Chapter 6
Modeling Electrical Machines

This chapter introduces, develops and explains generalized mathematical model of electrical machines. It explains the need for modeling, introduces and explains approximations and neglected phenomena, and formulates generalized model as a set of differential and algebraic equations.

Working with electrical machines requires mathematical representation of the process of electromechanical conversion. It is necessary to determine equations which correlate the electrical quantities of the machine, such as the voltages and currents, with mechanical quantities such as the speed and torque. These equations provide the link between the electrical access to the machine (terminals of the windings) and the mechanical access to the machine (shaft). The two accesses to the machine are shown in Fig. 6.1, which illustrates the process of electromechanical conversions and presents the principal losses and the energy accumulated in magnetic field. Equations of the mathematical model are used to calculate changes in electromagnetic torque, electromotive forces, currents, speed, and other relevant variables. Besides, the model helps calculating conversion losses in windings, in magnetic circuits, and in mechanical parts of the machine.

The set of equations describing transient processes in electrical machines contains differential and algebraic equations, and it is also called *dynamic model*. Operation of a machine in steady states is described by the steady state equivalent scheme, which gives relations between voltages and currents at the winding terminals, and by mechanical characteristic, which describes relation between the torque and speed at the mechanical access. In the following subsection, an introduction to the modeling of electrical machines is presented. Figure 6.1 presents a diagram showing power of the source P_e, mechanical power P_m, winding losses P_{Cu}, losses in the coupling field P_{Fe}, mechanical losses due to rotation $P_{\gamma m}$ (motion resistance losses), and energy of the coupling field W_m.

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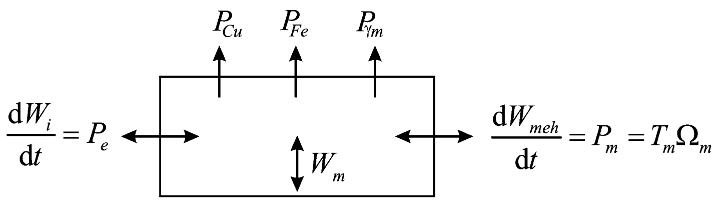

Fig. 6.1 Power flow in an electromechanical converter which is based on magnetic coupling field

6.1 The Need for Modeling

A good knowledge of electrical machines is a prerogative for successful work of electrical engineers. Knowledge of the equivalent schemes in steady states and mechanical characteristics is required for selecting a machine which would be adequate for a particular application, for designing systems containing electrical machines, as well as for solving the problems which may arise in industry and power engineering. Knowledge of the dynamic model of electrical machines is necessary for solving the control problems of generators and motors, for designing protection and monitoring systems, for determining the structures and control parameters in robotics, as well as for solving the problems in automation of production, electrical vehicles, and other similar applications.

In all cases mentioned above, one should know the basic concepts concerning the size, mass, construction, reliability, and coefficient of efficiency of electrical machines.

Further on, a general model of electrical machines is presented in this section. Along with the model, common approximations made in the course of modeling are listed, explained, and justified. The main purpose of studying the general model is to determine the dynamic model for commonly used electrical machines and to obtain their steady state equivalent schemes and mechanical characteristics.

The diagram shown in Fig. 6.1 presents power P_e which the electromechanical system receives from the source, power P_m which is transferred via shaft to the mechanical subsystem, losses in the electrical subsystem P_{Cu}, iron losses P_{Fe}, as well as the losses due to friction $P_{\gamma m}$ in the mechanical subsystem. It is necessary to develop corresponding mathematical model which describes the phenomena within the electromechanical converter shown in Fig. 6.1.

What is a *good model*? How to obtain a *good model*?

Generally speaking, a *model* is a mathematical representation of the system which is under consideration. In most cases, less significant interactions are neglected, and then, a simplified representation is obtained, yet still adequate for the purposes and uses. In electrical engineering, a model is usually a set of differential equations describing behavior of certain system. In some cases, like steady state operation, these equations can be reduced to an equivalent electrical circuit (expressions *replacement scheme*, *replacement circuit*, and *equivalent scheme* are also used).

The phenomena and systems of interest for an electrical engineer are usually complex and include some interactions which are not of uttermost importance and should not be taken into account. Considering gas pressure pushing the head of a piston in the cylinder of an endothermic motor, the action is the result of continuous collisions of a large number of gas particles with the surface of the piston. Strictly speaking, the force is not constant, but it consists of a very large number of strikes (pulses) per second. However, in the analysis of the torque which the motor transfers to the work machine, only the average value of the force is of interest. Therefore, the effects of micro strikes are neglected, and the force is considered to be proportional to the surface of the piston head and to the gas pressure in the cylinder.

In electrical engineering, passive components, such as resistors, capacitors, and inductive coils (chokes), are very often mentioned and used. Strictly speaking, models of real chokes, capacitors, and resistors are more complex compared to widely accepted models that are rather simple. At high frequencies, influences of parasitic inductance and equivalent series resistance of a capacitor become noticeable. Under similar conditions, parallel capacitance of a choke cannot be neglected at very high frequencies. Similar conclusion can be made for a resistor. In a rigorous analysis, real components would have to be modeled as networks with distributed parameter. Yet the frequencies of interest are often low. At low frequencies, parasitic effects are negligible, and the well-known elements R, L, and C are considered as lumped parameter circuit elements described by relations $Ri = u$, $u = L\mathrm{d}i/\mathrm{d}t$, and $i = C\mathrm{d}u/\mathrm{d}t$. Therefore, parasitic effects and distributed parameters do not have to be taken into account when solving problems and tasks at low frequencies. In cases when all the parasitic and secondary effects are modeled, the considered *RLC* networks become rather complex, and their analysis becomes difficult. Drawing conclusions or making design decisions based upon too complex models becomes virtually impossible.

Therefore, a *good model* is not the one that takes into account all aspects of dynamic behavior of a system, but the one which is simplified by justifiable approximations, thus facilitating and improving the process of making conclusions and taking design decisions, while retaining all relevant (significant) phenomena within the system. It is not possible to develop an analytical expression which would help in defining the *relevance*, but in making approximations, it is necessary to distinguish the essential from nonessential on the basis of a deeper knowledge of the system and material and through the use of experience. A tip to apprentices in modeling conventional electrical machines is to take into account all the phenomena up to several tens of kHz and to neglect the processes at higher frequencies.

6.1.1 Problems of Modeling

In the process of modeling, it is required to neglect insignificant phenomena in order to obtain a simple, clear-cut, and usable model. For successful modeling and use of the model, it is necessary to make correct judgment as regards phenomena that can be neglected. If important phenomena are neglected or overlooked, the result of modeling will not be usable. Here, an example is presented which shows that taking correct decisions often relies on a wider knowledge of the considered objects and phenomena which is acquired by engineering practice.

Consider a capacitor which consists of two parallel plates with a dielectric of thickness d in between. It is customary to consider that the field in dielectric is homogeneous and equal to $E = U/d$. If this capacitor is used in a system where its high-frequency characteristics are not significant, it can be considered ideal and its interelectrode field homogeneous. Consider the case when a pulse-shaped voltage having a very large slope $\mathrm{d}V/\mathrm{d}t$ is connected to the plates. At low frequencies, the capacitor can still be considered ideal since the simplification made cannot be of any influence to the low-frequency response. However, it is possible that a very high $\mathrm{d}V/\mathrm{d}t$ values result in breakdown of dielectric, even in cases where the steady state field strength $E = U/d$ is considerably smaller than the *dielectric strength*. The term dielectric strength corresponds to the maximum sustainable electric field in dielectric, exceeding of which results in breakdown of the dielectric. High $\mathrm{d}V/\mathrm{d}t$ contributes to a transient nonuniform distribution of electric field between the plates, with $E < U/d$ in the middle and $E > U/d$ next to the plates. Namely, what actually happens in the process of feeding the voltage to the plates is, in fact, propagation of an electromagnetic wave which comes from the source and is directed by conductors (*waveguides*) toward the plates. The propagation of the electromagnetic wave continues in the dielectric; therefore, the highest intensity of the electric field is in the vicinity of the plates, whereas in the space between the plates, it is lower. Uneven initial distribution of the field is established within a very short interval of time, which is dependent on dimensions and is of the order of nanoseconds.

As a consequence of this uneven field distribution, the process of an abrupt voltage rise (very high slope $\mathrm{d}V/\mathrm{d}t$) may lead to a situation when, for a short time, the field exceeds dielectric strength of the material in close vicinity of the capacitor plates, even in cases when $E = U/d$ is very small. A breakdown results in destruction of the structure and chemical contents of the dielectric, but nevertheless, it has local character. The damaged zones of the dielectric are next to the plates, whereas toward inside the dielectric is preserved. However, if described incidents occur frequently (say 10,000 times per second), damaged zones tend to spread, and they change characteristics of the capacitor. Prolonged operation in the prescribed way eventually leads to dielectric breakdown between the plates, and it puts the capacitor out of service. Similar phenomenon occurs in insulation of AC motors fed from three-phase transistor inverters, commonly used to provide the so-called U/f frequency control. Three-phase inverters provide variable voltage by feeding a train of voltage pulses to the motor. The pulse frequency is next to 10 kHz. The width of the pulses is altered so as to obtain the desired change in the average voltage (*PWM – pulse width modulation*). The voltage pulses are of very sharp edges, with considerable values of $\mathrm{d}V/\mathrm{d}t$, and they bring up an additional stress to the insulation of the windings. Hence, certain high-frequency phenomena cannot be neglected when analyzing PWM-supplied electrical machines.

The example considered above requires a deeper understanding of the process and is founded on experience. The knowledge required for thorough understanding of given example is not a prerequisite for further reading. However, it is of interest to recognize the need to enrich the studies by laboratory work, practice, written papers, and projects, acquiring in this way the experience necessary for a successful engineering practice. A successful engineer combines the theoretical knowledge, skill in solving analytically solvable problems, but also the experience in modeling processes and phenomena. In order to make the most out of the knowledge and skill, it is necessary primarily to use the experience and deeper understanding of the process of electromechanical conversion and reduce a complex system to a mathematical model to be used in further work.

6.1.2 Conclusion

A good model is the simplest possible model still representing the relevant aspects of the dynamic behavior of a system – process – machine in a satisfactory way.

In the process of generating models, justifiable approximations are made in order to make a simple model suitable for recognizing relevant and significant phenomena, for making conclusions, and for taking engineering and design decisions. When introducing the approximations, care should be taken that these do not jeopardize the accuracy to the extent that makes the model useless.

This book is the first encounter with cylindrical electromechanical converters with magnetic coupling field for a number of readers. Therefore, initial steps in machine analysis and modeling are made with certain approximations. Among the four principal approximations, the losses in magnetic circuits or iron losses are also neglected. Omission of these losses makes the basic models of electrical machines easier to understand. In most electrical machines, iron losses are marginalized by lamination, and the mentioned approximation is partially justifiable. It is necessary, however, to have in mind that at higher frequencies and larger magnetic induction, the iron losses can be considerable and should be taken into account in calculating the total losses and coefficient of efficiency, as well as in designing the corresponding cooling systems.

6.2 Neglected Phenomena

In the course of developing a model, it is justifiable to neglect less significant phenomena, the omission of which does not cause significant deviations of the obtained results. The four most common approximations are:

- The system is considered as a lumped parameter network.
- Parasitic capacitances are neglected.

- The iron losses are neglected.
- Ferromagnetic materials are considered linear

6.2.1 Distributed Energy and Distributed Parameters

Electrical machines are usually considered as networks with lumped parameters and represented by circuits comprising discrete inductances and resistances. Considering actual L and C elements, the coil energy resides in spatially distributed magnetic field, while the capacitor energy resides in spatially distributed electrical field. It is well known[1] that changes in magnetic field give a rise to induced electrical field and *vice versa*. The induced field is proportional to the rate of change of the inducting field, that is, to the operating frequency. Hence, a coil with an AC current is surrounded by magnetic field, but also with an induced electrical field, the strength of which depends on the operating frequency. Similar conclusion can be drawn for a capacitor. The presence of both fields contributes to parasitic capacitance of the coil and parasitic inductance of the capacitor.

Lumped parameter approach neglects the spatial distribution of the coil and capacitor energy. It is assumed that both energies reside within discrete elements and that the amounts ½Li^2 and ½Cu^2 do not reside in space. The coils and capacitors are considered ideal, *lumped parameter L-C* elements. The adopted models are $u_L = L\mathrm{d}i/\mathrm{d}t$ and $i_C = C\mathrm{d}u/\mathrm{d}t$, neglecting the secondary effects such as capacitance of a coil or inductance of a capacitor. With the induced fields being proportional to the operating frequency, lumped parameter approach introduces a negligible error at relatively low frequencies that are in use in typical applications of electrical machines.

One of the consequences of neglecting distributed energy and distributed parameters is concealing the energy transfer. In a lumped parameter network, a pair of conductors with electrical current i and voltage u transmits the energy at a rate of $p = ui$. This expression involves macroscopic quantities like voltage and current and suggests that the energy passes through conductors. In reality, the energy is transmitted through the surrounding space with the presence of electrical and magnetic field.

6.2.2 Neglecting Parasitic Capacitances

For electrical machines operating on the basis of a magnetic coupling field, the effects of parasitic capacitances of the windings and the amounts of energy accumulated in the electrical field are negligible. Since the spatial density of

[1] Consider Maxwell equations, such as rot $\vec{E} = -\partial\vec{B}/\partial t$.

magnetic field is considerably higher than that of electrical field ($\mu H^2 \gg \varepsilon E^2$), it is justified to neglect the capacitances between insulated conductors and capacitances between the windings and magnetic circuit.

6.2.3 Neglecting Iron Losses

It is considered that the losses due to hysteresis and losses due to eddy currents are considerably smaller compared to the power of conversion; thus, they can be neglected. Specific losses in ferromagnetic materials (iron losses) are dependent on magnetic induction and operating frequency and they can be represented by the following expression:

$$p_{Fe} = \frac{P_{Fe}}{m} = p_H + p_V = \sigma_H \cdot f \cdot {B_m}^2 + \sigma_V \cdot f^2 \cdot {B_m}^2.$$

Since the losses due to eddy currents are dependent on squared frequency and squared magnetic induction, the iron losses are to be reconsidered in cases when electrical machine operates with elevated frequencies. In such cases, it is necessary to check whether neglecting the iron losses can be justified.

6.2.4 Neglecting Iron Nonlinearity

The characteristic of magnetization of magnetic materials is considered linear. Therefore, the effects of saturation of the ferromagnetic material (iron) are neglected. Permeability B/H is considered constant and equal to differential permeability $\Delta B/\Delta H$ at all operating points of the magnetization characteristic. In applications where induction exceeds 1.2T, it is necessary to check whether this is justified.

General model of electrical machine based on magnetic coupling field is developed hereafter relying on the four basic approximations mentioned above. It is assumed that the converter has N windings which can be either short-circuited or connected to a source. The windings are mounted on the rotor or stator.

6.3 Power of Electrical Sources

Figure 6.2 shows a converter having N magnetically coupled contours (windings) which could be either connected to a power source or separated from it and brought into short circuit. Windings of electrical machines can be fed from current or voltage sources. Real voltage sources have finite internal resistance (impedance), whereas current sources have finite internal conductance (admittance). With no loss in generality, in further text, it is assumed that electrical sources are ideal.

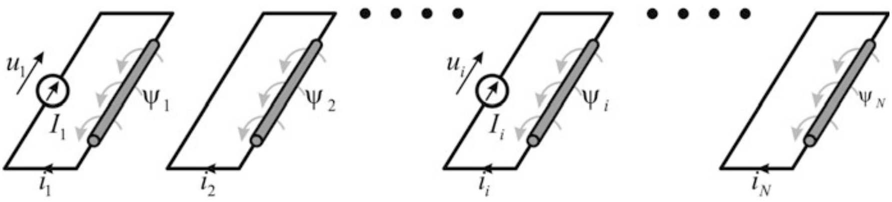

Fig. 6.2 Model of electromechanical converter based on magnetic coupling field with N contours (windings). Contours 1 and i are connected to electric sources, while contours 2 and N are short circuited thus voltages at their terminals are zero

In windings connected to a current source, winding current is constant and determined by the current of the source. If the winding is short-circuited, then the voltage balance in the winding is given by expression $u = Ri + \mathrm{d}\Psi/\mathrm{d}t = 0$. If resistance of the winding is negligible, then $\mathrm{d}\Psi/\mathrm{d}t = 0$, and flux in the short-circuited winding is constant. In the case where the winding terminals are connected to a voltage source, voltage of the source determines the change of flux ($u \approx \mathrm{d}\Psi/\mathrm{d}t$).

Electrical power delivered by the sources to the electromechanical converter is determined by (6.1), where $\underline{u}$ and $\underline{i}$ are vector-columns with their elements being voltages and currents of individual windings. The expression for the power the sources deliver to the machine does not depend on whether the windings are connected to current sources and voltage sources or are short-circuited.

$$
\begin{aligned}
P_e &= \sum_{j=1}^{N} u_j i_j = \underline{i}^T \cdot \underline{u}, \\
\underline{i}^T &= [i_1,\ i_2,\ \ldots\ i_i,\ \ldots\ i_{N-1},\ i_N], \\
\underline{u}^T &= [u_1,\ u_2,\ \ldots\ u_i,\ \ldots\ u_{N-1},\ u_N].
\end{aligned} \tag{6.1}
$$

6.4 Electromotive Force

Voltage balance in the winding is given by (6.2), where u is the voltage across winding terminals, i is the winding current, and $\Psi = N\Phi$ is the winding flux. Parameter R denotes the winding resistance.

$$
u = Ri + \frac{d\Psi}{dt} = Ri + e_{CEMF} \tag{6.2}
$$

The considered winding is shown in Fig. 6.3. Flux derivative determines the electromotive force induced in the winding. When making the equivalent scheme, the electromotive force can be represented as an ideal voltage generator attaching the sign + pointed downward, in accordance with the adopted reference direction

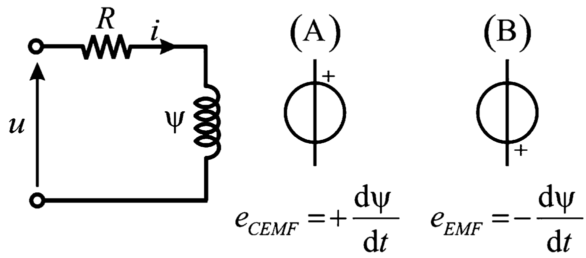

Fig. 6.3 The electromotive and counter-electromotive forces

for the current. Then, the force is $e_{EMF} = -d\Psi/dt$. Quantity $-d\Psi/dt$ is the electromotive force induced in the winding, as shown in Fig. 6.3, in the part denoted by (B).

On the other hand, it is possible to alter the reference direction and sign ($e = +d\Psi/dt$), as shown in the part (A) of Fig. 6.3. Quantity $e_{CEMF} = +d\Psi/dt$ is the counter-electromotive force induced in the winding.

Approach (B) is used in majority of courses in Electrical Engineering Fundamentals and Electromagnetics, since it undoubtedly illustrates the circumstance that in each contour, the induced electromotive force and current are opposing the change of flux. For example in the case that intensity of the current decreases, flux through the contour decreases, and a positive value $e = -d\Psi/dt$ appears. Taking into account that sign + is pointed downward, it is concluded that the induced electromotive force supports current in the circuit opposing the change of flux.

Approach (A) results in an equivalent scheme where the reference positive terminals of the voltage and electromotive force are pointed upward. Taking that $e = +d\Psi/dt$, current in the circuit can be determined as ratio $(u-e)/R$. Defined in this way, the electromotive force opposes the voltage; thus, it is called *counter-electromotive force*. Approach (A) is often applied when solving electrical circuits containing electromotive forces, as is the case of replacement schemes of electrical machines. The question of choice of the reference direction of the electromotive force is not of essential significance since the choice does not lead to essential changes in voltage balance equation in the winding. In the Anglo-Saxon, German, and Russian literatures, the approaches are different, which should confuse the reader. In practice, both approaches are accepted, provided that the adopted reference direction corresponds to the sign taken for the electromotive force ($e = +/- d\Psi/dt$).

Electromotive force and counter-electromotive force induced in a contour are discussed further on, in Chap. 10, *"Electromotive Forces Induced in the Windings."*

6.5 Voltage Balance Equation

Voltage balance in each winding is given by (6.2). For a system having N windings, equilibrium of k-th winding is given by expression

$$u_k = R_k i_k + \frac{d\,\Psi_k}{d\,t}, \tag{6.3}$$

where u_k, i_k, R_k, and Ψ_k are the voltage, current, resistance, and flux of the kth winding, respectively. Flux Ψ_k in kth winding is the sum of all the fluxes that pass through the winding, whatever the cause of relevant magnetic field. This flux is a consequence of the electrical current in kth winding itself, as well as the currents in other windings that are magnetically coupled to k-th winding. The part of the flux Ψ_k caused by the current i_k is equal to $L_{kk}i_k$. The coefficient L_{kk} is also called *self-inductance* of the winding. Self-inductance is expressed in H = Wb/A, and it is strictly positive. In cases where the current in the winding i_k is the only originator of magnetic field, the flux in the winding is $\Psi_k = L_{kk}i_k$. Electrical currents in remaining windings (Fig. 6.2.) can also contribute to the flux Ψ_k. Current i_j in the turns of the winding j changes the flux Ψ_k proportionally to the coefficient of mutual inductance between windings k and j, L_{kj}. Parameter L_{kj} can be positive, negative, or zero. Spatial orientation of the two windings may be such that a positive current in one of the windings contributes to a negative flux in the other.

Voltage balance of a system with N windings is described by a set of N differential equations. A shorter and more clear-cut record of these equations can be obtained by introducing vectors of the voltages and currents

$$\begin{aligned} \underline{i}^T &= [i_1,\ i_2,\ \ldots\ i_k,\ \ldots\ i_{N-1},\ i_N] \\ \underline{u}^T &= [u_1,\ u_2,\ \ldots\ u_k,\ \ldots\ u_{N-1},\ u_N], \end{aligned} \tag{6.4}$$

by defining vectors of winding fluxes

$$\underline{\Psi}^T = [\Psi_1,\ \Psi_2,\ \ldots\ \Psi_k,\ \ldots\ \Psi_{N-1},\ \Psi_N], \tag{6.5}$$

as well as by introducing matrix of resistances $\underline{R}$ in (6.6), which contains resistances of the windings along the main diagonal. Voltage balance equations in matrix form are given by (6.7), which represents N differential equations of the form (6.2). Voltage balance equations define dynamics of the electrical part of an electromechanical converter, that is, dynamics of the electrical subsystem.

$$\underline{R} = \begin{bmatrix} R_1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & R_2 & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & R_k & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & \ldots & R_N \end{bmatrix} \tag{6.6}$$

$$\underline{u} = \underline{R} \cdot \underline{i} + \frac{\mathrm{d}\Psi}{\mathrm{d}t} \tag{6.7}$$

Flux vector-column $\underline{\Psi}$ is determined by the winding currents, self-inductances, and mutual inductances. Flux of kth winding is determined by the coefficient of self-inductance of the winding k, as well as by the coefficients of mutual inductance

L_{kj} between the kth winding and remaining windings, as given by the following equation:

$$\Psi_k = L_{k1} i_1 + L_{k2} i_2 + \ldots + L_{kk} i_k + \ldots + L_{kN} i_N.$$

Since the above expression applies for each winding, the flux vector can be obtained by multiplying the inductance matrix $\underline{L}$ (6.8) and current vector-column $\underline{i}$, in the way shown by (6.9):

$$\underline{L} = \begin{bmatrix} L_{11} & L_{12} & \ldots & L_{1k} & \ldots & L_{1N} \\ L_{21} & L_{22} & \ldots & L_{2k} & \ldots & L_{2N} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ L_{k1} & L_{k2} & \ldots & L_{kk} & \ldots & L_{kN} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ L_{N1} & L_{N2} & \ldots & L_{Nk} & \ldots & L_{NN} \end{bmatrix} \tag{6.8}$$

$$\underline{\Psi} = \underline{L} \cdot \underline{i} \tag{6.9}$$

Along the main diagonal of inductance matrix, there are self-inductances of individual windings, while the remaining coefficients residing off the main diagonal are mutual inductances. Since $L_{ij} = L_{ji}$, the inductance matrix is symmetrical, that is, $\underline{L} = \underline{L}^T$.

Elements of the inductance matrix can be variable. Variations of the self-inductances and mutual inductances can be due to relative movement of the moving parts of the electromechanical converter (rotor) with respect to the immobile parts (stator). Windings may exist in both parts; thus, the movement causes changes in relative positions of individual windings. For each winding, it is possible to define the winding axis (Sect. 5.5). Considering a pair of windings, rotation of one with respect to the other changes the angle between their axes. Consequently, their mutual inductance is also changed. The rotor motion can also change self-inductances. Self-inductance of a winding depends on the magnetic resistance R_μ. Considering a stator winding, its flux passes through the stator magnetic circuit, passes through the air gap, and proceeds through the rotor magnetic circuit. There are cases where the rotor has unequal magnetic resistances in different directions. In such cases, the rotor motion changes the equivalent magnetic resistance R_μ of the stator winding and changes the self-inductance of the winding. An example where movement changes self-inductance of the winding is given in Fig. 2.6, where the magnetic resistance decreases and self-inductance increases by inserting a piece of iron in the magnetic circuit of the coil.

6.6 Leakage Flux

With the current i_k being the sole originator of magnetic field, the flux in kth winding is $\Psi_k = L_{kk}\, i_k$. One portion of this flux passes to other windings as well, and it is called *mutual flux*. The remaining flux encircles only the kth winding

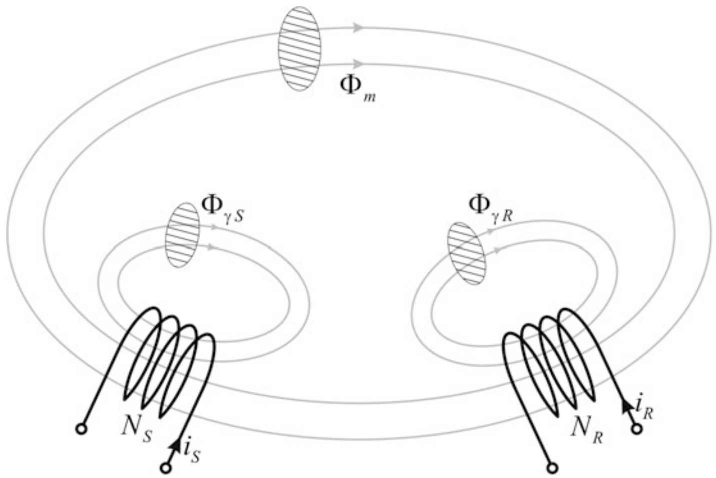

Fig. 6.4 Definitions of the leakage flux and mutual flux

and does not pass to any other winding. As this component, in a way, "misses the opportunity" to effectuate magnetic coupling, it is also called *leakage flux*. Figure 6.4 depicts the mutual and leakage fluxes of a system having two windings, one on stator and the other on rotor. The flux in one turn is denoted by Φ. Fluxes of individual windings are obtained by multiplying flux in each turn Φ by the number of turns.

Self-inductances of the stator and rotor windings are equal to the ratio of the flux established due to the winding current and the current intensity. In other words, self-inductance is the ratio of the flux and current of the winding in cases where the flux does not get affected by external magnetic fields or by currents in other windings, but it is the consequence of the electrical current in the winding itself. In a system comprising a number of coupled windings, self-inductance of the considered winding can be determined by dividing the flux and current in conditions when all the remaining windings are with zero current. Self-inductances of practical windings are *strictly positive*, whereas mutual inductances could be negative. Mutual inductance of the stator and rotor windings determines the measure of the contribution of stator current to the total flux of the rotor winding. Since $L_{SR} = L_{RS} = L_m$, the impact of stator currents on rotor flux is the same as the impact of rotor currents on stator flux. By rotation, relative positions of the two windings may become such that positive current in one winding reduces the flux in the other, thus resulting in a negative value of the mutual inductance. For the system of windings shown in Fig. 6.4, the matrix of inductances is of dimensions 2 × 2. Along the main diagonal of the matrix, there are positive coefficients of self-induction L_S and L_R. At the remaining places of the matrix are mutual inductances that may be positive, negative, or equal to zero, which is the case when the winding axes are displaced by the angle of π/2. The mutual inductance is dependent of the coupling coefficient

$$k = \frac{L_m}{\sqrt{L_S L_R}}.$$

The coefficient k is a measure of magnetic coupling of the stator and rotor windings. In cases where the leakage flux is negligible, all of the flux is mutual, and it encircles both windings. In such cases, $k = 1$. As the leakage flux increases, the mutual flux decreases as well as the coefficient k. In cases when the two windings do not have any mutual flux, $k = 0$. It is important to notice that k cannot exceed 1.

Relation between the mutual inductance L_m and coefficient k can be illustrated by the example where the stator and rotor windings have the same axis and the same number of turns N. If the magnetic resistance R_μ is the same, the self-inductances are equal, $L_S = L_R = N^2/\ R_\mu$, while the mutual inductance is $L_m = kL_S = kL_R$. Since $k < 1$, mutual inductance is smaller than the self-inductance. With the introduced assumptions, the difference,

$$L_{\gamma S} = L_S - L_m = (1 - k)L_S$$

is called *leakage inductance* of the stator, a measure of the *leakage flux* which encircles the stator winding and does not reach to the rotor winding. A stronger magnetic coupling between the windings means the higher coupling coefficient and the smaller leakage flux and leakage inductance. The example outlined above assumes that $N_S = N_R$. It is of interest to consider the leakage flux and leakage inductance in the case when the windings have different magnetic circuits and different numbers of turns.

Fluxes through one turn of the stator and rotor windings are shown in Fig. 6.4 and given by expressions

$$\Phi_S = \Phi_{\gamma S} + \Phi_m,$$
$$\Phi_R = \Phi_{\gamma R} + \Phi_m.$$

The mutual flux has a component which is a consequence of the stator current (Φ^S) and a component which is a consequence of the rotor current (Φ^R),

$$\Phi_m = \Phi_m^S + \Phi_m^R.$$

Fluxes of the windings are obtained by multiplying flux through one turn by the number of turns:

$$\Psi_S = N_S \Phi_S = N_S \Phi_m + N_S \Phi_{\gamma S} = N_S \Phi_m + \Psi_{\gamma S},$$
$$\Psi_R = N_R \Phi_R = N_R \Phi_m + N_R \Phi_{\gamma R} = N_R \Phi_m + \Psi_{\gamma R}.$$

Flux $\Psi_{\gamma S}$ is the *leakage flux* of the stator winding, while $\Psi_{\gamma R}$ is the leakage flux of the rotor winding. Leakage flux in each winding is proportional to the current. The coefficient of proportionality is *the leakage inductance* of the winding. For the windings shown in Fig. 6.4, the leakage inductances are given by expressions

$$L_{\gamma S} = \frac{\Psi_{\gamma S}}{i_S}, \quad L_{\gamma R} = \frac{\Psi_{\gamma R}}{i_R}.$$

The mutual inductance is determined by expression

$$L_m = L_{SR} = L_{RS} = \frac{N_S \Phi_m^R}{i_R} = \frac{N_R \Phi_m^S}{i_S}.$$

In order to define the winding self-inductance, it is necessary to identify the component of the winding flux which is caused by the electrical currents of the same winding. Self-inductance is the quotient of this flux component ($L_S i_S$) and the current intensity (i_S). One part of the flux component ($L_S i_S$) is partially mutual (that is, encircling both windings) and partially leakage (encircling only the stator winding). Self-inductances of the stator and rotor are

$$\begin{aligned} L_S &= \frac{N_S \Phi_m^S + N_S \Phi_{\gamma S}}{i_S} = \frac{N_S \Phi_m^S + \Psi_{\gamma S}}{i_S} \\ &= \frac{N_S}{N_R} L_{RS} + L_{\gamma S} = \frac{N_S}{N_R} L_m + L_{\gamma S}, \end{aligned}$$

$$\begin{aligned} L_R &= \frac{N_R \Phi_m^R + N_R \Phi_{\gamma R}}{i_R} = \frac{N_R \Phi_m^R + \Psi_{\gamma R}}{i_R} \\ &= \frac{N_R}{N_S} L_{SR} + L_{\gamma R} = \frac{N_R}{N_S} L_m + L_{\gamma R}. \end{aligned}$$

Therefore, the leakage inductance is a part of the self-inductance of the winding. The leakage inductance is higher when the magnetic coupling between the coupled windings is weaker. In the case when the numbers of turns of the stator and rotor are equal, as well as in the case when the rotor quantities are scaled (transformed) to the stator side, previous equations take the following form:

$$\begin{aligned} L_S &= L_m + L_{\gamma S}, \\ L_R &= L_m + L_{\gamma R}. \end{aligned}$$

6.7 Energy of the Coupling Field

The coupling field has a key role in the process of electromechanical conversion. The energy obtained from the source can be accumulated in the coupling field and then taken from the field and converted to mechanical work. It is of significance to determine the relation between the energy of this field, winding currents, and parameters such as the self-inductances and mutual inductances. Spatial density of energy accumulated in the coupling magnetic field is

$$w_m = \int \vec{H} \cdot \mathrm{d}\vec{B}.$$

In linear media, permeability $\mu = B/H$ is constant. Therefore, the energy density in the coupling field is equal to $w_m = ½BH = ½\mu H^2$. Total energy can be obtained by integrating the density w_m over the domain where the field exists. The field energy can be expressed as function of electrical currents and winding inductances such as L_S, L_R, L_m, and similar. Mutual inductance between coils L_1 and L_2 is denoted by L_m or by $L_{12} = L_m$. For a system with two coupled windings, the field energy is equal to $f(i_1, i_2, L_1, L_2, L_{12}) = ½L_1 i_1^2 + ½L_2 i_2^2 + L_{12}\, i_1\, i_2$. A rigorous proof of this statement is omitted at this point. Instead, it is supported by considerations which indicate that the spatial integral of energy density w_m corresponds to $f(i_1, i_2, L_1, L_2, L_{12})$. Spatial integration of the energy density involves the sum of minute energy portions $½BH/\mathrm{d}V$ comprised in infinitesimal volumes $\mathrm{d}V$. Taking into account that $\mathrm{d}V = \mathrm{d}S/\mathrm{d}x$, the problem can be reduced to calculating integral $(½\, BH)\, \mathrm{d}S\, \mathrm{d}x = ½\, (B\, \mathrm{d}S)\, (H\, \mathrm{d}x)$. In general, integration of $(B\mathrm{d}S)$ results in a *flux*, whereas integration of $(H\mathrm{d}x)$ results in a magnetomotive force Ni, that is, in electrical current (ampere-turns). Therefore, the formula for the field energy contains members of the form Φi or Li^2.

For a system containing N coupled windings, energy of the coupling field is

$$W_m = \int_V w_\mathrm{m} \mathrm{d}V = \int_V \left(\int \vec{H} \cdot \mathrm{d}\vec{B} \right) \mathrm{d}V = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} L_{ij} i_i i_j.$$

In the above expression, elements L_{ii} correspond to self-inductances of the windings, and they are strictly positive. Elements L_{ij} are mutual inductances, and they can be positive or negative. A more illustrative expression for the coupling field energy is obtained by introducing the flux and current vectors

$$\underline{i}^T = [i_1,\ i_2,\ \ldots\ i_k,\ \ldots\ i_{N-1},\ i_N],$$
$$\underline{\Psi}^T = [\Psi_1,\ \Psi_2,\ \ldots\ \Psi_k,\ \ldots\ \Psi_{N-1},\ \Psi_N],$$

resulting in (6.10), where $\underline{L}$ is matrix of inductances of the considered system of windings:

$$W_m = \frac{1}{2} \underline{i}^T \underline{L} \underline{i} \tag{6.10}$$

Question (6.1): Consider two windings having self-inductances L_1 and L_2. Is it possible for the coefficient of mutual inductance to exceed $(L_1 \cdot L_2)^{0.5}$?

Answer (6.1): Mutual inductance of the two windings is $L_{12} = k \cdot (L_1 \cdot L_2)^{0.5}$, where k is coupling coefficient. Maximum value of k is 1, and it exists in cases without any leakage, when the total flux of one winding goes through the other winding as well. Since the coupling coefficient cannot be greater than 1, mutual inductance cannot be greater than $(L_1 \cdot L_2)^{0.5}$.

Question (6.2): Is it possible that expression for the field energy

$$2W_m = \sum_j \sum_k L_{jk} i_j i_k$$

gives a negative result? Derive the proof taking the example of a system having two coupled windings.

Answer (6.2): The above expression gives magnetic field energy, and therefore, it cannot have a negative value. In the case of two windings, the expression takes the form

$$W_m = {}^1\!/\!_2 L_1 i_1{}^2 + {}^1\!/\!_2 L_2 i_2{}^2 + L_{12} i_1 i_2 = {}^1\!/\!_2 \left[L_1 i_1{}^2 + L_2 i_2{}^2 + 2k \cdot (L_1 \cdot L_2)^{0.5} i_1 i_2 \right].$$

By introducing notation $a = (L_1)^{0.5} i_1$ and $b = (L_2)^{0.5} i_2$, the expression takes the form $2W_m = a^2 + b^2 + 2k \cdot a \cdot b$. It is required to prove that this expression cannot take a negative value, whatever the current intensities i_1 and i_2 might be. Since only the third member of the sum may assume a negative value, and this happens in the event when current intensities are of opposing signs, it is necessary to prove that $2W_m \geq 0$ for $k = 1$. If so, the statement holds for any $k < 1$. With $k = 1$, $2W_m = (a + b)^2$, which completes the proof.

6.8 Power of Electromechanical Conversion

For the considered system of N windings coupled in a magnetic field, it is required to determine the power at the electrical and mechanical accesses, power losses, and power of the electromechanical conversion. Power of the source is supplied through the electrical access of the machine, and it is determined by the sum of powers $p_k = u_k i_k$ supplied to each individual winding.

$$p_e = \sum_k u_k i_k = \underline{i}^T \underline{u} = \underline{u}^T \underline{i} \tag{6.11}$$

Since the voltage vector is expressed by the voltage balance equations (6.7), given in matrix form, power of the source can be expressed as function of the current vector, resistance matrix, and inductance matrix:

$$\begin{aligned} p_e &= \underline{i}^T \left(\underline{R}\, \underline{i} + \frac{d\underline{\Psi}}{dt} \right) = \underline{i}^T \left(\underline{R}\, \underline{i} + \frac{d}{dt} (\underline{L}\, \underline{i}) \right) \\ &= \underline{i}^T \underline{R}\, \underline{i} + \underline{i}^T \frac{d\underline{L}}{dt} \underline{i} + \underline{i}^T \underline{L} \frac{d\underline{i}}{dt}. \end{aligned} \tag{6.12}$$

Power losses in the coupling field are neglected at this point. The losses in the windings can be expressed in matrix form as well. In a winding of resistance R_k, with electrical current i_k, the losses are determined by expression $R_k {i_k}^2$. Total winding losses of a system containing N windings are given by expression

$$p_{Cu} = \sum_k R_k i_k^2 = \underline{i}^T \underline{R}\ \underline{i} \tag{6.13}$$

where $\underline{R}$ is a square matrix of dimensions $N \times N$ having winding resistances along the main diagonal, while the remaining elements are zeros.

One part of work supplied from the source is accumulated in the coupling field. Since the energy of the coupling field is given by (6.10), the power p_{wm} depicting the rate of change of energy accumulated in the field is given by (6.14):

$$\begin{aligned} p_{wm} &= \frac{\mathrm{d}W_m}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\underline{i}^T\underline{L}\underline{i}\right) \\ &= \frac{1}{2}\left(\frac{\mathrm{d}\underline{i}^T}{\mathrm{d}t}\right)\underline{L}\underline{i} + \frac{1}{2}\underline{i}^T\left(\frac{\mathrm{d}\underline{L}}{\mathrm{d}t}\right)\underline{i} + \frac{1}{2}\underline{i}^T\underline{L}\left(\frac{\mathrm{d}\underline{i}}{\mathrm{d}t}\right). \end{aligned} \tag{6.14}$$

Expression for power p_{wm} can be written in a more convenient form. It should be noted that expression (6.14) represents a sum of three scalar quantities, each obtained by multiplying the vector of electrical currents and the matrix of system inductances. It can be shown that values of the first and third member are equal. For any scalar quantity $\underline{s} = s$ (i.e., for matrices of dimensions 1×1), it can be written that $\underline{s} = \underline{s}^T$. At the same time, the inductance matrix is symmetric ($L_{jk} = L_{kj}$, $\underline{L} = \underline{L}^T$). Therefore, it can be shown that

$$\left(\frac{\mathrm{d}\underline{i}^T}{\mathrm{d}t}\right)\underline{L}\underline{i} = \left[\left(\frac{\mathrm{d}\underline{i}^T}{\mathrm{d}t}\right)\underline{L}\underline{i}\right]^T = \underline{i}^T\underline{L}^T\left(\frac{\mathrm{d}\underline{i}}{\mathrm{d}t}\right) = \underline{i}^T\underline{L}\left(\frac{\mathrm{d}\underline{i}}{\mathrm{d}t}\right).$$

By introducing this substitution to (6.14), one obtains (6.15):

$$p_{wm} = \frac{1}{2}\underline{i}^T\left(\frac{\mathrm{d}\underline{L}}{\mathrm{d}t}\right)\underline{i} + \underline{i}^T\underline{L}\left(\frac{\mathrm{d}\underline{i}}{\mathrm{d}t}\right). \tag{6.15}$$

Equation 6.15 contains first time derivatives of the current $\underline{i}$ and inductance $\underline{L}$. Variations in the matrix occur due to the relative motion of the rotor with respect to the stator. This motion leads to variation of mutual inductance between the rotor and stator windings. In certain conditions, rotor movement may cause variation of self-inductances of individual windings. Derivative of the current vector $\underline{i}$ in (6.15) is a vector whose elements are derivatives of the currents of individual windings. In cases where a winding is connected to an ideal current source which provides constant current, derivative of the winding current is zero. Derivative of a winding current can take nonzero values if the winding is short-circuited, or connected to an ideal voltage source, or connected to real current or voltage sources.

The part p_{wm} of the power p_e determines the increase in energy W_m accumulated in the coupling field. The part p_{Cu} is lost in winding conductors due to Joule effect. What remains of the source power is $p_e - p_{Cu} - p_{wm} = p_c$. The remaining power p_c is converted to mechanical through electromagnetic processes involving conductors, magnetic circuit and coupling magnetic field. An integral of p_c represents the mechanical work. The power p_c is also called *power of electromechanical conversion* or *conversion* power. In the motor mode (Fig. 5.8), reference direction for power is such that the power p_c is positive. A positive power of electromechanical conversion means that electrical energy is being converted into mechanical work. In the generator mode (Fig. 5.9), direction of the converter power is reversed, and the power p_c, as defined above, assumes a negative value.

Since

$$p_C = p_e - p_{Cu} - p_{wm},$$

and considering (6.12), (6.13), and (6.15), p_C is expressed as

$$p_C = \left(\underline{i}^T \underline{R}\, \underline{i} + \underline{i}^T \frac{\mathrm{d}\underline{L}}{\mathrm{d}t} \underline{i} + \underline{i}^T \underline{L} \frac{\mathrm{d}\underline{i}}{\mathrm{d}t}\right) - \left(\underline{i}^T \underline{R}\, \underline{i}\right) - \left(\frac{1}{2} \underline{i}^T \left(\frac{\mathrm{d}\underline{L}}{\mathrm{d}t}\right) \underline{i} + \underline{i}^T \underline{L} \left(\frac{\mathrm{d}\underline{i}}{\mathrm{d}t}\right)\right).$$

By simple rearrangement of this expression, it is obtained that

$$p_C = \frac{1}{2} \underline{i}^T \frac{\mathrm{d}\underline{L}}{\mathrm{d}t} \underline{i}. \tag{6.16}$$

According to the later expression, electromechanical conversion is possible only in cases where at least one element of the inductance matrix $\underline{L}$ changes. Variation of the self-inductance or mutual inductance is generally a consequence of changing the rotor position relative to the stator. In rotating machines, the rotor displacement θ_m is tied to mechanical speed of rotation $\Omega_m = \mathrm{d}\theta_m/\mathrm{d}t$, and the expression for conversion power takes the form (6.17)

$$p_C = \frac{1}{2} \underline{i}^T \frac{\mathrm{d}\underline{L}}{\mathrm{d}t} \underline{i} = \frac{1}{2} \underline{i}^T \frac{\mathrm{d}\underline{L}}{\mathrm{d}\theta_m} \underline{i} \cdot \frac{\mathrm{d}\theta_m}{\mathrm{d}t} = \frac{\Omega_m}{2} \underline{i}^T \frac{\mathrm{d}\underline{L}}{\mathrm{d}\theta_m} \underline{i}. \tag{6.17}$$

Equation 6.17 shows that electromechanical conversion in rotating machines relies on variation of one or more elements of the inductance matrix in terms of the rotor movement θ_m.

In the case when a converter operates in the motor mode, power of electromechanical conversion is transferred to the mechanical subsystem. Within the mechanical subsystem, a small part of mechanical power p_c is dissipated on covering mechanical losses such as friction, while the remaining power is, via shaft, transferred to a work machine (mechanical load). In the generator mode, mechanical power of the driving turbine is, via shaft, transferred to electromechanical converter, where one part of turbine power is dissipated on covering mechanical losses. The remaining power is converted to electrical power and, reduced by the losses in the electrical subsystem, transferred to electrical receivers connected to the winding terminals.

6.9 Torque Expression

Rotating electrical machine consists of a still stator and a moving rotor, both of cylindrical shapes having a common axis. The rotor is rotating relative to the stator with speed Ω_m, and its position with respect to the stator is determined by angle θ_m at each instant. Electrical machine has N windings, and some of them are positioned on the stator while the remaining ones are on the rotor. Since self-inductances and mutual inductances depend on relative position between the stator and the rotor, elements of the inductance matrix $\underline{L}$ are also functions of the same angle.

Speed of rotation is

$$\Omega_m = \frac{\mathrm{d}\theta_m}{\mathrm{d}t},$$

and the inductance matrix can be represented by expression

$$\underline{L} = f_1(t) = f_2(\theta_m).$$

The magnetic coupling field acts on both stator and rotor and creates electromagnetic forces. Coupled forces create electromagnetic torque. Torque T_{em} acts upon rotor, while torque $-T_{em}$, of equal amplitude and opposite direction, acts upon stator. Since the stator is not mobile, it is only the rotor which can move. Torque T_{em} is, via rotor and shaft, transferred to work machine or driving turbine.

In motor mode, torque acts in the direction of movement. It tempts to increase the speed of rotation Ω_m. Therefore, the power $p_C = T_{em}\Omega_m$ is positive. The torque T_{em} acts in the direction of motion. It is transferred to a work machine via shaft, and it tends to start up or to accelerate its movement.

In generator mode, torque T_{em} is acting in the direction opposite to the movement; thus, the power $p_C = T_{em}\Omega_m$ is negative. Negative value of the power of electromechanical conversion indicates that the direction of conversion is changed, that is, mechanical work of the driving turbine is converted to electrical energy. Acting in the direction opposite to the movement, the torque T_{em} is, via shaft, transferred to the driving turbine and resists its rotation, tending to lower the speed of rotation.

It is of interest to determine the expression for the electromagnetic torque. Since the power is equal to the product of the torque and speed, the conversion power of expression (2.84) can be represented by equation

$$p_C = \Omega_m \left(\frac{1}{2} \underline{i}^T \frac{\mathrm{d}\underline{L}}{\mathrm{d}\theta_m} \underline{i} \right) = T_{em} \Omega_m,$$

where T_{em} is the electromagnetic torque determined by (6.18)

$$T_{em} = \frac{1}{2} \underline{i}^T \frac{\mathrm{d}\underline{L}}{\mathrm{d}\theta_m} \underline{i}, \tag{6.18}$$

This result can be verified by using the example of a rotational converter with N windings connected to ideal current sources. The winding currents are determined by the source currents and therefore constant. On the basis of (5.6), which applies in cases where magnetic field exists in linear medium, work of the source is evenly distributed between the mechanical work and the increase of the field energy; thus,

$$\mathrm{d}W_m = \mathrm{d}W_{meh} = T_{em}\, \mathrm{d}\theta_m,$$

which results in the torque expression

$$T_{em} = \frac{\mathrm{d}W_m}{\mathrm{d}\theta_m}.$$

The energy of magnetic field is determined by (6.10). Therefore, the torque expression assumes the following form:

$$T_{em} = \frac{\mathrm{d}}{\mathrm{d}\theta_m} \left(\frac{1}{2} \underline{i}^T \underline{L} \underline{i} \right).$$

In accordance with the above assumptions, the winding currents are fed from ideal current sources. Therefore, the current intensities are constant. For this reason, the inductance matrix is the only factor in the above expression that may change as a function of angle θ_m. Therefore, the torque expression takes the form

$$T_{em} = \frac{1}{2} \underline{i}^T \frac{\mathrm{d}\underline{L}}{\mathrm{d}\theta_m} \underline{i}.$$

Electromagnetic torque can exist if at least one element of the induction matrix varies as function of angle θ_m. This could be variation of the winding self-inductance or variation of the mutual inductance between two windings. Variable inductances change as the rotor changes its position relative to the immobile stator. For a system of N windings, the electromagnetic torque is given by (6.19):

$$T_{em} = \frac{1}{2} \sum_{k=1}^{N} \sum_{j=1}^{N} \left(i_k i_j \frac{\mathrm{d}L_{jk}}{\mathrm{d}\theta_m} \right). \tag{6.19}$$

Question (6.3): Determine the course of change of the mutual inductance between two windings, one of them being on the stator and the other on the rotor.

Answer (6.3): Since the rotor revolves, angle θ_m between the stator and rotor reference axes varies. Without the lack of generality, it can be assumed that the case $\theta_m = 0$ corresponds to the rotor position where the magnetic axes of the two windings are collinear. In such case, the lines of the magnetic field created by the stator winding are perpendicular to the surface delineated by the turns of the rotor winding. The part of the stator flux passing through rotor winding is at a maximum. Mutual inductance between the two windings is the highest for $\theta_m = 0$. When the rotor moves, angle between the field lines and the rotor surface is no longer $\pi/2$ but takes value of $\pi/2-\theta_m$. Since the flux is determined by the sine of this angle, variation of the mutual inductance is determined by function $\cos\theta_m$. Therefore, $L_{SR} = L_m \cos\theta_m$.

Question (6.4): Give an example of a cylindrical machine with one of the stator windings having the self-inductance that varies with the rotor position.

Answer (6.4): Self-inductance of stator winding depends on the number of turns and magnetic resistance. Resistance of the magnetic circuit consists of the resistance of the iron core of the stator, resistance of the magnetic circuit of the rotor, and magnetic resistance of the air gap, where flux from the stator magnetic circuit passes to the rotor magnetic circuit and *vice versa*. Magnetic circuit of the rotor is mainly cylindrical, and has a circular cross section. By removing some iron on the rotor sides, the cross section becomes elongated and resembles an ellipse. The elliptical rotor and cylindrical stator produce a variable air gap. Therefore, the flux extending along the larger rotor diameter will encounter magnetic resistance much smaller than the flux oriented along the shorter diameter of the elliptical rotor. For this reason, the stator flux meets a variable magnetic resistance as the rotor revolves. In turn, the self-inductance $L_S = N_S^2/R_\mu$ is variable as well.

6.10 Mechanical Subsystem

Moving parts of a rotating electrical machine are magnetic and current circuits of the rotor, shaft, and bearings. Bearings are usually mounted on both shaft ends. They hold the rotor shaft firm and collinear with the axis of the stator cylinder. There are electrical machines having special fans built in the rotor for the purpose of enhancing the air flow and facilitate the cooling (*self-cooling machines*). In addition, rotor often has built-in sensors for performing measurements of the speed of rotation, position, and temperature of the rotor. In some cases, rotor may contain permanent magnets or semiconductor diodes. When modeling mechanical subsystem of an electrical machine, these details are not taken into account, and the rotor is modeled as a homogeneous cylinder of known mass and dimensions. Owing to the action of electromagnetic forces, torque T_{em} acts upon the rotor. The rotor is

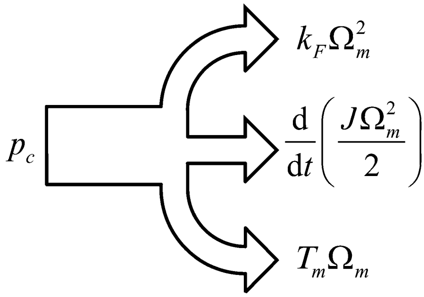

Fig. 6.5 Balance of power in mechanical subsystem of rotating electrical machine. Obtained mechanical power p_c covers the losses in mechanical subsystem and the increase of kinetic energy and provides the output mechanical power $T_{em}\Omega_m$

connected via shaft to a work machine or a driving turbine. The shaft transfers the mechanical torque T_{em}.

In the mechanical subsystem, there are losses due to friction and ventilation. A certain amount of energy is accumulated as kinetic energy in the rotating parts of the machine. For this reason, mechanical torque T_m existing at the shaft output is not equal to the electromagnetic torque T_{em} acting on the rotor. The power of electromechanical conversion p_c is divided into the losses of mechanical subsystem, accumulation, and output power $p_m = T_m\Omega_m$, as shown in Fig. 6.5.

6.11 Losses in Mechanical Subsystem

Losses in mechanical subsystem consist of the energy required to overcome the resistance due to air friction experienced by the rotor and to overcome the friction in bearings, as well as the motion resistances of other nature and of secondary importance. Power losses in mechanical subsystem vary as a function of speed (Fig. 6.5). This variation may be a complex function of speed. Since the losses in mechanical subsystem are usually small, it is not of interest to introduce a complex model, but most often, the assumption is introduced that the friction torque is proportional to the speed and can be modeled by expression $k_F\Omega_m$, resulting in the expression for power losses in the mechanical subsystem

$$p_{\gamma m} = k_F\Omega_m^2. \tag{6.20}$$

This model of losses appears in a number of books and articles dealing with electrical machines. There are, however, electrical machines and applications where this model of losses in the mechanical subsystem is inadequate.

Model of the losses in a mechanical subsystem, shown in Fig. 6.5, has been developed at the time when majority of electrical machines were mainly DC machines, which will be discussed in more detail in the subsequent chapters. Stator of these machines creates a still magnetic field, and it accommodates revolving rotor.

Variation of magnetic induction in the rotor magnetic circuit is determined by the speed of rotation Ω_m. Losses due to eddy currents in the rotor magnetic circuit are

$$p_{Fe} = k_V \Omega_m^2.$$

Dividing the losses p_{Fe} by the speed Ω_m, the torque $T_{Fe} = p_{Fe}/\Omega_m$ is obtained which resists the rotor motion and tends to diminish the speed. This torque is equal to

$$T_{Fe} = \frac{p_{Fe}}{\Omega_m} = k_V \Omega_m.$$

The obtained motion resistance T_{Fe} corresponds to the model $p_{\gamma m} = k_F {\Omega_m}^2$, often used in the literature.[2]

In electrical machines operating at speeds above 1,000 rad/s per second, there is a substantial air resistance. Forces resisting movement through the air are proportional to the square speed; thus, power of losses due to air friction should be modeled by expression

$$p_{\gamma m} = k_F \Omega_m^3.$$

6.12 Kinetic Energy

Accumulation of energy in the rotating masses is dependent on rotor inertia J. Rotor can be represented as a homogeneous cylinder of uniform specific mass in all its parts. With radius R and mass m, resulting moment of inertia is $J = ½mR^2$. Kinetic energy W_k of a rotor with moment of inertia J and speed Ω_m is $W_k = ½\, J{\Omega_m}^2$. In order to increase kinetic energy, it is necessary to supply the power $\mathrm{d}(W_k)/\mathrm{d}t = J\Omega_m\, \mathrm{d}\Omega_m/\mathrm{d}t$. Therefore, increasing the speed of rotation involves adding an amount of energy into revolving masses, while in order to slow down a speed of rotation, the energy should be taken away (by supplying a negative power).

[2] Losses p_{Fe} in the magnetic field of the rotor of DC machines p_{Fe} belong to the losses in magnetic circuit, that is, to iron losses. Nevertheless, the motion resistance torque T_{Fe} arises due to losses p_{Fe}. It is of interest to emphasize that in AC machines having permanent magnets, motion resistance T_{Fe} also appears even in cases with no electric current in the windings. Motion resistance T_{Fe} does not belong to mechanical losses, since it is not caused by friction in the bearings nor friction with the air, but it is a specific *electrical friction*. There is, therefore, dilemma whether power p_{Fe} should be classified as motion resistance losses or losses in the magnetic field. If all the losses that oppose to motion and diminish the speed are classified as motion resistance losses, whether their cause is mechanical friction or not, then losses in the rotor magnetic circuit of DC machines should be classified as motion resistance losses as well. Similar dilemma arises in the classification of the stator iron losses of synchronous machines having permanent magnets in their rotors.

As a consequence of this, the torque T_m obtained at the shaft of the machine is smaller than electromagnetic torque T_{em} during acceleration intervals because one part of power p_C and one part of the torque $T_{em} = p_C/\Omega_m$ are used to increase kinetic energy of revolving masses. For the same reason, the torque T_m might exceed the electromagnetic torque T_{em} during deceleration intervals.

Balance of power shown in Fig. 6.5 can be represented in analytical form. Considering a most common electromechanical converter with only one mechanical access (that is, only one shaft), the mechanical power is given by expression

$$p_m = T_m\Omega_m.$$

Kinetic energy is given by

$$W_k = \frac{1}{2}J\Omega_m^2,$$

and the rate of change of the kinetic energy is

$$\frac{\mathrm{d}W_k}{\mathrm{d}t} = J\Omega_m\frac{\mathrm{d}\Omega_m}{\mathrm{d}t}. \tag{6.21}$$

Starting from the power of electromagnetic conversion p_C, one part of this power is dissipated on the losses in the mechanical subsystem (6.20), and the other part changes kinetic energy and alters the speed of rotation (6.21); thus, the mechanical power available at the shaft (the output power) is given by (6.22):

$$\begin{aligned} p_c &= \Omega_m T_{em} = \frac{\mathrm{d}W_k}{\mathrm{d}t} + p_{\gamma m} + p_m \\ &= J\Omega_m\frac{\mathrm{d}\Omega_m}{\mathrm{d}t} + k_F\Omega_m^2 + T_m\Omega_m. \\ p_m &= p_c - J\Omega_m\frac{\mathrm{d}\Omega_m}{\mathrm{d}t} - k_F\Omega_m^2. \end{aligned} \tag{6.22}$$

6.13 Model of Mechanical Subsystem

Equation 6.22 can be divided by the rotor angular speed Ω_m to obtain (6.23) which determines the torque T_m. This torque is transferred to work machine via shaft.

$$T_m = T_{em} - J\frac{\mathrm{d}\Omega_m}{\mathrm{d}t} - k_F\Omega_m \tag{6.23}$$

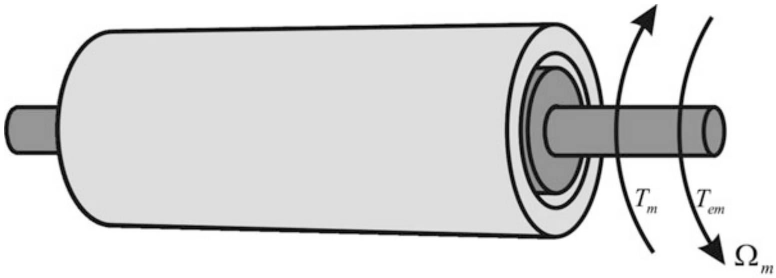

Fig. 6.6 Reference directions for electromagnetic torque and speed of rotation

In motor mode, the electrical machine acts on the work machine by the torque T_m in order to support its motion. At the same time, the work machine reacts by the torque $-T_m$ that opposes to rotor motion. Action and reaction torques are of the same magnitude, and they have opposite directions. Reference directions of the torque and speed are presented in Fig. 6.6. Former equation can be presented in the form

$$J \frac{d\Omega_m}{dt} = T_{em} - T_m - k_F \Omega_m, \tag{6.24}$$

which represents Newton equation for rotational motion. This equation is the model of mechanical subsystem of an electrical machine. In this equation, torque T_m is opposed to electromagnetic torque T_{em} as well as the friction torque. In the case when electromagnetic torque prevails ($T_{em} > T_m + k_F\Omega_m$), the speed Ω_m increases. If $T_{em} < T_m + k_F\Omega_m$, the speed decreases. In steady state, electromagnetic torque T_{em} is equal to the sum of all the torques that oppose to motion. Steady state is described by the equations

$$\frac{d\Omega_m}{dt} = 0, \quad T_{em} = T_m + k_F \Omega_m.$$

Figure 6.6 shows reference directions of the electromagnetic torque T_{em} and torque T_m of the work machine which opposes the motion. The torque with positive sign with respect to assigned reference directions corresponds to the motor mode. In the case of the *generator* mode, when mechanical work is converted to electrical energy, the meaning and signs of the above two torques are reversed. Namely, in the generator mode, the torque T_{em} has a negative sign, and it resists the motion, while the torque T_m tends to support the motion. In such cases, torque T_m is obtained from a turbine and supplied via shaft to the generator, making the rotor turn. In practice, reference directions of the torques T_{em} and T_m can be different than those shown in Fig. 6.6. Within this book, theoretical considerations and problem solving are written in accordance with directions presented in Fig. 6.6. Therefore, in the motor mode $T_{em} > 0$, $T_m > 0$, while in the generator mode $T_{em} < 0$, $T_m < 0$. As an exception, it is possible to define *generator* torque $T_G = -T_{em}$, which assumes a positive value in the generator mode.

The analysis and modeling of the mechanical subsystem apply to electromechanical converters having only one mechanical access. Moving parts in most electrical machines are rotors with only one degree of freedom. They revolve around the axis of the cylindrical stator. There is only one shaft attached to the rotor and positioned along the axis of rotation. The rotor motion is characterized by unique speed, and it depends on driving torques and motion resistance torques. It is possible to imagine, design, and produce electromechanical converters whose mobile parts could move with more than one degree of freedom, involving more different speeds and a corresponding number of forces and torques acting in different directions. These converters are not studied in this book.

6.14 Balance of Power in Electromechanical Converters

Diagram presented in Fig. 6.7 shows the power flow in a rotational electromechanical converter having N windings located on immobile cylindrical stator and on revolving cylindrical rotor. It is assumed that converter operates with magnetic coupling field. The relevant powers presented in the figure are explained in the following sequence.

Power at electrical access of the machine, also called input[3] power, or electrical power transferred by the source to the converter is

$$p_e = \underline{i}^T \underline{u}.$$

The power lost in the windings due to Joule effect represents losses in the electrical subsystem. This power is called *power of losses in copper*, and it is equal to

$$p_{Cu} = \underline{i}^T \underline{R}\,\underline{i} = \sum_k R_k i_k^2.$$

Power which determines the increase of energy of the coupling field is

$$p_{wm} = \frac{\mathrm{d}W_m}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{i}^T \underline{L}\,\underline{i}\right).$$

Power of losses in the magnetic circuit, also called power of iron losses, amounts

$$p_{Fe} = \sigma_H B^2 f + \sigma_V B^2 f^2,$$

and it is neglected in preliminary considerations.

[3] For electrical motors, electrical power is supplied to the motor, and it is considered an input. Mechanical power is obtained on the shaft, and it represents an output. In the case that machine operates in the generator mode, mechanical power is considered an input, while electrical power is output.

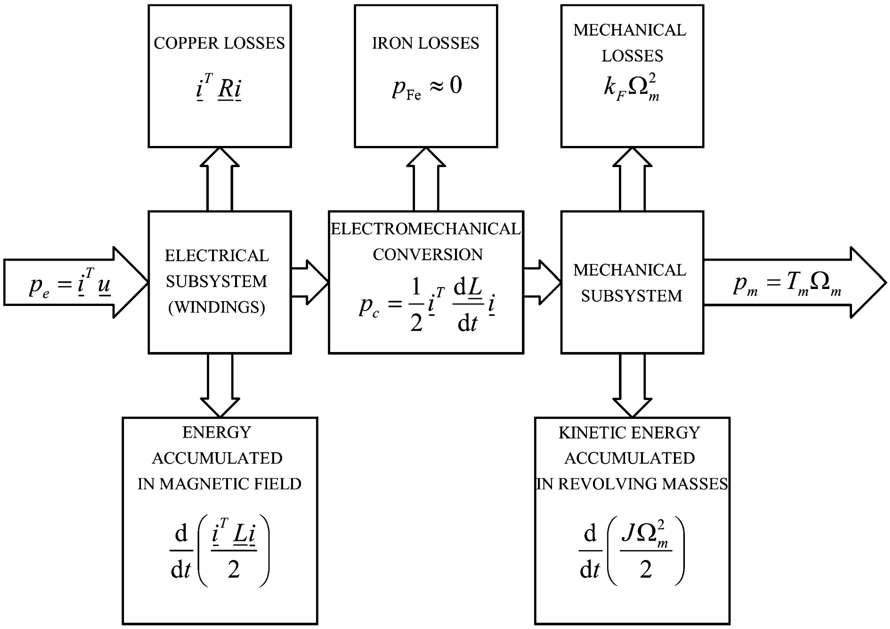

Fig. 6.7 Block diagram of the electromechanical conversion process

Power of electromechanical conversion is

$$P_c = \frac{1}{2} \underline{i}^T \frac{\mathrm{d}\underline{L}}{\mathrm{d}t} \underline{i}.$$

Power which determines the increase in kinetic energy of revolving parts represents the accumulation in the mechanical subsystem, and it is equal to

$$p_{wk} = \frac{\mathrm{d}W_k}{\mathrm{d}t} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(J\Omega_m^2\right).$$

Power of losses in the mechanical subsystem (power of losses due to rotation or motion resistance losses) is equal to

$$p_{\gamma m} = k_F \Omega_m^2.$$

Power at the mechanical access of the machine is also called output power or shaft power, and it is equal to

$$p_m = T_m \Omega_m.$$

6.15 Equations of Mathematical Model

On the basis of the preceding considerations and introduced approximations, the mathematical model of an electrical machine with N windings contains:

1. N differential equations expressing the voltage balance (6.7)
2. Inductance matrix which establishes relation between the currents and winding fluxes (6.9)
3. Expression for electromagnetic torque (6.19)
4. Newton equation of movement (6.24)

Differential equations of the voltage balance are given by expression

$$\underline{u} = \underline{R} \cdot \underline{i} + \frac{\mathrm{d}\underline{\Psi}}{\mathrm{d}\,t}. \tag{6.25}$$

The relation between fluxes and currents is given by the nonstationary inductance matrix

$$\underline{\Psi} = \underline{L}(\theta_m) \cdot \underline{i}. \tag{6.26}$$

The electromagnetic torque is determined by the following equation:

$$T_{em} = \frac{1}{2}\underline{i}^T \frac{\mathrm{d}\underline{L}}{\mathrm{d}\theta_m}\underline{i} = \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N}\left(i_k i_j \frac{\mathrm{d}L_{jk}}{\mathrm{d}\theta_m}\right). \tag{6.27}$$

According to (6.24), transient phenomena in the mechanical subsystem are determined by Newton differential equation of motion. The change of the speed of rotation is determined by expression

$$J\frac{\mathrm{d}\Omega_m}{\mathrm{d}t} = T_{em} - T_m - k_F\Omega_m. \tag{6.28}$$

The four above expressions define general model of a rotational electromechanical converter based on the magnetic coupling field. The model is derived including the four previously mentioned approximations. Among the approximations are the assumptions that ferromagnetic materials are linear and that iron losses are negligible.

The inductance matrix is a nonstationary matrix. In general, elements of the matrix may be functions of the angle, time, as well as of the flux and current, which could change the self-inductances and mutual inductances due to nonlinearities in ferromagnetic materials and due to magnetic saturation. Within the following considerations, it is considered that the ferromagnetic material is linear and that the inductance matrix and its elements are dependent only on the angle θ_m. This approximation is justified in the majority of cases and will not present an obstacle in understanding the operation of electrical machines and deriving their characteristics.

It should be noted in 6.27 that the electromechanical conversion can be accomplished only in cases where at least one element of the inductance matrix changes with the angle θ_m, either self-inductance of a winding or mutual inductance between two windings.

In cases where an electrical machine has N windings, expression (6.25) contains N differential equations of voltage balance, expression (6.26) gives relation between the winding currents and corresponding fluxes, expression (6.27) gives electromagnetic torque, and expression (6.28) is Newton differential equation defining the speed change. Therefore, the model contains $N + 1$ differential equations and the same number of *state variables*.