 0$. One of them is ($S_1 = S_3 =$ ON and $S_2 = S_4 =$ OFF), while the other is ($S_2 = S_4 =$ ON and $S_1 = S_3 =$ OFF). They are not considered in further discussion, so as to keep the introduction to pulse-width modulation principles as simple as possible. It has to be noticed, though, that there exist practical reasons to use these *zero-voltage* states in practical implementation.

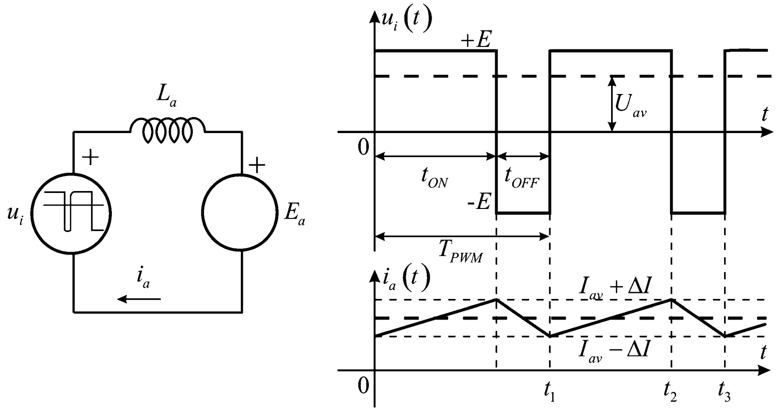



**Fig. 12.15** Change of the armature current during one switching period

By continuous variation of the pulse width over the range $0 < t_{ON} < T$, the average value of the output voltage varies from –*E* to +*E*. Since $U_a \approx k_e \Phi_f \Omega_m$, variation of the pulse width $t_{ON}$ can be used to change the rotor speed within the range $-E/(k_e\Phi_f)$ up to $+E/(k_e\Phi_f)$. Variation of the pulse width is called *pulse-width modulation* (PWM). Switching bridge in Fig. 12.14 plays the role of a power amplifier whose operation is controlled by the variable $t_{ON}$. It provides the output voltage $u_i$ with an average value determined by the pulse width $t_{ON}$. Whenever there is a need to make a continuous change of the armature voltage, this can be accomplished by changing the pulse width $t_{ON}$ in a continuous manner. The switching bridge allows variation of the voltage with almost no losses.

#### 12.13.4.4 AC Components of the Output Voltage

In addition to the average value, the armature voltage depicted in Fig. 12.15 also has an AC component. The voltage shape is periodic, and it contains a number of harmonic components. The basic frequency component, that is, the one with the lowest frequency, has the period *T* and frequency $f = 1/T$. The period *T* is the time interval comprising one positive voltage pulse and one negative voltage pulse. Repetition of such periods makes the pulse train providing the output voltage. Frequency *f* can be close to 10 kHz.

DC machines require the armature voltage that can change continuously. The instantaneous value of the armature voltage does not satisfy this requirement, as it takes one of the two discrete values, either +*E* or $-E$. The voltage fed to the brushes is pulse-shaped voltage which, in addition to the average value, comprises parasitic AC components. It is necessary to envisage the consequences of such AC components of the voltage and analyze whether the switching bridge is a suitable power supply for electrical machines. If the AC component of the supply voltage does not have any significant effect on the armature current, electromagnetic torque, and the rotor speed, and it does not contribute to losses, then the operation of an electrical machine supplied by the pulse-shaped voltage corresponds to the operation of the same machine fed from an ideal voltage source providing the voltage $u_a(t) = (2t_{ON} - T)E/T$ which is pulse-free and does not have any AC components.

#### 12.13.4.5 Low-Pass Nature of Electrical Machines

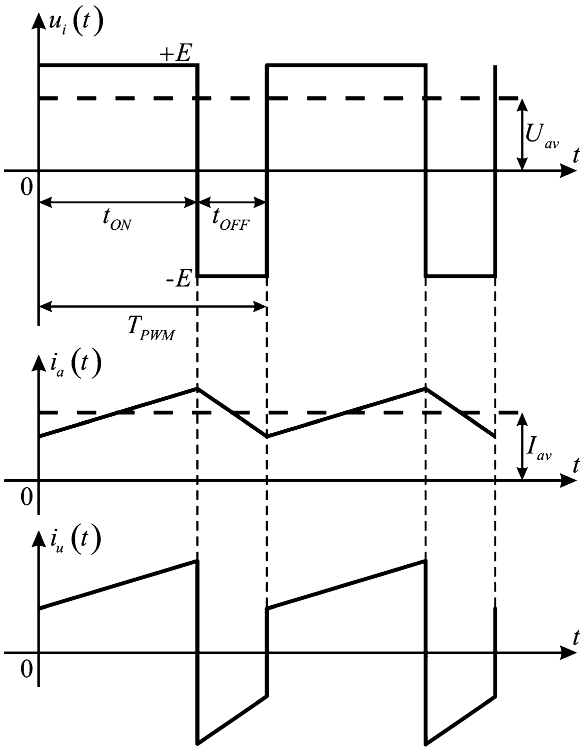
The windings of electrical machines have certain inductance, determined by the number of turns and the magnetic resistance. Impedance of the winding to electrical currents of the angular frequency $\omega$ is $X = L\omega$. In cases where an AC voltage is fed to the winding, the amplitude of AC electrical currents caused by such voltage decreases at elevated angular frequencies $\omega$. Therefore, at very large frequencies, the impact of AC voltages on armature current is negligible.

Differential equation $u_a(t) = R_a i_a(t) + L_a \mathrm{d}i_a(t)/\mathrm{d}t + E_a$ describes transient processes in the armature winding. Laplace transform can be applied to obtain the equation $U_a(s) = R_a I_a(s) + sL_a I_a(s) + E_a(s)$ with complex images of voltages and currents. The armature current is obtained as $I_a(s) = (U_a(s) - E_a(s))/(R_a + sL_a)$. The function $W(s) = 1/(R_a + sL_a)$ represents *transfer function* of the armature winding, and it describes the response of the armature current to changes in the armature voltage. The transfer function is obtained by dividing the complex images of relevant currents and voltages. Since the armature voltage is the cause while the armature current change is the consequence, the voltage is considered an input and the current an output. The function $W(s)$ is used to establish the response of armature current to the excitation by armature voltage, wherein the latter comprises certain harmonic components. The ratio of currents $I(\mathrm{j}\omega)$ and voltages $U(\mathrm{j}\omega)$ having certain angular frequency $\omega = 2\pi/T$ is obtained by replacing $s = \mathrm{j}\omega$ in $W(s)$:

$$\frac{I(\mathrm{j}\omega)}{U(\mathrm{j}\omega)} = W(\mathrm{j}\omega) = \frac{1}{R_a + \mathrm{j}\omega L_a} \approx \frac{1}{\mathrm{j}\omega L_a}. \tag{12.20}$$

At frequencies of the order of several kHz, it is justified to assume that $R_a \ll \omega L_a$; thus, the ratio of AC components of currents and voltages is determined as $1/(L_a\omega)$. Therefore, the voltage components at higher frequencies produce corresponding components of the armature current with rather small amplitude. At frequencies of the order of 10 kHz, reactance $L_a\omega$ is so high that the response of the armature current is negligible. Therefore, the pulsating nature of the supply voltage has no significant effect on the armature current. Therefore, for all practical uses, the presence of AC components in the armature voltage can be neglected. Therefore, the analysis of operation of DC machines fed from PWM-controlled switching bridge supplies can be simplified by modeling the switching bridge in a way that omits the high-frequency AC components. If the switching frequency $f = 1/T$ is sufficiently high, the switching supply shown in Fig. 12.12 can be

**Fig. 12.16** Change of armature voltage, armature current, and source current for a DC machine supplied from PWM-controlled switching bridge




represented by an ideal voltage source providing adjustable output voltage, free from AC components. By varying the pulse width $t_{ON}$, the voltage of such source is changed according to the law $U_{av} = (2t_{ON} - T)E/T$.

### 12.13.5 Current Ripple

In the preceding subsection, it is shown that, at sufficiently high frequencies $f = 1/T$, it is justifiable to neglect the AC component of the train of voltage pulses fed to the armature winding. In this section, the changes of the armature current and voltage are analyzed by taking into account the high-frequency aspects. Moreover, electrical current of the primary source supplying the switching bridge is analyzed as well. In Fig. 12.12, the current of the source $E$ is denoted by $i_u$. These quantities are shown in Fig. 12.16.

Variation of current in armature winding is determined by differential equation

$$\frac{di_a}{dt} = \frac{1}{L_a}(u_a - R_a i_a - E_a) \approx \frac{1}{L_a}(u_a - E_a). \tag{12.21}$$

The voltage drop due to resistance $R_a$ of the armature winding is neglected; thus, the current change is determined by the ratio of difference $(u_a - E_a)$ and inductance $L_a$. In cases where the electromotive force $E_a$ is equal to the supply voltage, there is no change of current. If these conditions persist, the system enters the steady state. When the armature winding is fed from a switching power supply feeding the voltage pulses of variable pulse width, the instantaneous value of the armature voltage assumes one of the two distinct states, either $+E$ or $–E$. The power supply voltage gets equal to the electromotive force $E_a$ only in the exceptional cases where $E_a = +E$ or $E_a = -E$. In majority of cases, the electromotive force is smaller than the supply voltage:

$$|E_a|<|E|.$$

Within each period $T$, the armature voltage $+E$ is greater than the electromotive force during the interval $t_{ON}$, where the first derivative of the current is positive and the current linearly increases in accordance with (12.21). When the interval $t_{ON}$ elapses, the armature voltage $-E$ is smaller than the electromotive force, which leads to a linear decrease of the current during the interval $T - t_{ON}$. Figure 12.15 shows the equivalent circuit which contains the pulsating voltage source $u_i$, inductance $L_a$ of the armature winding, and electromotive force of the armature winding. Resistance of the armature winding is neglected. It is assumed that the electromotive force remains constant within switching period $T$. The amplitude of armature current oscillations is denoted by $\Delta I$, and it is called *current ripple*. During each voltage pulse, the change of the current is linear, as determined by (12.21). In order to determine the amplitude of these oscillations $\Delta I$, one starts from instant $t_1$, shown in Fig. 12.15, when a positive voltage pulse commences. At this instant, the armature current is $i_a(t_1) = I_{av} - \Delta I$. Over the interval $[t_1 .. t_2]$, the output voltage of the switching supply is equal to $u_i = +E$, and the current $i_a$ increases with the slope $(E - E_a)/L_a$. The current change is linear, and it reaches the value of $i_a(t_2)$ at instant $t_2 = t_1 + t_{\mathrm{ON}}$, which marks the end of the positive voltage pulse:

$$i_a(t_2) = i_a(t_1) + t_{ON}\frac{E - E_a}{L_a}. \tag{12.22}$$

Over the interval $[t_2 .. t_3]$ in Fig. 12.15, the power supply feeds negative voltage pulse. The output voltage of the switching supply is $u_i = -E,$ and the current $i_a$ decreases linearly with the slope of $(-E - E_a)/L_a$. During this interval, the current decreases linearly. At instant $t_3 = t_2 + t_{\mathrm{OFF}} = t_2 + T - t_{\mathrm{ON}} = t_1 + T,$ which marks the end of the negative voltage pulse, the armature current reaches

$$\begin{aligned} i_a(t_3) &= i_a(t_2) + (T - t_{ON})\frac{-E - E_a}{L_a} \\ &= i_a(t_1) + t_{ON}\frac{E - E_a}{L_a} + (T - t_{ON})\frac{-E - E_a}{L_a} \\ &= i_a(t_1) + E\frac{2t_{ON} - T}{L_a} - E_a\frac{T}{L_a}. \end{aligned} \tag{12.23}$$

Prolonged operation in the prescribed way implies a constant width $t_{ON}$ and a constant average value $I_{av}$ of the armature current. Under circumstances, this operation can be characterized as the steady state, notwithstanding the fact that the current exhibits periodic oscillations $\Delta I$. In such state, the instantaneous values of the armature current at the end of each negative pulse must be equal. Therefore, at steady state, $i_a(t_1) = i_a(t_3)$. On the basis of (12.23), the steady state is reached when

$$E\frac{2t_{ON} - T}{L_a} - E_a\frac{T}{L_a} = 0,$$

when the electromotive force is equal to

$$E_a = E\frac{2t_{ON} - T}{T}. \tag{12.24}$$

On the basis of (12.19), the previous expression represents the average value of the output voltage provided by the switching power supply. With the assumption of $R_a \approx 0$, the steady state in armature circuit is reached when the average value of the supply voltage is equal to the electromotive force $E_a$. This steady state is represented in Fig. 12.15. The armature current oscillates around an average value which is maintained constant. The amplitude of oscillations $\Delta I$ depends on the switching frequency, supply voltage, and armature inductance.

The ratio of the positive pulse width and period $t_{ON}/T$ is called *modulation index*, and it is denoted by $m$. By replacing $m$ in expressions (12.19) and (12.24), the following steady-state relation is obtained:

$$U_{av} = E_a = E(2m - 1). \tag{12.25}$$

The amplitude of oscillations of the armature current around its average value can be calculated from the modulation index, winding inductance, supply voltage, and switching frequency. According to Fig. 12.15, the current is changed by $i_a(t_2) - i_a(t_1) = 2\Delta I$ over interval $[t_1 .. t_2]$. From (12.22) it follows that

$$2 \cdot \Delta I = i_a(t_2) - i_a(t_1) = t_{ON}\frac{E - E_a}{L_a} = \frac{TE}{L_a}m(2 - 2m).$$

which leads to

$$\Delta I = \frac{TE}{L_a}\left(m - m^2\right). \tag{12.26}$$

The analysis shows that steady state in the armature circuit is established when the electromotive force and modulation index satisfy condition $(2\,m - 1)E = E_a$. Then, the average value of current does not change between the successive switching periods. The instantaneous value of the current oscillates around its average value with the period $T$ and amplitude $\Delta I$ given in expression (12.26). The operating frequency of the switching bridge $f = 1/T = \omega/(2\pi)$ determines the frequency of oscillations of the armature current. The frequency $f$ and angular frequency $\omega = 2\pi f$ are called *commutation frequency* and also *switching frequency*. When commutation frequency is sufficiently high, the current ripple is very small. Then, it is justified to neglect the AC component of the pulse-width-modulated train of pulses and consider that machine is fed from an ideal power source feeding a variable voltage $U_{av} = (2\,m - 1)E$, determined by the modulation index and free from AC components.

A close estimate of the ripple can be obtained by considering only the voltage and current components at the switching frequency $f = 1/T$. The ratio between the AC current of frequency $\omega = 2\pi/T$ and the voltage of the same frequency is determined by the inductance of the armature winding. Namely, $|I(j\omega)/U(j\omega)| \approx 1/(L_a\omega)$. Considering the train of voltage pulses where the values $+E$ and $-E$ repeat with period $T$ and with duration of the positive pulse $t_{ON} = T/2$, the amplitude of the harmonic component of the frequency $f = 1/T$ is $V_1 = (4/\pi)E$. Corresponding harmonic component of the armature current has the amplitude of $I_1 = V_1/(L_a\omega) = (4/\pi)E/(2\pi L_a f) = (2/\pi^2)\, E/(L_a f) \approx 0.2026\, E/(L_a f)$. This approach neglects harmonic components at higher frequencies and overlooks the fact that the considered voltage does not change as a sinusoidal function. Instead, it is a train of pulses which comprises harmonic component at the frequency $f = 1/T$, but it also has harmonic components at frequencies that are odd multiples of $f$.

A more accurate estimate of the current ripple can be obtained by using expression (12.26). Ripple $\Delta I$ has the maximum value with modulation index of $m = 0.5$. In this case, the ripple is

$$\Delta I = \frac{TE_i}{4L_a}. \tag{12.27}$$

When using the above results, one should take into account that the preceding analysis assumes that the switching bridge in Fig. 12.12 uses only two switching states: the state with diagonal $S_1$-$S_4$ turned ON and the state with diagonal $S_2$–$S_3$ turned ON. In Table 12.1, there are two more available states, $S_1 = S_3 =$ ON and $S_2 = S_4 =$ ON, which both produce the output voltage of zero. Control of the switching bridge can be organized by using additional two states and inserting the time intervals when the voltage is equal to zero. In this case, the relevant expressions for $t_{ON}$ time change as well as the definition of the modulation index. In cases where DC machine requires a positive armature voltage, the train of voltage pulses is made by sequencing $+E$ and 0, providing an average value between these two values. Whenever a negative armature voltage is needed, the train of voltage pulses is made by sequencing $-E$ and 0. More detailed analysis of the pulse-width modulation technique is not studied in this book.

Input current taken from the source by the switching bridge is shown in Fig. 12.16, and it is denoted by $i_u$. This current depends on the instantaneous value of armature current $i_a$ and on the switching state. If diagonal $S_1$–$S_4$ is ON, positive terminal of the source $E$ is connected to brush A through the switch $S_1$, while negative terminal of the source is connected to brush B through the switch $S_4$. Therefore, in this switching state, $i_u = i_a$. If diagonal $S_2$–$S_3$ is ON, $i_u = -i_a$. As a consequence, the input current of the switching bridge has the shape of a train of pulses with an amplitude determined by the armature current and with the sign determined by the switching state of the bridge.

**Question (12.8):** Determine the change and the amplitude of oscillations of the armature current in the case when DC motor is supplied from the voltage shown in Fig. 12.26, with $E_a = 0$ and $R_a = 0$ and with known $L_a$, $T$, and $E$.

**Answer (12.8):** Since the average voltage has to match the electromotive force, duration of the positive voltage pulse is $t_{ON} = T/2$. During the first half period, $L_a \mathrm{d}i_a/\mathrm{d}t = +E$. Therefore, the current change is linear. The same applies for the second half period. The amplitude of oscillations of the armature current around its average value is $\Delta I$. Within the first half period, the current increases from $-\Delta I$ to $+\Delta I$. The change is linear, $\mathrm{d}i_a/\mathrm{d}t = 2\Delta I/(T/2) = E/L_a$, and therefore, $\Delta I = ET/(4L_a)$.

**Question (12.9):** Control of the switching bridge of Fig. 12.12 is carried out by keeping switch $S_4$ permanently closed and switch $S_3$ permanently open. At the beginning of period $T$, the switch $S_1$ is turned ON. After the on time $t_{ON} = mT$ elapses, the switch is turned OFF. In the remaining part of the period $T - t_{ON}$, the switch $S_2$ is turned ON. The switching bridge is supplied from a constant voltage source $E$. Determine the average value of the output voltage, and find the expression for the current ripple.

**Answer (12.9):** The output voltage $u_i$ is a periodic train of pulses that repeat with period $T$. In the first part of each period, during interval $t_{ON}$, the instantaneous value of the output voltage is $+E$. During the remaining part of the period $T - t_{ON}$, switches $S_2$ and $S_4$ are turned ON and the output voltage is equal to zero. The average value of the voltage is $U_{av} = [(t_{ON})\cdot E + (T - t_{ON}) \cdot 0]/T = (t_{ON}/T)E = mE$. The current ripple is determined by repeating the calculation included in the previous analysis, starting with (12.23) and ending with (12.26). Compared to the previous analysis, where the instantaneous values of the output voltage were $+E$ and $-E$, in this example, they are $+E$ and 0. With this in mind, the armature current ripple is

$$\Delta I = \frac{TE}{2L_a}\left(m - m^2\right).$$

## 12.13.6 Topologies of Power Converters

DC machines are to be supplied by continuously variable DC voltage. The voltage should be suited for the desired operating mode of the machine. The switching bridge in Fig. 12.12 illustrates the principle of operation of static power converters which allow lossless conversion of DC voltages and currents. Practical static converter may have other parts that are not shown in Fig. 12.12. It has electronic circuits that control the state of the power switches, microprocessor-based controller, auxiliary power supplies for electronic circuits, diode rectifier which converts the AC mains into DC voltage $E$, protection devices, communication devices, and other auxiliary parts. Primary source of electrical energy for supplying electrical machines is usually low-voltage distribution network with line-frequency AC voltages. The line frequency of AC distribution networks is 50 Hz. Three-phase connections have line-to-line voltages of 400 V rms. Low-power converters can be supplied from single-phase connection with phase voltage of 220 V rms. The voltages of AC distribution network do not correspond to the needs of DC machines. Therefore, it is necessary to use static power converters. Their task is to convert the electrical energy of AC voltages and currents to electrical energy of DC voltages and currents to be fed to the armature winding and the excitation winding. It is necessary to provide continuous change of DC voltages fed to the windings. In some cases, the primary source of electrical energy is a battery, and it gives a constant DC voltage. This is the case in autonomous vehicles and autonomous devices and systems that do not have connection to AC mains. Batteries provide constant DC voltages that cannot be adjusted to meet the needs of DC machines. In such cases, it is necessary to use static power converter that converts constant DC voltage of the battery to variable DC voltage to be fed to the armature winding. The latter is continuously changed according to the rotor speed.

Generally speaking, DC machines can receive electrical energy from primary sources which include batteries, single-phase AC supplies, and three-phase AC supplies but also other voltage and current systems and forms. Primary source voltages are rarely compatible with the machine needs and therefore should be adjusted. For this reason, it is necessary to use static power converter between the primary source connections and the terminals of electrical machine. The role of static power converter is to convert the voltages and currents of the primary source to the form and amplitude required by the actual operating mode of electrical machine.

Figure 12.17 shows a simplified scheme of a switching power converter with transistors, intended for feeding and controlling DC machines. This converter is often met in practice. Part (D) shows a switching bridge comprising four power transistors. The bridge is entirely the same as the one shown in Fig. 12.12 and analyzed previously. Each IGBT transistor has a diode in parallel, called *freewheeling diode*, aimed to conduct the switch currents in direction from emitter to collector. Diode rectifier, shown in part (A) of Fig. 12.17, converts three-phase system of AC voltages, provided from the mains, into DC voltage $E$. Part (B) of Fig. 12.17 contains a series inductance and parallel capacitor used for filtering the rectifier output voltage. This part of static power converter is called *intermediate DC circuit* or *DC link*. The voltage of the intermediate DC circuit is constant, and it represents the input voltage to the switching bridge, previously denoted by $E$.

Part (C) contains an additional transistor switch which, as required, may be turned ON and thus connects a resistor in parallel to the capacitor. By turning ON this fifth transistor, DC link voltage appears across the resistor. The resistor current acts toward reducing the DC link voltage and converting a certain amount of energy

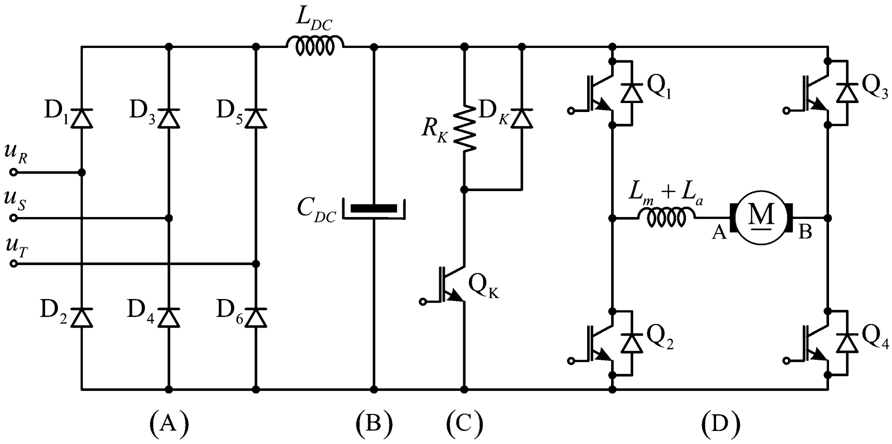



**Fig. 12.17** Topology of switching power converter with transistors

into heat. This may be required when DC machine operates in the second or fourth quadrant, namely, when the torque and speed have opposite directions and the machine operates as a generator. In generator mode, the armature winding does not consume the electrical energy. Instead, it acts as a source and passes the energy back toward the switching bridge. In other words, the power flow changes and, in Fig. 12.17, it goes from the right to the left.

In even quadrants, DC machine breaks and converts mechanical work into electrical energy. The energy obtained in this way is called *braking energy*, since it is obtained by deducting mechanical work and/or kinetic energy from the mechanical subsystem. Direction of the current in the circuit is reversed, and the switching bridge does not take the energy from the intermediate DC circuit anymore. Instead, it delivers power and supplies the current to the elements of the intermediate DC circuit. Due to the sign change of the average value of the current $i_u$, this current is directed from the switching bridge to the DC link capacitor. Therefore, the voltage across DC link capacitor increases. The obtained energy cannot be returned to the AC mains. For this to achieve, direction of the rectifier current should be changed. Semiconductor diodes of the three-phase rectifier (A in Fig. 12.17) conduct the current from anode to cathode and cannot have the currents in the opposite direction. Hence, the braking energy cannot return to the mains and remains in DC link circuit. The excess energy is accumulated in the capacitor, increasing its energy to $W_C = ½CE^2$. Excessive increase of $E$ may damage circuit elements. Therefore, the process of accumulation of the braking energy has to be stopped. By turning the fifth transistor ON (C), a high-power resistor is connected in parallel to the intermediate DC link circuit, and the excess of the braking energy is dissipated in the resistor. The elements used in the process are called *braking device* or *dynamic braking device*. In Fig. 12.17, dynamic braking device is denoted by (C).

Static power converters with power transistors have advantageous characteristics compared to other solutions. Therefore, they are widely used. Their use is limited

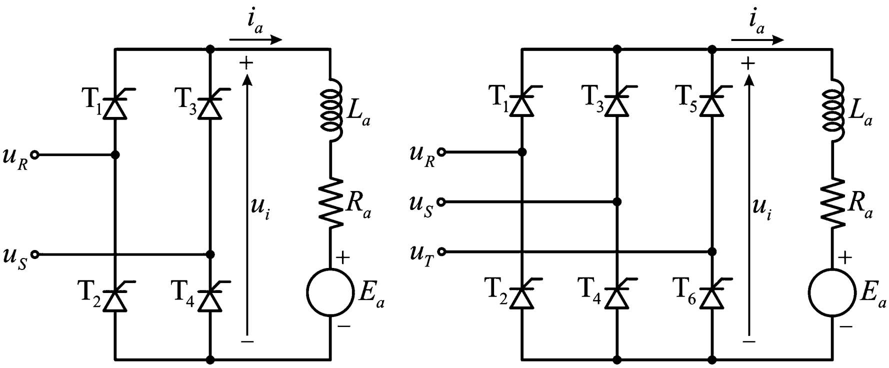



**Fig. 12.18** Topology of converters with thyristors: Single phase supplied (*left*) and three phase supplied (*right*)

only by the voltage and current rating of the available power transistors. At present, commercially available transistors cover the voltages up to 6 kV and currents up to 2–3 kA. Practical switching power converters with power transistors reach the power levels in excess of 1 MW, covering virtually all practical applications of DC machines.

A couple of decades ago, at the end of the twentieth century, power transistor technology was sufficient for building switching power converters up to of several tens of kilowatts. At that time, there were no power transistors with sufficient voltage and current ratings to cover larger powers. In order to build large power static power converters, thyristors were used, four-layer semiconductor devices invented and put to practical use years before power transistors. Now and then, the voltage and current ratings of available thyristors exceed greatly the ratings of available power transistors. Yet, designing static power converters with thyristors is not an easy task. While power transistors can be turned ON or OFF at will, thyristors behave differently. They can be turned ON by gate pulses, but they cannot be turned OFF[4] unless the anode-to-cathode current does not return to zero. For those reasons, electrical schematics of thyristor-based static power converters do not resemble the ones with power transistors.

Characteristic topologies of thyristor converters for supplying large DC machines are shown in Fig. 12.18. Although the thyristor topologies are primarily of historical significance, one can encounter previously installed systems based on thyristor converters in industrial and other applications requiring controlled DC machines of large power.

[4] Thyristors have three electrodes. Their anode and cathode conduct the switch current, while the third electrode, the gate, serves as the control electrode. A small positive pulse of gate current turns ON the thyristor, provided that $u_{AK} > 0$. Conventional thyristor cannot be turned OFF by operating the gate. There are *gate turn-OFF thyristors* made in such way that a very large spike of negative gate current may result in turn OFF. Yet, their use and the associated auxiliary circuits are rather involved. Therefore, their use is rather limited.