Chapter 15
Modeling of Induction Machines

This chapter introduces and explains mathematical model of induction machines. This model represents transient and steady-state behavior in electrical and mechanical subsystems of the machine. Analysis and discussion introduces and explains Clarke and Park coordinate transforms. The model includes differential equations that express the voltage balance in stator and rotor windings, inductance matrix which relates flux linkages and currents, Newton differential equation of motion, expression for the air-gap power, and expression for the electromagnetic torque. The model development process starts with replacing the three-phase machine with two-phase equivalent. Namely, the three-phase voltages, currents, and flux linkages are transformed in two-phase variables by appropriate transformation matrix which implements $3\Phi/2\Phi$ transform, also called Clarke coordinate transform. Two-phase model is formulated in stationary coordinate frame. The drawbacks and difficulties in using this model are the rationale for introducing and applying Park coordinate transform, which results in the machine model in synchronous *dq* coordinate frame. Necessary techniques and procedures of applying and using coordinate transforms are explained in detail, including representation of machine vectors by complex numbers. The operable model of induction machines is obtained in *dq* coordinate frame which revolves synchronously with the stator field. The merits and practical uses of the model in *dq* frame are explained at the end of the chapter.

15.1 Modeling Steady State and Transient Phenomena

The work with induction machines requires a sound knowledge of their behavior and principal characteristics. From the electrical access point, it is of interest to find relations between steady-state voltage and currents, so as to obtain an equivalent circuit of the machine, representing the steady-state operation. At the same time, it is important to study the torque–speed relations at the mechanical access of the machine.

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Analysis of induction machines at steady state is based on *mechanical characteristics* and *steady-state equivalent circuit*. Mechanical characteristic of an induction machine gives relation between steady-state values of the electromagnetic torque and the rotor speed. The mechanical characteristic is dependent on the frequency and amplitude of the stator voltage. Therefore, any change in stator voltage will affect the mechanical characteristic. At steady state, the voltages and currents are sinusoidal quantities of constant amplitude and frequency. For that reason, they can be represented by appropriate phasors.[1] Relations between voltages and currents can be presented by equivalent circuit. A steady-state equivalent circuit is a network consisting of resistances and reactances which serves for calculation of phasors of the stator and rotor currents in conditions with known supply conditions and specified rotor speed.

Working on problems of supplying and controlling induction machines requires a good knowledge of the *dynamic model*. This model comprises differential equations and algebraic expressions relating the machine variables and parameters during transient processes and also in the steady state. Relation between the voltages and currents during transients is given by differential equations describing the voltage equilibrium in the windings, also called *voltage balance equations*. The voltage balance equations describe the *electrical subsystem* of induction machine. The *mechanical subsystem* is described by Newton differential equation of motion. The set of differential equations and expressions describing behavior of the machine is called *mathematical model* or *dynamic model*.

In further considerations, the analyses of electrical and mechanical subsystems of induction machines are presented and explained, resulting in dynamic model. This model includes transforms of the state coordinates, also called *coordinate transforms*. They facilitate the analysis of transient processes in both synchronous and induction machines. Dynamic model is usually mostly used for transient analysis and for solving control problems, but it can also be used to resolve steady states. Starting from dynamic model, one can obtain the steady-state relations; mechanical characteristics; relations between voltages, currents, fluxes, torques, and speed in the steady state; as well as the steady-state equivalent circuit.

The readers with no interest in transient processes in induction machines and with no need to deal with problems of supply and control do not have to study dynamic model of induction machine. Such readers could skip entire Chap. 15 which develops mathematical model and deals with transient processes. The steady-state equivalent circuit can be also determined by using analogy with a transformer, as shown in Sect. 16.6. The analyses of steady-state equivalent circuits and the study of mechanical characteristics of induction machines can be continued in Chap. 16.

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[1] Phasor is a complex number which represents a sinusoidal AC voltage or current. The absolute value of phasor corresponds to the amplitude, while the phasor argument determines the initial phase of considered voltages and currents. Phasors can be used to represent other quantities that have sinusoidal change in steady state, such as the magnetomotive forces and fluxes.

15.2 The Structure of Mathematical Model

Within the introductory chapters, it is shown that the dynamic model of electrical machines comprises four basic parts. These are:

1. N differential equations of voltage equilibrium
2. Inductance matrix
3. Expression for the torque
4. Newton equation

Differential equations of voltage balance are given by expression

$$\underline{u} = \underline{R} \cdot \underline{i} + \frac{\mathrm{d}\underline{\Psi}}{\mathrm{d}t}. \tag{15.1}$$

Relation between the fluxes and currents is given by nonstationary inductance matrix

$$\underline{\Psi} = \underline{L}(\theta_m) \cdot \underline{i}. \tag{15.2}$$

The electromagnetic torque is determined by equation

$$T_{em} = \frac{1}{2}\underline{i}^T \frac{\mathrm{d}\underline{L}}{\mathrm{d}\theta_m}\underline{i} = \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1}^{N}\left(i_k i_j \frac{\mathrm{d}L_{jk}}{\mathrm{d}\theta_m}\right). \tag{15.3}$$

Transient phenomena in mechanical subsystem are determined by Newton differential equation of motion

$$J\frac{\mathrm{d}\Omega_m}{\mathrm{d}t} = T_{em} - T_m - k_F\Omega_m. \tag{15.4}$$

The four equations given above define general model applicable to any rotating electrical machine. The model is derived assuming four basic approximations:

1. The effects of distributed parameters are neglected.
2. The energy of electrical field is neglected along with parasitic capacitances.
3. The iron losses are neglected.
4. Magnetic saturation is neglected along with nonlinear $B(H)$ characteristic of ferromagnetic materials.

In the case when a machine has N windings, expression (15.1) contains N differential equations of voltage balance, expression (15.2) provides relation between the winding currents and their fluxes, expression (15.3) gives the electromagnetic torque, and expression (15.4) is in fact Newton differential equation describing variation of the rotor speed. Therefore, in the presented model, there are $N + 1$ differential equations and the same number of *state variables*.

15.3 Three-Phase and Two-Phase Machines

Most induction machines have a three-phase stator winding. Stator AC currents create the vector of magnetomotive force $F_S = F_a + F_b + F_c$. This vector has a radial direction within the machine, and it does not have any axial component. Therefore, it resides in the plane defined by radial and tangential unit vectors of cylindrical coordinate frame. The same plane can be represented by rectangular coordinate system of two orthogonal axes, hereafter denoted by α and β. In order to represent the vector in the α–β coordinate system, directions of axes α and β are defined by their corresponding unit vectors $\boldsymbol{\alpha_0}$ and $\boldsymbol{\beta_0}$.

While the machine has three-phase windings, spatially displaced by 2π/3, relevant vector will be displayed in α–β coordinate system. Therefore, there is a need to express the orientation of the magnetic axes of individual phases in terms of unit vectors $\boldsymbol{\alpha_0}$ and $\boldsymbol{\beta_0}$:

$$\begin{aligned}\vec{a}_0 &= \vec{\alpha}_0,\\ \vec{b}_0 &= -\frac{\vec{\alpha}_0}{2} + \frac{\sqrt{3}}{2}\vec{\beta}_0,\\ \vec{c}_0 &= -\frac{\vec{\alpha}_0}{2} - \frac{\sqrt{3}}{2}\vec{\beta}_0.\end{aligned} \tag{15.5}$$

With symmetrical set of three-phase voltages, the stator currents can be expressed as

$$\begin{aligned}i_a &= I_m \cos\omega_e t,\\ i_b &= I_m \cos(\omega_e t - 2\pi/3),\\ i_c &= I_m \cos(\omega_e t - 4\pi/3),\end{aligned} \tag{15.6}$$

and they result in the following magnetomotive forces:

$$\begin{aligned}\vec{F}_a &= N i_a \vec{\alpha}_0,\\ \vec{F}_b &= N i_b \left(-\frac{1}{2}\vec{\alpha}_0 + \frac{\sqrt{3}}{2}\vec{\beta}_0\right),\\ \vec{F}_c &= N i_c \left(-\frac{1}{2}\vec{\alpha}_0 - \frac{\sqrt{3}}{2}\vec{\beta}_0\right).\end{aligned} \tag{15.7}$$

The sum of the three magnetomotive forces results in

$$\begin{aligned}\vec{F}_s &= \vec{F}_a + \vec{F}_b + \vec{F}_c = N\left[\vec{\alpha}_0\left(i_a - \frac{i_b}{2} - \frac{i_c}{2}\right) + \vec{\beta}_0 \frac{\sqrt{3}}{2}(i_b - i_c)\right],\\ \vec{F}_s &= \frac{3}{2} N I_m \left[\vec{\alpha}_0 \cos\omega_e t + \vec{\beta}_0 \sin\omega_e t\right],\end{aligned} \tag{15.8}$$

the vector which revolves at the speed $\Omega_e = \omega_e$ and maintains the amplitude $F_{Sm} = 3/2\ NI_m$. Modeling three-phase winding encounters certain difficulties. One of them is the fact that the phase currents are not independent variables. They are restrained by relation $i_a + i_b + i_c = 0$, which comes from the circumstance that the windings are star connected. For delta connection, this problem has different nature. Namely, the sum of the phase voltages of delta-connected winding is equal to zero. Considering star-connected winding, conclusion is drawn that only two out of three stator currents are independent variables. Therefore, in all differential equations, the current i_c has to be replaced by $(-i_a - i_b)$, making the equations clumsy and difficult to work with. In addition, angular displacement between the magnetic axis of the phase windings is $2\pi/3$, resulting in nonzero values of mutual inductances. An increased number of nonzero elements in the inductance matrix increases the number of factors in voltage balance equations, making them more involved and less intuitive. On the other hand, hypothetical two-phase machine can be envisaged with only two currents and zero mutual inductance. The mathematical model of an induction machine is more simple if considered machine has two-phase windings on the stator, one of them oriented along unit vector α_0 and the other oriented along unit vector β_0. The stator winding has only two electrical currents, i_α and i_β, and they are independent. Due to orthogonal magnetic axes of the windings, their mutual inductance is zero, simplifying a great deal the voltage balance equations.

In a two-phase machine with stator windings oriented along unit vectors $\boldsymbol{\alpha_0}$ and $\boldsymbol{\beta_0}$, the mathematical model becomes more usable because the winding currents correspond to projections of the magnetomotive force vector on the axes α and β. Namely, the magnetomotive force component along the axis α is $F_\alpha = Ni_\alpha$, which is also projection of the vector F_S on axis α. The magnetomotive force component along the axis β is $F_\beta = Ni_\beta$, equal to projection of the vector $\boldsymbol{F_S}$ on axis β. The same conclusions can be derived for the flux vector. The flux in the phase winding α is equal to the projection of the flux vector $\boldsymbol{\Phi_S}$ on the axis α. Correspondence between the phase quantities and projections of relevant vectors on axes α and β facilitates understanding and using the two-phase model.

One and the same magnetomotive force can be obtained with both three-phase and the two-phase windings. The three-phase system of phase windings of Fig. 15.1 can be replaced by the two-phase system of phase windings, given in Fig. 15.2.

On the basis of (15.8) and assuming that the number of turns is unchanged ($N_{abc} = N_{\alpha\beta}$), the stator magnetomotive force vector $\boldsymbol{F_S}$ retains the same orientation and amplitude provided that the electrical currents in the two-phase system are

$$\begin{aligned} i_{\alpha s} &= i_a - \frac{i_b}{2} - \frac{i_c}{2} = \frac{3}{2}\ i_a = \frac{3}{2}\ I_m \cos\omega_e t \\ i_{\beta s} &= \frac{\sqrt{3}}{2}(i_b - i_c) = \frac{3}{2}\ I_m \sin\omega_e t. \end{aligned} \tag{15.9}$$

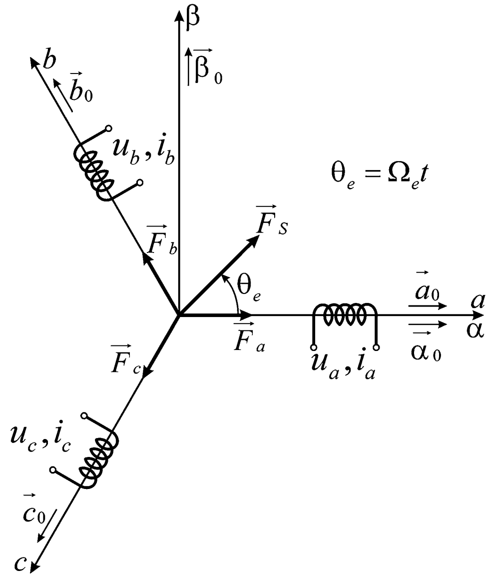

Fig. 15.1 Positions of the phase windings in orthogonal αβ coordinate system

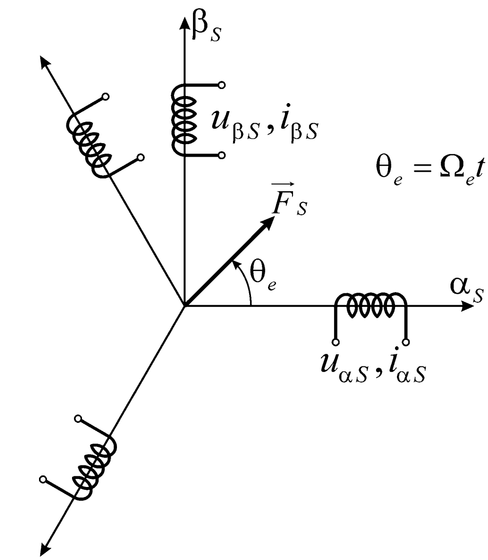

Fig. 15.2 Replacing three-phase winding by two-phase equivalent

Thought experiment of removing a three-phase winding from induction machine and replacing them with two orthogonal phase windings results in a two-phase induction machine. Provided that the electrical currents $i_\alpha(t)$ and $i_\beta(t)$ in two-phase windings correspond to (15.9), this modified machine would have the same vector of the stator magnetomotive force $\boldsymbol{F_S}$ as the original three-phase machine. Consequently, the stator flux vector $\boldsymbol{F_S}/R_\mu$ would be the same as well. Moreover, the same flux and speed would result in the same electromagnetic torque. Namely, in addition to the same flux amplitude and speed Ω_e and with the same rotor speed, the rotor would have the same slip frequency, electromotive forces, and currents as the original three-phase machine. This thought experiment results in the following conclusion: *The operation of three-phase induction machine would not change if the three-phase winding is replaced by the two-phase winding, provided that the latter provides the same stator magnetomotive force*. In other words, neither flux nor torque or power of the machine is changed by replacing the three-phase winding by the two-phase equivalent, provided that the magnetomotive force remains invariant. This conclusion will be used further on.

Question (15.1): Starting from the described thought experiment, where three-phase winding system is replaced by two-phase winding system and where the number of turns N_{abc} in each phase of the former is equal to the number of turns $N_{\alpha\beta}$ in each phase of the latter, compare the phase voltages of the two. Is it possible to make a two-phase equivalent of the original machine that would have different voltages and currents? (See Fig. 15.2).

Answer (15.1): One should recall that the maximum value of the electromotive force induced in one turn is $e_1 = \omega_e \Phi_m$, while the maximum value of the winding electromotive force is $e = \omega_e \Psi_m$, while the voltage balance equation for the phase a is $u_a = R_a i_a + \mathrm{d}\Psi_a/\mathrm{d}t \approx \mathrm{d}\Psi_a/\mathrm{d}t = -\omega_e \sin(\omega_e t)\Psi_m$. Assumption is that both the original three-phase machine and the equivalent two-phase machine have the same magnetomotive force, flux, torque, and power. Therefore, the stator flux is in both cases of the same amplitude, and it revolves at the same speed. For that reason, the electromotive force induced in one turn is unchanged. Since the number of turns in each phase winding is the same, the voltages u_{abc} and $u_{\alpha\beta}$ are of the same amplitude, and they have the same rms values. Notice that the ratio u/i changes as the original machine is replaced by the equivalent. This ratio has dimension of impedance. Although the voltages are proven to be the same, electrical currents $i_{\alpha\beta}$ of the two-phase machine have their amplitude and rms value larger than currents i_{abc} by 50% (see (15.9)).

Generally, a three-phase stator winding can be replaced by a two-phase stator winding with $N_{\alpha\beta} = mN_{abc}$ turns. In such cases, phase voltages of the two-phase equivalent would be $u_{\alpha\beta} = m\, u_{abc}$. The magnetomotive force F_S would remain unaltered provided that electrical currents of the two-phase equivalent are obtained according to $i_{\alpha\beta} = (3/2) \cdot (i_{abc}/m)$. Hence, the right-hand side of (15.9) should be divided by m.

* * *

Although the two-phase equivalent of induction machine is simple, unambiguous, and intuitive, induction machines are nevertheless manufactured, deployed, and used as three-phase machines with three-phase windings on the stator. Magnetic axes of the stator phases are displaced by 2π/3. There are practical advantages of the three-phase systems over the two-phase systems which resulted in the former being widely used.

Considering the number of conductors required to connect an induction machine to the grid, a three-phase stator winding gets connected to the mains by three lines (wires). The three line-to-line voltages have the same rms value, 0.4 kV for mains supplied low-voltage machines. At steady state, each of the three supply lines has the line current of the same rms values. The number of conductors in the high-voltage transmission lines is also three. Hypothetical two-phase system does not have the same advantages.

Question (15.2): A two-phase induction machine is fed from two voltage sources having the voltages of the same amplitude, phase shifted by $\pi/2$. It is necessary to connect these sources to the machine and use only three supply lines (i.e., three wires). Determine the voltages between the conductors, and compare the rms values of their line currents.

Answer (15.2): A two-phase machine can be fed from the two voltage sources by using four wires to connect each end of the two supplies (and there are two of them) to the winding terminals α1, α2, β1, and β2 of the two-phase system. It is possible to reduce the number of wires (lines) by using one and the same return path for the two windings. The two return lines, say α2 and β2, can be merged and replaced by a single line α2β2. Then, the number of conductors can be only three. Yet, in this case, the currents in these three lines would not have the same amplitude. The current in the return conductor α2β2 is $i_\alpha(t) + i_\beta(t) = (3/2)\, I_m(\cos\omega_e t + \sin\omega_e t)$. It has $2^{0.5}$ times higher amplitude and rms value than the current in remaining two lines. This asymmetry exists in line voltages as well. The voltage between the line α1 and the return conductor α2β2 corresponds to the phase voltage U_α. The voltage between the line β1 and the return conductor α2β2 corresponds to the phase voltage U_β and has the same amplitude as the previous one. On the other hand, the voltage between conductors α1 and β1 is equal to $U_\alpha - U_\beta$, and it has $2^{0.5}$ times higher amplitude.

* * *

Uneven voltages and currents in three-wired two-phase systems are one of the reasons it never had any wider practical use. The complexity in wiring such system is considerable, since the line conductors cannot be exchanged. On the other hand, equal voltages and currents of the three-phase, three-wire system make the connection process much easier. Connection of the three-phase induction machine to the three-phase mains is much easier since all the wires have the same rms value of electrical current and the same rms value of their line-to-line voltages. The worst consequence of making a random connection is the possibility that the machine would rotate in wrong direction.[2] Nowadays, all the power lines and distribution networks operating with line frequency AC voltages are symmetrical three-phase

[2] When this is the case, it is sufficient to exchange any two of the three connections for the machine to change direction and revolve correctly. The two line conductors to be exchanged can be arbitrarily chosen. It is an understatement that this action must be performed in no voltage conditions.

systems. Therefore, even AC machines, such as induction machines and synchronous machines, are made with three-phase stator windings. For the purposes of modeling and analysis, three-phase machines are represented by their two-phase equivalent, so as to achieve clear and usable models, equivalent circuits, and other mathematical representations of machine.

15.4 Clarke Transform

A three-phase machine can be represented by its two-phase equivalent (Fig. 15.3). If the two-phase equivalent produces the same magnetomotive force F_S as the original machine, then the equivalent machine has the same flux, torque, and power as the original three-phase machine. Invariant magnetomotive force is obtained provided that the two-phase equivalent has the number of turns $N_{\alpha\beta}$ and currents $i_\alpha(t)$ and $i_\beta(t)$ that result in the same amplitude and spatial orientation of the vector F_s. In cases where $N_{\alpha\beta} = N_{abc}$, respective stator currents are related by (15.9).

It is not necessary to actually make the two-phase equivalent in order to use the benefits of the two-phase model. Instead, mathematical operation similar to (15.9) can be applied to all the relevant variables. This operation is, as a matter of fact, *coordinate transform* suited to provide the user with a simple, clear, and intuitive model. Relation between the original variables (i_{abc}, u_{abc}, Ψ_{abc}) and their *transformed* counterparts, the two-phase equivalents ($i_{\alpha\beta}$, $u_{\alpha\beta}$, $\Psi_{\alpha\beta}$), is called coordinate transform, and it is expressed by relations similar to (15.9). In the considered case, the three-phase/two-phase transform is applied, named *Clarke transform* after the author.

Generally speaking, the actual state of each system subject to analysis or control is described by *state variables*. The set of state variables uniquely defines the *state* of dynamical system. Putting aside the usual approximations, the set of state variables provides enough information about the system so as to determine its further behavior. A state variable cannot be expressed in terms of other state variables. As an example, only two out of the three-phase currents in a three-phase

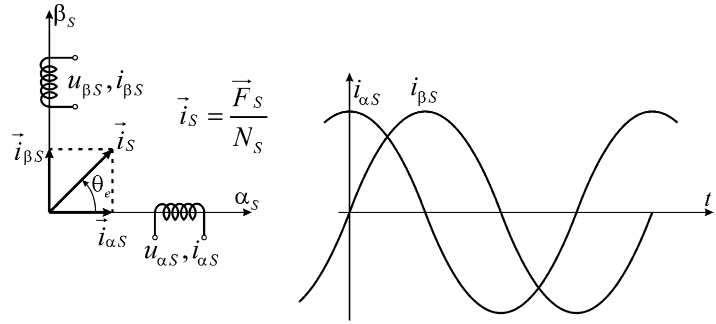

Fig. 15.3 Two-phase equivalent of a three-phase winding

winding are the actual state variables, as the third one is determined from the sum of the other two.

The benefit of coordinate transform can be demonstrated by a simple example. State of an object that moves in three-dimensional space can be described by coordinates x, y, and z in the orthogonal Cartesian coordinate system, as well as by the fist derivative of these coordinates dx/dt, dy/dt, and dz/dt, representing the speed. On the other hand, an observer may have a need to concerning distance r, elevation φ, and azimuth θ in spherical coordinate system. Coordinates x, y, and z can be expressed in terms of spherical coordinate system r, φ, and θ. The function which translates one set of coordinates into another set is called *coordinate transform*. Differential equations of motion can be written by using either the first or the second set of coordinates. The first set of equations would be called *model in x–y–z coordinate system*, and the second *model in spherical coordinate system*. Modeling the system in Cartesian or spherical coordinate system resembles looking into the windowed room through one or the other window. The room remains the same, but the image representing the room changes. Generally, selection of another coordinate system reflects only the observer viewpoint and does not have any impact on the object or system to be modeled. Selection of the appropriate coordinate system and corresponding transform of the state variables may have significant impact on mathematical model. Such model becomes simple, clear, and more intuitive, facilitating decision making regarding control and exploitation of the system. The model in Cartesian coordinate frame is more suitable when modeling an object that moves along the x axis, due to $\mathrm{d}y/\mathrm{d}t = 0$ and $\mathrm{d}z/\mathrm{d}t = 0$. An attempt to represent the same motion in spherical coordinate frame results in rather involved changes in coordinates r, φ, and θ. On the other hand, spherical coordinate frame is more suited to describe rotation around the origin or radial motion. Similarly, a three-phase machine can be represented in the original, three-phase domain but also by its equivalent two-phase machine. The latter proves more suitable to study machine properties and characteristics and to specify and design supplies and controls.

Electrical currents of the equivalent two-phase system which represents the three-phase winding are given by (15.10), which is the matrix form of (15.9). The matrix is multiplied by coefficient K_I. In the case when the two-phase equivalent and the three-phase winding have the same number of turns, the value of $K_I = 1$ is required to secure invariant magnetomotive forces F_S. It should be noted that a three-phase winding can be represented by a two-phase equivalent having different number of turns. In such case, for the vector of the stator magnetomotive force F_s to remain unchanged, coefficient K_I must have a different value:

$$\begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \end{bmatrix} = K_I \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}. \tag{15.10}$$

The question arises whether the three variables such as i_a, i_b, and i_c can be replaced by only two, $i_{\alpha s}$ and $i_{\beta s}$. Due to $i_a + i_b + i_c = 0$, only the two-phase currents of the original machine are independent state variables, which provides the rational for the transform expressed by (15.10).

15.5 Two-Phase Equivalent

Transform of the state variables of an existing three-phase machine can be understood as a thought experiment which represents a three-phase machine by an imaginary two-phase machine. There is also possibility to actually replace an existing three-phase stator winding by a two-phase stator winding that provides the same magnetomotive force, flux, torque, and power as the original three-phase machine. It is of interest to compare the two induction machines that have the same behavior. One of them is the original three-phase machine, and the other is the two-phase equivalent. It is assumed that two-phase induction machine, called M2, has the same magnetic circuits and the same rotor as the original three-phase machine, M3. If the individual phases of M2 and M3 have the same number of turns, electrical currents in respective stator windings must correspond to the following relation:

$$\begin{aligned} i_{\alpha s} &= i_a - \frac{i_b}{2} - \frac{i_c}{2} = \frac{3}{2}\, i_a \\ i_{\beta s} &= \frac{\sqrt{3}}{2}(i_b - i_c). \end{aligned} \tag{15.11}$$

so as to provide the same magnetomotive force. With the same magnetomotive force and identical magnetic circuits, both machines have the same flux. Electromotive force in one turn is proportional to the flux and the angular frequency ω_e. Therefore, each turn in machines M2 and M3 has electromotive force of the same amplitude. With $N_{abc} = N_{\alpha\beta}$, electromotive forces induced in phases a, b, c, α, and β have the same peak and rms values. With the assumption that the voltage drop Ri is negligible with respect to the electromotive force, conclusion is drawn that, in the considered case, the phase voltages u_{abc} and $u_{\alpha\beta}$ have the same peak and rms values. Specifically, the voltage across the phase a of machine M3 has the same peak value as the voltage across the phase α of machine M2. On the other hand, considering (15.9), the peak and rms values of the phase currents $i_{\alpha\beta}$ are 3/2 times larger with respect to i_{abc} currents. The above-mentioned considerations show that a three-phase machine can be converted into a two-phase machine by rewinding the stator, yet preserving the same magnetomotive force, flux, torque, and power. Maintaining the same number of turns, the phase voltages remain the same, while the phase currents increase by factor 3/2.

Common practice in applying coordinate transforms to electrical machines includes applying one and the same transformation formula to all the relevant variables, whether voltages, current, or flux linkages. Benefits of this approach will be discussed within subsequent chapters. For the example involving machines M2 and M3, the transformation matrix for electrical currents is given in (15.11). Applying the same formula to voltages, one obtains (15.12) which gives the phase voltage $u_{\alpha s}$ obtained by using the same three-phase to two-phase transform as the one used for currents:

$$u_{\alpha s} = u_a - \frac{u_b}{2} - \frac{u_c}{2} = \frac{3}{2}\ u_a. \tag{15.12}$$

Apparent problem arises from the fact that the actual phase voltage $u_{\alpha s}$ of the rewound machine M2 does not correspond to the value obtained in (15.12). It seems that there is no way to actually make the equivalent two-phase machine unless the transformation matrices used for voltages are different than those used for currents. Further on, the answer to Question (15.3) proves the opposite. It shows that, with the proper choice of K_I, it is possible to devise a three-phase to two-phase transform that corresponds to two-phase induction machine that can actually be made.

Coordinate transforms do not have to correspond to actual physical systems in order to prove their usefulness in modeling. An example is Park transform, discussed and used in subsequent chapters, which proves very useful in deriving dynamic model and steady-state equivalent circuit and yet results in state variables that correspond to virtual electrical machine that cannot be made.

Considered example includes the three-phase machine M3 which is transformed into two-phase equivalent M2. Machines M2 and M3 have identical magnetic circuit and the same number of turns per phase. The problem that arises with the machine M2 is that it has voltages that do not correspond to those obtained by applying the transform matrix (15.14) on the original voltages u_{abc}. Notice that the currents are transformed according to (15.10), adopting $K_I = 1$. Generally, the voltages can be transformed by using

$$\begin{bmatrix} u_{\alpha s} \\ u_{\beta s} \end{bmatrix} = K_U \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}.$$

Therefore, with $K_U = K_I = 1$, the phase voltages and currents obtained by applying the transform on machine M3 do not correspond to the voltages and currents actually measured on machine M2. On the other hand, coefficients K_I and K_U do not have to be equal to 1. Other values can be applied as well. The only practical restriction is $K_I = K_U$, which maintains the ratio between the voltages and currents and secures that all the impedances of the original machine retain their value after the transformation. Not even this restriction is obligatory, yet it is often imposed due to practical reasons. Assuming that the phase current is transformed according to (15.10), with $K_I = 1$, while the voltages are transformed by using $K_U = 2/3$, the voltages and currents derived from the transform will correspond to those measured on two-phase machine M2 which has the same number of turns per phase as three-phase original M3. Drawback of this approach is that the ratio of voltage and current of machines M2 and M3 will not be the same. Thus, proposed transform will affect the impedances. They will not be invariant. Parameter R_S of the three-phase machine would have to be multiplied by 2/3 in order to get the parameter R_S of the two-phase machine. Generally, impedances of the original three-phase machine should be multiplied by K_U/K_I in order to obtain impedances of the two-phase equivalent.

Up to now, discussion was focused on devising a 3-phase to 2-phase transform that corresponds to physical prototypes M2 and M3. In general, transformed quantities can but do not have to correspond to a practical two-phase machine. It is acceptable to adopt $K_I = 1$ and $K_U = 1$ and obtain correct mathematical model. This transform does provide the voltages that can be measured on M2, but it has the advantage of being impedance invariant. On the other hand, transform with $K_I = 1$ and $K_U = 1$ is not invariant in terms of power, namely, $P_{abc} \neq P_{\alpha\beta}$. Nonetheless, such model can be advantageously used. The lack of power invariance has to be kept in mind and taken care of.

Three-phase to two-phase transform with $K_I = K_U = 2/3$ is frequently encountered. It is impedance invariant, but it brings in the relation $P_{abc} = 3/2\ P_{\alpha\beta}$. For better understanding, before listing the properties of Clarke transform, the values of frequently used coefficients K_I and K_U will be described in brief.

15.6 Invariance

If Clarke transform preserves the ratio between voltages and currents of the three-phase original and the two-phase equivalent, it is invariant in terms of impedance. If the ratio between fluxes and currents remains the same, the transform is invariant in terms of inductance. If the expression for power $P_{\alpha\beta}$ of the two-phase equivalent corresponds to the power of the original three-phase machine, then the transform is invariant in terms of power.

It is necessary to point out that transforms which are not power invariant can be advantageously used, provided that the user of mathematical model respects the ratio $P_{abc} = K \cdot P_{\alpha\beta}$.

First-time user of coordinate transforms may nurture doubts whether the mathematical model is correct, considering that it calculates apparently incorrect power due to $P_{abc} \neq P_{\alpha\beta}$. To resolve such doubts, it is important to recall that the state variables obtained by using coordinate transform do not have to correspond to any machine that could be actually made. However, this does not minimize practical values of the mathematical model. As an example, one can start with mathematical model of a simple resistor, $u = Ri$. By performing *coordinate transform* $u_1 = 2u$

and $i_1 = 2i$, one obtains the model having electrical power four times higher than the actual resistor. However, this model is still useful. Given the current, one can calculate the voltage according to $u_1 = Ri_1$. The user should recall the power invariance and calculate the actual power as $u_1 i_1/4$. Representation of u_1, i_1 of a resistor does not represent real resistor, but it stands as a usable model.

Most practical uses of Clarke transform retain the impedance invariance and the inductance invariance, while the lack of power invariance is often acceptable. This means that transforms of the currents, voltages, and fluxes are carried out by using the same transform matrix for all the variables. The transformation matrix is given in (15.13), while the coefficients for voltage and flux transform are given by $K_I = K_U = K_\Psi$:

$$\begin{bmatrix} i_\alpha \\ i_\beta \end{bmatrix} = K_I \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}. \tag{15.13}$$

It is of interest to use (15.13) with $K_I = 1$ and derive the phase currents of the two-phase equivalent having the same number of turns ($N_{\alpha\beta} = N_{abc}$) as the three-phase original. The three-phase winding with symmetrical set of phase currents $i_a(t) = I_m \cos\omega_e t$, $i_b(t) = I_m \cos(\omega_e t - 2\pi/3)$, and $i_c(t) = I_m \cos(\omega_e t - 4\pi/3)$ can be transformed by using $K_I = 1$ and (15.13). The two-phase equivalent is obtained with $i_\alpha(t) = i_a(t) - i_b(t)/2 - i_c(t)/2 = 3/2\ I_m \cos\omega_e t$ and $i_\beta(t) = 3^{0.5}/2 \cdot (i_b(t) - i_c(t)) = 3/2\ I_m \sin\omega_e t$. Hence, the two-phase equivalent has phase currents shifted by $\pi/2$, which corresponds to the spatial shift between magnetic axes of corresponding windings.

In addition to the phase currents, the voltages and fluxes should also be transformed in $\alpha\beta$ coordinate frame. Clarke transform for the voltages and fluxes is given by

$$\begin{bmatrix} u_\alpha \\ u_\beta \end{bmatrix} = K_U \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}, \tag{15.14}$$

$$\begin{bmatrix} \Psi_\alpha \\ \Psi_\beta \end{bmatrix} = K_\Psi \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \Psi_a \\ \Psi_b \\ \Psi_c \end{bmatrix}. \tag{15.15}$$

In general, coefficients K_U and K_Ψ in the above expressions can be arbitrarily selected and do not have to be equal to K_I. The choice $K_I = K_U = K_\Psi$ has the advantages that contribute to legibility and usability of mathematical model that results from transforms.

Selecting $K_I = K_U$, one obtains impedance invariant Clarke transform. Namely, all the resistances, impedances, and other accounts where the ratio u/i appears remain unaltered by the transform. Hence, parameters such as R_S retain their value even in $\alpha\beta$ coordinate frame. Deciding otherwise would create the need to scale all the impedances by the ratio K_U/K_I.

Selection $K_I = K_\Psi$ results in inductance invariant Clarke transform. The self-inductances, mutual inductances, and leakage inductances of all the windings remain unaltered by the transform. Hence, deciding otherwise would create the need to scale all the inductances by the ratio K_Ψ/K_I.

Selection $K_U = K_\Psi$, relation between electromotive forces and fluxes, remains $e = \mathrm{d}\Psi/\mathrm{d}t$. Deciding otherwise would require the two-phase model to include relations such as $e = (K_U/K_\Psi)\,\mathrm{d}\Psi/\mathrm{d}t$.

Throughout this book, it is assumed that $K_I = K_U = K_\Psi$. Other choices are rarely met in reference literature. They result in mathematical models that are correct but more difficult to use than the models obtained by invariant transform.

The freedom of choice is often a problem. An example to that is the attempt to replace the original three-phase machine M3 by actual two-phase prototype M2, already discussed before. Considered is the case where the phase windings of both M3 and M2 have the same number of turns per phase, $N_{\alpha\beta} = N_{abc}$. Clarke transform of the three voltages (u_a, u_b, u_c) calculates the values (u_α, u_β) of the two-phase equivalent. In order to obtain the values of (u_α, u_β) that correspond to voltages actually measured on the machine M2, the coefficient K_U has to be equal to 2/3. This conclusion is already explained, and it relies on the fact that both machines have the same amplitude of electromotive forces induced in single turn. Due to $N_{\alpha\beta} = N_{abc}$, the phase windings of machines M2 and M3 have the same amplitudes of winding electromotive forces. With $e \approx u$, the same holds for the phase voltages as well. Hence, for the Clarke transform to provide the same voltages (u_α, u_β) and currents (i_α, i_β) as the actual prototype M2, with $N_{\alpha\beta} = N_{abc}$, it is necessary to use $K_U = 2/3$ and $K_I = 1$. Correspondence between the obtained two-phase equivalent and the actual machine M2 increases the user confidence. However, there are problems created by the choice $K_I \neq K_U$. The use of different transforms for voltages and currents leads to different ratios u/i in *abc* and $\alpha\beta$ coordinate systems. In other words, transform is not impedance invariant. Parameters such as resistance R or reactance X have different values in *abc* and $\alpha\beta$ frames. Any transition from *abc* to $\alpha\beta$ frame requires impedances to be scaled by 3/2. This does not mean that the model is inaccurate, but it compromises clarity and augments the chances of making errors.

Previous discussion demonstrates that the choice $K_U = 2/3$ and $K_I = 1$ results in Clarke transform that provides two-phase voltage and currents in full correspondence with the actual two-phase machine M2. Yet, such transform is not impedance invariant, and it brings difficulties in using the model. For that reason, decision $K_I = K_U = K_\Psi$ is used throughout this book, although it does not correspond to voltages and currents of the machine M2.

As a rule, while selecting transform of the state variables, it is considered that resistances (R) and inductances (L) should stay invariant. Therefore, invariability of impedances and inductances is set as a prerequisite. In other words, legibility and usability of the model are considered more important than similarity to the actual two-phase prototype M2.

In the analysis and modeling of electrical machines, all the transforms of the state variables are made so as to maintain the ratio of voltages, currents, and fluxes. In this way, transforms are impedance invariant and inductance invariant.

- The same voltage and current transform matrices result impedance invariability.
- The same flux and current transform matrices result in invariable self-inductances, mutual inductances, and leakage inductance.
- The same voltage and flux transform matrices maintain relation $e = \mathrm{d}\Psi/\mathrm{d}t$. With $K_U \neq K_\Psi$, this relation becomes relation $e = (K_U/K_\Psi)\ \mathrm{d}\Psi/\mathrm{d}t$.

Question (15.3): Is it possible to make an actual two-phase machine with $N_{\alpha\beta} = mN_{abc}$ turns which has the same stator magnetomotive force F_s as the original three-phase machine and which has, at the same time, the voltages, currents, and fluxes which correspond to values obtained by Clarke transform performed with $K_U = K_I = K_\Psi$?

Answer (15.3): The actual two-phase equivalent of the original three-phase machine must have the same magnetomotive force, flux, torque, and power as the original machine. A three-phase stator winding can be replaced by a two-phase winding having $N_{\alpha\beta} = mN_{abc}$ turns. Invariability of F_s requires that windings of the two-phase equivalent carry currents $i_{\alpha\beta} = (3/2)\cdot(i_{abc}/m)$. At the same time, the two-phase equivalent has the same electromotive force in a single turn and m times more turns per phase. Therefore, the phase voltages of the two-phase equivalent will be $u_{\alpha\beta} = mu_{abc}$. Finally, with $K_U = K_I$, the ratio of voltages and currents of the original machine (u_{abc}/i_{abc}) is equal to the ratio of voltages and currents of the two-phase equivalent ($u_{\alpha\beta}/i_{\alpha\beta}$). Summarizing the above statements,

$$\frac{u_{\alpha\beta}}{i_{\alpha\beta}} = \frac{m\,u_{abc}}{(3/2)(i_{abc}/m)} = \frac{2m^2}{3}\frac{u_{abc}}{i_{abc}} = \frac{u_{abc}}{i_{abc}}$$
$$\Rightarrow \quad m = \sqrt{\frac{3}{2}} \;\Rightarrow\; K_U = K_I = K_\Psi = \sqrt{\frac{2}{3}}.$$

Hence, when three-phase machine is replaced by two-phase equivalent which has $(3/2)^{0.5}$ times more turns per phase, the voltages and currents of the actual two-phase machine correspond to these obtained from the Clarke transform performed with leading coefficient of $(2/3)^{0.5}$. This Clarke transform calculates the voltages, currents, and fluxes in $\alpha\beta$ domain by applying the same transformation matrices to the original voltages, currents, and fluxes in the *abc* domain. Hence, $K_U = K_I = K_\Psi = (2/3)^{0.5}$. This transform is invariant in terms of impedance, inductance, magnetomotive force, torque, and power. The amplitude of fluxes in windings α and β, the amplitudes and rms values of corresponding voltages, and the amplitudes and rms values of currents are $(3/2)^{0.5}$ times higher compared to the original variables in *abc* domain. The presence of irrational number $(3/2)^{0.5}$ in calculations is the reason to avoid the Clarke transform with coefficient $K = (2/3)^{0.5}$, notwithstanding its positive sides.

Question (15.4): Prove that Clarke transform with $K_U = K_I = K_\Psi = (2/3)^{0.5}$ is invariant in terms of power.

Answer (15.4): Without lack of generality, it is possible to assume that the machine operates in steady state. Electrical power in each phase winding can be calculated as product of rms values of winding current, voltage, and power factor. For purposes of proving the power invariance, the latter can be considered constant or even equal to 1. Total power of the machine is obtained as the sum of individual phase powers. Consider a three-phase machine with rms values of the phase voltages and currents U_{abc} and I_{abc}; electrical power of the three-phase machine is found to be 3 $U_{abc}I_{abc}$. By applying Clarke transform with coefficients $K_U = K_I = K_\Psi = (2/3)^{0.5}$, one obtains currents and voltages in $\alpha\beta$ domain with amplitudes $(3/2)^{0.5}$ times higher. Therefore, the phase α power is equal to $[(3/2)^{0.5}U_{abc}]\cdot[(3/2)^{0.5}I_{abc}] = 3/2\ U_{abc}I_{abc}$. The same power is obtained in phase β, resulting in total power of $3U_{abc}I_{abc}$, which confirms invariability in terms of power.

Question (15.5): Prove that the application of Clarke transform with $K_U = K_I = K_\Psi = 1$ is not invariant in terms of power but results in $P_{\alpha\beta} = (3/2)P_{abc}$.

Answer (15.5): Assume that voltages and currents of the three-phase machine are known and equal to u_a, u_b, u_c, i_a, i_b, and i_c. By using expression (15.13) for the three-phase/two-phase transform, it is required to determine variables u_α, u_β, i_α, and i_β. By replacing the corresponding variables from the original *abc* domain in equation $P_{\alpha\beta} = u_\alpha\, i_\alpha + u_\beta\, i_\beta$, one obtains that $P_{\alpha\beta} = (3/2)P_{abc}$.

15.6.1 Clarke Transform with K = 1

In the case when $K_U = K_I = K_\Psi = 1$, the two-phase equivalent machine should have the same number of turns in order to provide the same magnetomotive force. Transformed variables include 3/2 times larger amplitudes of voltages and currents. Transform is invariant in terms of impedance and inductance, but it is not power invariant. Hence,

$$N_{\alpha\beta} = N_{abc}, \tag{15.16}$$

$$\left|\vec{i}_{\alpha\beta}\right| = \frac{3}{2}\cdot i_{abc}^{\max}, \tag{15.17}$$

$$\left|\vec{u}_{\alpha\beta}\right| = \frac{3}{2}\cdot u_{abc}^{\max}. \tag{15.18}$$

The transform is:

- Invariant in terms of impedance
- Invariant in terms of inductance
- Not invariant in terms of power since

$$P_{\alpha\beta} = \frac{3}{2} P_{abc} \tag{15.19}$$

15.6.2 Clarke Transform with K = sqrt(2/3)

In the case when $K_U = K_I = K_\Psi = (2/3)^{0.5}$, the two-phase equivalent machine should have $(3/2)^{0.5}$ times increased number of turns so as to provide the same magnetomotive force. Transformed variables include $(3/2)^{0.5}$ times larger amplitudes of phase voltages and currents. Transform is invariant in terms of impedance, inductance, and power. Hence,

$$N_{\alpha\beta} = \sqrt{\frac{3}{2}} N_{abc} \tag{15.20}$$

$$\left| \vec{i}_{\alpha\beta} \right| = \sqrt{\frac{3}{2}} \cdot i_{abc}^{\max} \tag{15.21}$$

$$\left| \vec{u}_{\alpha\beta} \right| = \sqrt{\frac{3}{2}} \cdot u_{abc}^{\max} \tag{15.22}$$

The transform is:

- Invariant in terms of impedance
- Invariant in terms of inductance
- Invariant in terms of power

15.6.3 Clarke Transform with K = 2/3

In the case when $K_U = K_I = K_\Psi = 2/3$, the two-phase equivalent machine should have 3/2 times increased number of turns so as to provide the same magnetomotive force. Transformed variables include the same amplitudes of phase voltages and currents. Transform is invariant in terms of impedance and inductance, but it is not power invariant. Hence,

$$N_{\alpha\beta} = \frac{3}{2} N_{abc} \tag{15.23}$$

$$\left| \vec{i}_{\alpha\beta} \right| = i_{abc}^{\max} \tag{15.24}$$

$$\left| \vec{u}_{\alpha\beta} \right| = u_{abc}^{\max} \tag{15.25}$$

The transform is:

- Invariant in terms of impedance
- Invariant in terms of inductance
- Not invariant in terms of power since

$$P_{\alpha\beta} = \frac{2}{3} P_{abc} \tag{15.26}$$

15.7 Equivalent Two-Phase Winding

Preceding sections summarize the needs for representing a three-phase machine by its two-phase equivalent. Clarke $3\Phi/2\Phi$ transform is introduced and explained. The choice of transform coefficients is discussed, along with consequences in terms of impedance, inductance, and power invariance of the transform. In further analysis, the following $3\Phi/2\Phi$ transform is adopted and used.

Clarke $3\Phi/2\Phi$ transform of voltages, currents, and fluxes is performed in a unified way, by using the same transform matrix for all the variables having the leading coefficient of $K = 2/3$. The symbol V in the expression (15.27) represents voltage, current, or flux in one phase winding:

$$\begin{bmatrix} V_\alpha \\ V_\beta \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}. \tag{15.27}$$

As a consequence, the applied transform is invariant in terms of impedances and inductances. Therefore, parameters such as R_S, R_R, L_m, L_S, and all other inductances and resistances retain their original values.

The peak and rms values of variables in $\alpha\beta$ frame are equal to the peak and rms values of original *abc* variables. Identities $u_\alpha(t) \equiv u_a(t)$, $i_\alpha(t) \equiv i_a(t)$, and $\Psi_\alpha(t) \equiv \Psi_a(t)$ apply too.

Selected transform with $K = 2/3$ provides $\alpha\beta$ variables that cannot be reproduced by any actual two-phase prototype. Namely, it is not possible to make an actual two-phase stator winding which replaces the three-phase winding, provides the magnetomotive force, and has the stator voltages and currents which correspond to the values obtained by the transform. Invariability of F_s requires $N_{\alpha\beta} = 3/2\, N_{abc}$, which leads to $u_{\alpha\beta} = 3/2\, u_{abc}$, owing to equal electromotive forces induced in one turn. On the other hand, selected transform results in $u_{\alpha\beta} = u_{abc}$.

Power of the two-phase equivalent $P_{\alpha\beta} = u_\alpha i_\alpha + u_\beta i_\beta$ is equal to $2/3P_{abc}$. Namely, since phase quantities in $\alpha\beta$ domain have the same values as the original counterparts in *abc* domain, the power per phase is equal, resulting in $P_{\alpha\beta} = 2/3$ P_{abc}. While using the model, it should be recalled that numerical value $P_{\alpha\beta} = u_\alpha i_\alpha + u_\beta i_\beta$ should be multiplied by 3/2 in order to get the power of the original three-phase machine. Therefore,

$$P_{abc} = \frac{3}{2} P_{\alpha\beta}. \tag{15.28}$$

The question arises whether the model in $\alpha\beta$ frame with $P_{\alpha\beta} = 2/3P_{abc}$ stands as an adequate representation of the induction machine having 3/2 larger power. It has to be recalled that the coordinate transforms result in a mathematical model with variables that do not necessarily correspond to any actual machine. Any attempt to envisage a practical two-phase equivalent and to interpret the variables u_α, i_α, u_β, and i_β as voltages and currents of practical α and β windings may be helpful in understanding and using the model. Yet, the virtual machine in $\alpha\beta$ frame is actually a mathematical fiction, and therefore, relations such as $P_{\alpha\beta} = 2/3P_{abc}$ do not invalidate the model. Recall that any resistor with voltage u and current i can be represented by mathematical model $u_1 = Ri_1$, where the new voltage and current are obtained by coordinate transform $u_1 = 2u$ and $i_1 = 2i$. The model has electrical power 4 times larger than the actual resistor and been used to represent the basic properties of the resistor. Due to lack of the power invariance of the transform $V_1 = 2\ V$, the user should recall to calculate the actual power as $u_1 i_1/4$.

15.8 Model of Stator Windings

By applying Clarke transform, a three-phase machine can be represented by a two-phase equivalent. Axes of virtual phase windings α_S and β_S are still with respect to the stator. The axis α_S is collinear with the magnetic axis of the phase winding *a* of the original machine. The model where the currents, voltages, and fluxes of the stator are represented by their α_S and β_S components is called *model in stationary coordinate frame*. Given the voltage, current, and flux vectors of the stator winding, their α_S and β_S components can be found as projections of relevant vectors on the axes of α_S-β_S coordinate frame.

Figure 15.4 shows an induction machine represented by a two-phase stator winding and a two-phase rotor winding. Angle θ_m represents the rotor position,

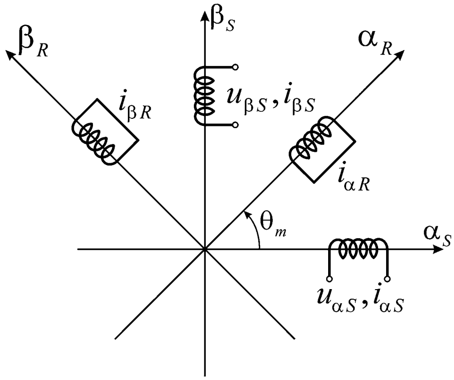

Fig. 15.4 Two-phase equivalent

$$\theta_m = \theta_{m0} + \int_0^t \Omega_m \mathrm{d}\tau. \qquad (15.29)$$

By reducing the three-phase stator to the two-phase equivalence, the model obtains two virtual stator windings, α_S and β_S, which create the magnetomotive force and flux along axes α_S and β_S of the still coordinate system called *stationary* or *stator* coordinate system. Currents and voltages of these windings are denoted by $u_{\alpha S}$, $u_{\beta S}$, $i_{\alpha S}$, and $i_{\beta S}$, in order to distinguish them from the rotor variables which are introduced later and which make use of subscript R.

Question (15.6): Direction and amplitude of the vector of magnetomotive force of the stator F_s are known. Determine currents $i_{\alpha S}$ and $i_{\beta S}$.

Answer (15.6): The required currents are determined by projections of vector F_s on axes α_S and β_S of the still coordinate system.

15.9 Voltage Balance Equations

Voltage equilibrium in three-phase winding is given by expressions comprising phase voltages u_a, u_b, and u_c; phase currents i_a, i_b, and i_c; and total flux linkages of the phase windings Ψ_a, Ψ_b, and Ψ_c. The flux Ψ_a in the phase winding a has component $L_S i_a$, produced by the current in the same winding. Coefficient L_S is the self-inductance of the stator winding. Other windings on stator and rotor may contribute to the flux Ψ_a. Their contribution is proportional to electrical currents in those windings and also to the coefficient of mutual inductance. For winding denoted by x, the flux contribution is $L_{ax} i_x$, where L_{ax} is mutual inductance between the phase winding a and the winding x, while i_x is the corresponding current. Phase windings have the same number of turns, the same resistance $R_a = R_b = R_c = R_S$, and the same coefficients of self-inductance $L_a = L_b = L_c = L_S$. For any windings, the voltage, current, and flux are tied by relation $u = Ri + \mathrm{d}\Psi/\mathrm{d}t$. Therefore,

$$\begin{aligned} u_a &= R_S i_a + \mathrm{d}\Psi_a/\mathrm{d}t, \\ u_b &= R_S i_b + \mathrm{d}\Psi_b/dt, \qquad \Rightarrow \\ u_c &= R_S i_c + \mathrm{d}\Psi_c/\mathrm{d}t. \end{aligned}$$

$$\begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} = \begin{bmatrix} R_S & 0 & 0 \\ 0 & R_S & 0 \\ 0 & 0 & R_S \end{bmatrix} \begin{bmatrix} i_a \\ i_c \\ i_c \end{bmatrix} + \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \Psi_a \\ \Psi_b \\ \Psi_c \end{bmatrix}. \tag{15.30}$$

By applying Clarke transform which uses the same transformation matrix for the voltages, currents, and fluxes, the voltage balance equations can be transferred to α_S–β_S coordinate frame and expressed in terms of α_S and β_S projections of the voltage, current, and flux vectors. Quantities $\Psi_{\alpha S}$ and $\Psi_{\beta S}$ are projections of the stator flux vector on axes α_S and β_S. They can be calculated by applying the three-phase/two-phase transform to the total fluxes of the phase windings Ψ_a, Ψ_b, and Ψ_c. Moreover, the transformation matrix can be applied to the whole right side of (15.30), obtaining in this way:

$$\begin{aligned} u_{\alpha S} &= R_S i_{\alpha S} + \mathrm{d}\Psi_{\alpha S}/\mathrm{d}t, \\ u_{\beta S} &= R_S i_{\beta S} + \mathrm{d}\Psi_{\beta S}/\mathrm{d}t. \end{aligned} \tag{15.31}$$

The above equation represents the voltage equilibrium in the two-phase equivalent of the stator winding. In addition to modeling the stator, it is required to model the short-circuited rotor cage. Voltage equilibrium equations in the rotor circuit will complete the model of the electrical subsystem of the induction machine.

15.10 Modeling Rotor Cage

The rotor cage contains a relatively large number of conductors which are short circuited by the front and rear rings. An example of the rotor cage separated from the rotor magnetic circuit is shown in Fig. 15.5a. For rotor cage with $N_R = 28$ conductors, it is possible to identify 14 short-circuited turns, each created by one pair of diametrically positioned conductors. Therefore, it is possible to make a model of the rotor comprising 14 short-circuited turns with mutual magnetic coupling, also coupled with α_S and β_S stator windings, as shown in Fig. 15.5b. However, such model would be of little practical value. Its inductance matrix will have dimensions 16×16. Therefore, another approach is needed to model the rotor cage.

As the first step, it is of interest to observe the part (c) in Fig. 15.5 and assume that the rotor flux pulsates along the vertical axis. At this point, it is of interest to

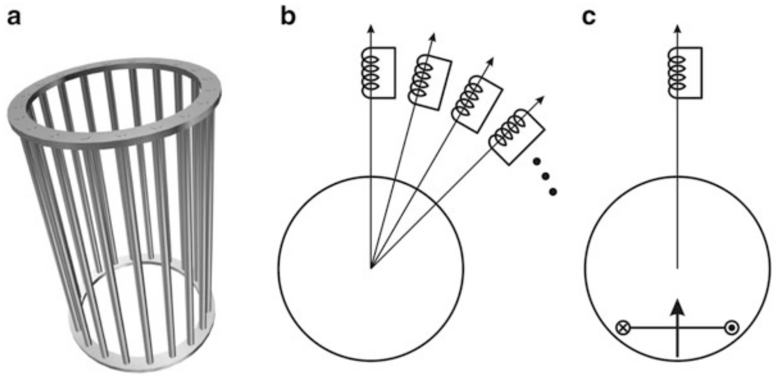

Fig. 15.5 Modeling the rotor cage

derive a model of the rotor cage that would reflect the induction of the rotor electromotive forces and currents in this particular case. The rotor conductors can be considered as a set of turns where the conductors making one turn are positioned symmetrical with respect to the vertical axis, as shown in Fig. 15.5c. With the assumption that pairs of rotor conductors which constitute one turn reside on the same horizontal line, and also assuming that there are no other connections between the turns, electrical currents in such turns would create the rotor flux along vertical axis. Under assumptions, they cannot make any flux in horizontal direction. Therefore, such rotor windings can be denoted by the symbol of coil placed on the vertical axis, as indicated in Fig. 15.5. This symbol represents the rotor turns that can be envisaged as one short-circuited rotor phase with vertical magnetic axis. For the time being, this approach overlooks the circumstance that the front and rear rings make a short circuit for all conductors. Assuming that an external stator flux pulsates along vertical axis, the flux within individual rotor turns would change, resulting in electromotive forces and consequential currents in short-circuited rotor turns. This thought experiment proved that the rotor winding connected according to the part (c) in Fig. 15.5 provides the short-circuiting effect of the rotor cage for vertical pulsations of the external flux. The rotor currents induced due to changes in the external flux act toward suppressing the flux changes. Namely, they contribute to the rotor flux in the direction opposite to the original flux change.

However, the same setup cannot represent the short-circuiting effect of the rotor cage in cases where the changes of the external flux have their horizontal component. For flux pulsations along horizontal direction, there are no induced electromotive forces and no rotor currents since the only phase winding of the rotor has vertical magnetic axis. Hence, it does not react to changes in horizontal flux component. Recall that the setup in Fig. 15.5c reacts only to changes in vertical component of the flux. Therefore, the rotor model comprising only one short-circuited phase winding cannot serve as an accurate representation of phenomena occurring in short-circuited rotor cage.

An actual rotor cage which is short circuited by the front and rare conductive rings exhibits its short-circuiting effects in arbitrary direction. As a matter of fact,

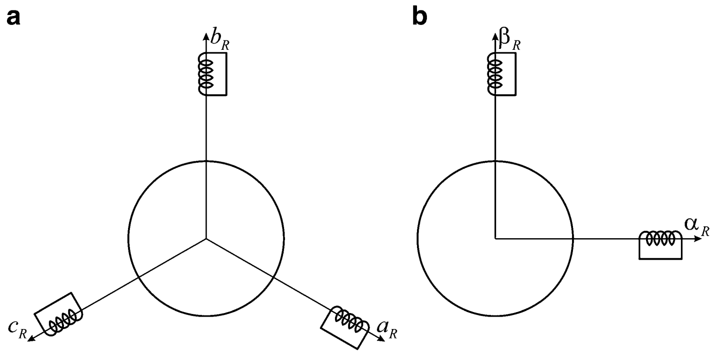

Fig. 15.6 Three-phase rotor cage and its two-phase equivalent

the end rings provide the short circuit between all the rotor bars and make a short-circuited turnout of any pair of rotor bars. Thus, variation of the external flux in any arbitrary orientation induces electromotive forces and currents in rotor turns that have their magnetic axis aligned with the vector of the flux change. The rotor cage is symmetrical, and it has a number of conductors. For this reason, the cage can be modeled as a three-phase winding, with individual rotor phases being short circuited and shifted by 2π/3, as shown in Fig. 15.6a. It is also possible to model the rotor as a two-phase, short-circuited winding, as shown in Fig. 15.6b. Validity of the two-phase model of the rotor cage can be verified by considering an arbitrary variation of the flux and analyzing the rotor reaction. Variation of the external flux with an arbitrary orientation can be represented by two orthogonal flux components lying along horizontal and vertical axis, which correspond to magnetic axis of the representative two-phase winding. Since parameters like resistance and inductance of the phase windings are identical, the short-circuiting effect of the rotor is the same for both flux components. In both horizontal and vertical axis, the rotor reacts to the flux changes by induced electromotive forces and consequential rotor currents. Hence, the two-phase representation of the short-circuited cage provides the model of the rotor reaction which does not depend on spatial orientation of the external flux changes. Therefore, two-phase representation of short-circuited rotor cage is an adequate model of the rotor winding, whatever the number of rotor conductors, provided that all the conductors are the same and that they are equally spaced around the rotor circumference.[3]

In the course of rotor motion, the rotor changes its position θ_m with respect to the stator. Therefore, magnetic axes of the two-phase rotor winding change their relative positions with respect to magnetic axes of the stator winding. In

[3] The rotor winding cannot be represented by two-phase equivalent in cases when the rotor cage is damaged. If one or more conductors are broken or disconnected from the short-circuiting rings, the rotor reaction to flux changes will be different in some directions. These cases are out of the scope of this book.

Fig. 15.4, the rotor axes are denoted by α_R and β_R. The voltages in short-circuited phases of the rotor are equal to zero; hence, $u_{\alpha R} = u_{\beta R} = 0$. The rotor currents $i_{\alpha R}$ and $i_{\beta R}$ represent electrical currents induced in the rotor cage, and they create the rotor magnetomotive force $\boldsymbol{F_R}$ whose amplitude and direction depend on currents $i_{\alpha R}$ and $i_{\beta R}$ but also on the rotor position θ_m. In the case when $i_{\alpha R} > 0$ and $i_{\beta R} = 0$, the vector $\boldsymbol{F_R}$ lies along α_R axis. For a given vector $\boldsymbol{F_R}$, currents of the two-phase model of the rotor cage can be determined from projections of this vector on the rotor axes α_R and β_R. The coefficient of proportionality between these currents and magnetomotive force is determined by the number of turns of the two-phase model of the rotor cage. The rotor phases α_R and β_R are virtual phases, that is, they are mathematical fiction that represent the rotor cage. Therefore, the number of turns of such virtual windings can be arbitrarily chosen. It should be noted that the short-circuiting effect of the rotor cage can be modeled by two-phase equivalent with large number of turns made comprising conductors with a lower cross section but also with lower number of turns made of conductors with larger cross section, even with $N_R = 1$. The original cage is aluminum cast, and it has one conductor per slot. For convenience, it is frequently assumed that the two-phase equivalent winding representing the rotor has the same number of turns as the stator phases. In this manner, transformation of rotor variables to the stator side is implied, and all the rotor variables and parameters that appear in the model are already scaled by the appropriate transformation ratio N_S/N_R.

15.11 Voltage Balance Equations in Rotor Winding

Two-phase representation of the stator and rotor windings reduces the mathematical model of the electrical subsystem of an induction machine to a set of four coupled phase windings. One pair of phase winding resides on the stator and the other pair on rotor. Due to rotor motion, the phase windings change their relative position. The voltage balance equation that applies to each of these windings is $u = Ri + \mathrm{d}\Psi/\mathrm{d}t$, where u, R, i, and Ψ denote the voltage across terminals of the considered phase winding, the winding resistance, electrical current, and total flux, respectively. The rotor winding is short circuited; thus, the voltage balance equations take the form

$$
\begin{aligned}
u_{\alpha S} &= R_S i_{\alpha S} + \frac{\mathrm{d}\Psi_{\alpha S}}{\mathrm{d}t}, \\
u_{\beta S} &= R_S i_{\beta S} + \frac{\mathrm{d}\Psi_{\beta S}}{\mathrm{d}t}, \\
0 &= R_R i_{\alpha R} + \frac{\mathrm{d}\Psi_{\alpha R}}{\mathrm{d}t}, \\
0 &= R_R i_{\beta R} + \frac{\mathrm{d}\Psi_{\beta R}}{\mathrm{d}t}.
\end{aligned} \tag{15.32}
$$

15.12 Inductance Matrix

Electrical subsystem of an induction machine is described by 4 differential equations of voltage balance comprising 4 currents and 4 fluxes. Among these 8 variables, there are only 4 state variables. Namely, if electrical currents $i_{\alpha S}$, $i_{\beta S}$, $i_{\alpha R}$, and $i_{\beta R}$ are promoted to the state variables, then the 4 fluxes $\Psi_{\alpha S}$, $\Psi_{\beta S}$, $\Psi_{\alpha R}$, and $\Psi_{\beta R}$ can be expressed in terms of currents. Relation between the fluxes and currents is generally nonlinear, due to nonlinearity of ferromagnetic materials and magnetic saturations. Under assumptions adopted in modeling electrical machines, which include the assumption that ferromagnetic materials have linear *B*-*H* characteristic, the flux linkages and electrical currents are in linear relation, defined by the inductance matrix. For the induction machine under consideration, the inductance matrix is given by expression (15.33). Along the main diagonal of the inductance matrix, there are coefficients of self-inductances of the phase windings. Coefficient $L_{11} = L_S$ is self-inductance of the stator winding α_S. Given the magnetic resistance R_μ of the stator flux circuit and the number of turns N_S, self-inductance of the stator can be determined as $L_{11} = L_{22} = L_S = N_S{}^2/R_\mu$. The stator phases have the same number of turns. At the same time, the air gap does not change along the machine circumference, and therefore, the magnetic resistance is also the same. For that reason, both stator phases have the same self-inductance L_S. Coefficients $L_{33} = L_{44} = L_R$ are self-inductances of rotor phases α_R and β_R. Assuming that the rotor phase windings have the same number of turns as the stator phase windings, the difference in L_S and L_R depends on magnetic resistances encountered by the stator and rotor flux linkages. The field lines of the rotor flux pass through the same air gap as the lines of the stator flux. Therefore, magnetic resistance to the stator flux is approximately equal to the magnetic resistance to the rotor flux. Small difference between L_S and L_R can be seen due to different leakage flux path and different leakage inductances:

$$\begin{bmatrix} \Psi_{\alpha s} \\ \Psi_{\beta s} \\ \Psi_{\alpha R} \\ \Psi_{\beta R} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & L_m \cos\theta_m & -L_m \sin\theta_m \\ 0 & L_{22} & L_m \sin\theta_m & L_m \cos\theta_m \\ L_m \cos\theta_m & L_m \sin\theta_m & L_{33} & 0 \\ -L_m \sin\theta_m & L_m \cos\theta_m & 0 & L_{44} \end{bmatrix} \cdot \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \\ i_{\alpha R} \\ i_{\beta R} \end{bmatrix}. \tag{15.33}$$

15.13 Leakage Flux and Mutual Flux

If rotor comes to position where magnetic axis of one stator phase coincides with magnetic axis of one rotor phase, then mutual inductance between them assumes maximum value. Figure 15.7 defines the mutual flux and the leakage flux. Flux linkage in one turn of the stator phase and flux linkage in one turn of the rotor phase are given by equations

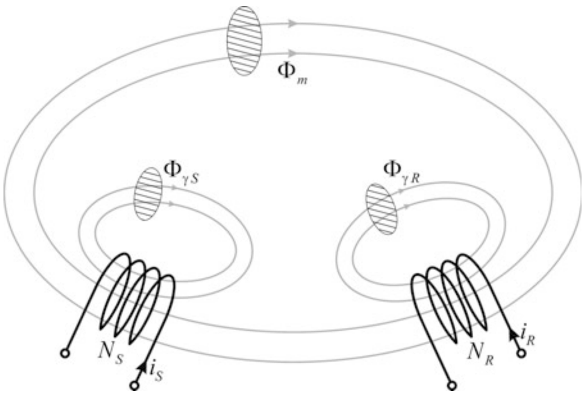

Fig. 15.7 Mutual flux and leakage flux

$$\Phi_S = \Phi_{\gamma S} + \Phi_m,$$
$$\Phi_R = \Phi_{\gamma R} + \Phi_m. \tag{15.34}$$

Each flux has mutual component, common for both stator and rotor turns, related to the lines of magnetic field that embrace both windings. In addition, there are leakage flux components. The stator leakage flux is related to magnetic field that encircles only the stator winding. It does not pass through the air gap and does not reach the rotor turns. Both stator and rotor currents contribute to the mutual flux. Mutual flux Φ_m in one turn has a component generated by the stator current ($\Phi^S{}_m$) and a component generated by the rotor current ($\Phi^R{}_m$),

$$\Phi_m = \Phi_m^S + \Phi_m^R. \tag{15.35}$$

The flux in phase windings depends on the flux in one turn and on the number of turns per phase. Therefore,

$$\Psi_S = N_S\Phi_S = N_S\Phi_m + N_S\Phi_{\gamma S} = N_S\Phi_m + \Psi_{\gamma S},$$
$$\Psi_R = N_R\Phi_R = N_R\Phi_m + N_R\Phi_{\gamma R} = N_R\Phi_m + \Psi_{\gamma R}. \tag{15.36}$$

In cases where $N_S = N_R$, the mutual flux components in stator and rotor windings are equal. Recall that N_R corresponds to the two-phase equivalent of the rotor winding, a mathematical fiction devised to model the rotor cage, while N_S corresponds to phase windings of the stator. Flux $\Psi_{\gamma S}$ is the *leakage flux* of the stator winding, while $\Psi_{\gamma R}$ is the leakage flux of the rotor winding. Leakage flux in each of the windings is proportional to the winding current. Coefficient of proportionality is *leakage inductance* of the winding. For the windings shown in Fig. 15.7, leakage inductances are given by expression

$$L_{\gamma S} = \frac{\Psi_{\gamma S}}{i_S}, \quad L_{\gamma R} = \frac{\Psi_{\gamma R}}{i_R}. \tag{15.37}$$

Mutual inductance of stator and rotor windings in aligned position equals

$$L_m = L_{SR} = \frac{N_S \Phi_m^R}{i_R} = L_{RS} = \frac{N_R \Phi_m^S}{i_S} \ . \tag{15.38}$$

Self-inductance of phase winding can be determined as the quotient of the winding flux and the winding current, wherein the flux is caused only by the current of the winding and does not get affected by other windings currents. This flux is mutual in one part, while the remaining part is leakage flux. Self-inductances of the stator and rotor are

$$\begin{aligned} L_S &= \frac{N_S \Phi_m^S + N_S \Phi_{\gamma S}}{i_S} = \frac{N_S \Phi_m^S + \Psi_{\gamma S}}{i_S} = \frac{N_S}{N_R} L_{RS} + L_{\gamma S} = \frac{N_S}{N_R} L_m + L_{\gamma S}, \\ L_R &= \frac{N_R \Phi_m^R + N_R \Phi_{\gamma R}}{i_R} = \frac{N_R \Phi_m^R + \Psi_{\gamma R}}{i_R} = \frac{N_R}{N_S} L_{SR} + L_{\gamma R} = \frac{N_R}{N_S} L_m + L_{\gamma R}. \end{aligned} \tag{15.39}$$

Therefore, leakage inductances make one part of self-inductances of phase windings. Leakage inductance is higher in the case when magnetic coupling of the two windings is weaker. In the case when the number of turns of the stator and rotor is equal, as well as in the case when the rotor quantities are *transformed* to the stator side, the preceding equation takes the form

$$\begin{aligned} L_S &= L_m + L_{\gamma S}, \\ L_R &= L_m + L_{\gamma R}. \end{aligned} \tag{15.40}$$

15.14 Magnetic Coupling

Leakage flux of the stator and leakage flux of the rotor exist in different magnetic circuits, and they may have different magnetic resistances. For that reason, even the leakage inductances can be different.

Gross part of the stator flux encircles both stator and rotor windings, but there are also some lines of magnetic field that encircle only the stator conductors. They do not cross the air gap and thus do not encircle the rotor conductors. These field lines belong to the leakage flux of the stator. Leakage flux of the stator is a smaller part of the stator flux $\boldsymbol{\Phi_S} = (L_S i_S)/N_S$. Leakage flux of the rotor is defined in similar. Different shapes of the stator and rotor slots as well as differences in the shape and cross section of conductors may result in different magnetic resistances on the path of the stator and rotor leakage fluxes.

Magnetic resistance encountered on the path of the mutual flux is one and the same for both stator and rotor phase windings. In both cases, mutual flux passes through the air gap, where the gross part of the magnetic resistance is encountered. Besides, mutual flux encircles both stator and rotor windings, passing through teeth, yoke, and other parts of stator and rotor magnetic circuits. On the other hand, magnetic resistance encountered along the path of leakage flux component is likely to be different on stator and rotor. Namely, the leakage flux path includes the width of the slots, and these are likely to be different. In general, a narrower slot opening results in a smaller magnetic resistance for the leakage flux and a larger leakage inductance, while a wide slot opening leads to a small leakage inductance.

In electrical machines, the power of electromechanical conversion and the electromagnetic torque depend on the magnetic coupling between the stator and rotor windings. Better coupling leads to more torque and power, hence the intention to keep the leakage as low as possible. Ideally, the coefficient of magnetic coupling of the stator and rotor $k = L_m/(L_S L_R)^{0,5}$ should reach unity. The leakage flux is proportional to difference $1 - k$, and in this case, it reaches zero as well as the leakage inductance coefficients $L_{\gamma S}$ and $L_{\gamma R}$. Practical machines cannot be designed to achieve the coupling coefficient of 1. Such a coupling would require the conductors of the two windings to be next to each other, so as to prevent any leakage flux, and this is not feasible due to practical reasons. The stator and rotor have to be separated by air gap for mechanical and electrical reasons. In machines designed for operation with higher voltages, insulation of individual conductors and windings has to sustain high-voltage stresses. For this reason, insulation layers are thicker, as well as distances between individual conductors. With increased distances between corresponding conductors, the space for the leakage flux is enlarged as well as the leakage flux. Provisional values of the coupling coefficient in low-voltage electrical machines (400 V, 50 Hz) are $k \sim [0.9 .. 0.98]$. In machines designed to operate with high voltages, the values of the coupling coefficient could be considerably lower, even $k < 0.9$.

15.15 Matrix L

Inductance matrix provides the link between the vector column with four total flux linkages and the vector column with four electrical currents. On the main diagonal, inductance matrix has the coefficients of self-inductances. Off the main diagonal, it has the mutual inductances. The mutual inductances describing magnetic coupling between stator and rotor phases are variable. They change in the course of motion.

Neglecting the differences in magnetic resistances for stator and rotor flux linkages, the ratio L_S/L_R depends on the number of stator and rotor turns, $L_{11} = L_{22} = L_S = N_S^2/R_\mu$, $L_{33} = L_{44} = L_R = N_R^2/R_\mu$. Self-inductances are strictly positive, while mutual inductances may assume negative values as well as positive. Mutual inductance L_{jk} determines the flux contribution brought into the phase winding k by the current i_j of the phase winding j. Magnetic coupling between the two windings is reciprocal, $L_{jk} = L_{kj}$; thus, the inductance matrix is symmetrical ($L = L^{\mathrm{T}}$). Mutual inductance of orthogonal windings is equal to zero[4]; thus, $L_{12} = L_{21} = L_{34} = L_{43} = 0$. Coefficient L_{13} of the matrix represents mutual inductance of windings α_{S} and α_{R}. Relative position of the considered windings changes as the rotor moves. With $\theta_m = 0$, the windings are placed one against the other, and their magnetic axes coincide. In this position, magnetic coupling peaks, and then current in one winding gives the highest change of flux in the other winding. With $\theta_m = \pi/2$, considered windings are orthogonal, and therefore, $L_{13} = 0$. With $\theta_m = \pi$, positive current in one winding gives negative flux in the other; thus, $L_{13} < 0$. Variation of coefficient L_{13} can be described by function $L_{13}(\theta_m) = L_m \cos(\theta_m)$, where $L_m = k\,(L_S L_R)^{0.5}$ is the maximum value of L_{13}, obtained in position $\theta_m = 0$. Other coefficients of the inductance matrix can be determined in a like manner. It should be noted that the matrix is not stationary. Some coefficients change with the angle $\theta_m = \Omega_m t$. Therefore, there is a nonzero derivative dL/dt. Recall at this point that the electromagnetic torque of electromechanical converter can be obtained as $T_{em} = ½\, i^T\, (\mathrm{d}L/\mathrm{d}\theta_m) i$.

$$\begin{bmatrix} \Psi_{\alpha s} \\ \Psi_{\beta s} \\ \Psi_{\alpha R} \\ \Psi_{\beta R} \end{bmatrix} = \begin{bmatrix} L_s & 0 & L_m \cos\theta_m & -L_m \sin\theta_m \\ 0 & L_s & L_m \sin\theta_m & L_m \cos\theta_m \\ L_m \cos\theta_m & L_m \sin\theta_m & L_R & 0 \\ -L_m \sin\theta_m & L_m \cos\theta_m & 0 & L_R \end{bmatrix} \cdot \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \\ i_{\alpha R} \\ i_{\beta R} \end{bmatrix} \tag{15.41}$$

15.16 Transforming Rotor Variables to Stator Side

Voltage balance equations for rotor windings are given in Sect. 15.9, and they comprise rotor currents $i_{\alpha R}$ and $i_{\beta R}$. The rotor currents are not directly accessible. They cannot be measured by accessing the rotor cage and inserting measurement devices. Moreover, the two-phase model replaces the rotor cage by an equivalent two-phase winding brought into the short circuit. Hence, the rotor currents $i_{\alpha R}$ and $i_{\beta R}$ are not the currents flowing through the rotor bars but the currents of the two-phase equivalent which has replaced the cage. The short-circuiting effect of the cage can be modeled by the two-phase equivalent having large number of conductors of small cross-sectional area or small number of conductors of large cross-sectional area. Therefore, the number of turns in the two phases depicting the rotor can be arbitrarily selected, as explained in the following example.

[4] This assumption is not valid for nonlinear magnetic circuits. Magnetic saturation contributes to so-called *cross saturation* in orthogonal windings, phenomenon where the flux in one of the windings changes the saturation level in common magnetic circuit and, hence, changes the flux of the other winding.

The magnetomotive force created by electrical currents of all the conductors placed in one rotor slot is equal to the sum of their currents. The slot may have only one conductor with current of 100 A or 100 conductors each carrying 1 A, and in both cases, the magnetomotive force will be 100 ampere-turns. This value corresponds to circular integral of magnetic field H along closed contour encircling the slot. Hence, the system with two-phase windings that models the rotor cage could have an arbitrary number of turns, as long as the product Ni of the rotor current and the number of turns equals the value created by the original short-circuited cage.

The freedom in choosing the number of turns of the two-phase rotor equivalent is most frequently used to introduce $N_R = N_S$. Assuming that the rotor has the same number of turns as the stator results in $L_S = N_S{}^2/R_\mu = N_R{}^2/R_\mu = L_R$ and gives $L_m = k\ (L_S L_R)^{0,5} = kL_S = kL_R$, while the leakage inductances of the stator and rotor become $L_{\gamma S} = L_S - L_m = (1 - k)L_S = (1 - k)L_R = L_{\gamma R}$. The obtained expressions are based on the assumption that differences in magnetic resistances for the stator and rotor fluxes are negligible. This assumption is valid in most cases.

The inductance matrix allows that each of the four flux linkages is expressed in terms of electrical currents. For example, the flux in phase α of the stator is

$$\Psi_{\alpha s} = L_s i_{\alpha s} + L_m \cos\theta_m\, i_{\alpha R} - L_m \sin\theta_m i_{\beta R}. \tag{15.42}$$

Question (15.7): Stator currents of an induction machine are $i_{\alpha S} = I_{mS}\cos\omega_e t$ and $i_{\beta S} = I_{mS}\sin\omega_e t$, where $\omega_e > 0$ and rotor currents are $i_{\alpha R} = I_{mR}\sin\omega_x t$ and $i_{\beta R} = I_{mR}\cos\omega_x t$ having the angular frequency $0 < \omega_x < <\omega_e$. The machine operates at steady state. By using (15.42) for flux $\Psi_{\alpha S}$, determine the rotor speed.

Answer (15.7): Currents $i_{\alpha S}$ and $i_{\beta S}$ produce the stator magnetomotive force and flux rotating in positive direction. Phase sequence of the given rotor currents is such that they create magnetic field which rotates relative to the rotor at the speed of $\omega_x = \Omega_x$ in negative direction. In steady state, the rotor and stator fields revolve synchronously. Therefore, it is concluded that $\omega_m = \omega_x + \omega_e$. The same conclusion can be obtained from the expression for flux, $\Psi_{\alpha S} = L_S I_{mS}\ \cos\omega_e t + L_m I_{mR}(\cos\omega_m t\ \ \sin\omega_x t - \sin\omega_m t\ \ \cos\omega_x t) = L_S I_{mS}\ \cos\omega_e t - L_m I_{mR}\ \sin(\omega_m t\text{-}\omega_x t)$. The elements of this must have the same frequency in steady-state conditions since the stator and rotor variables rotate at the same speed ω_e, maintaining the relative positions unchanged. This condition is met in cases $\omega_m - \omega_x = +\omega_e$ as well as $\omega_m - \omega_x = -\omega_e$, that is, for the speeds of rotation $\omega_m = \omega_x + \omega_e$ or $\omega_m = -\omega_e + \omega_x$. According to the assumed conditions, $0 < \omega_x \ll \omega_e$, and the solution is $\omega_m = \omega_x + \omega_e$. In this solution, the slip $\omega_{slip} = \omega_e - \omega_m = -\omega_x$ is negative; hence, the rotor is rotating faster than the field. The machine operates in generator mode.

Question (15.8): Starting from the inductance matrix of the system of windings α_S, β_S, α_R, and β_R, prove that the torque is equal to $T_{em} = (3/2)\ (\Psi_{\alpha S}\, i_{\beta S} - \Psi_{\beta S}\, i_{\alpha S})$.

Answer (15.8): The electromagnetic torque is given by expression $T_{em} = ½\, i^{\mathrm{T}}[\mathrm{d}\boldsymbol{L}(\theta_m)/\mathrm{d}\theta_m]i$ where $\boldsymbol{L}(\theta_m)$ is the inductance matrix whose elements are dependent on position θ_m of the rotor with respect to the stator. Variable elements of the inductance matrix are $L_{13} = L_{31}$, $L_{14} = L_{41}$, $L_{23} = L_{32}$, and $L_{24} = L_{42}$, while the remaining coefficients are constant and result in $\mathrm{d}L_{j\kappa}/\mathrm{d}\theta_m = 0$. The calculation can be simplified because $\boldsymbol{L}^{\mathrm{T}} = \boldsymbol{L}$; thus, the result can be obtained by doubling the contributions of coefficients L_{13}, L_{14}, L_{23}, and L_{24}. Finally, one obtains $½\, i^{\mathrm{T}}[\mathrm{d}\boldsymbol{L}(\theta_m)/\mathrm{d}\theta_m]i = -L_m \sin\theta_m\, i_{\alpha S}\, i_{\alpha R} - L_m \cos(\theta_m) i_{\alpha S}\, i_{\beta R} + L_m \cos(\theta_m) i_{\alpha R}\, i_{\beta S} - L_m \sin(\theta_m) i_{\beta R}\, i_{\beta S}$. The same result is obtained by starting from expression $(\Psi_{\alpha S}\, i_{\beta S} - \Psi_{\beta S}\, i_{\alpha S})$ and introducing the replacement where the fluxes are expressed from the first and second row of the inductance matrix. In expression $T_{em} = (3/2)\, (\Psi_{\alpha S}\, i_{\beta S} - \Psi_{\beta S}\, i_{\alpha S})$, coefficient 3/2 is the consequence of adopting the $3\Phi/2\Phi$ transform with $K_U = K_I = K_\Psi = 2/3$.

15.17 Mathematical Model

In subsequent considerations, mathematical model of induction machine is presented in terms of coordinates α_S, β_S, α_R, and β_R. The voltage balance equations and inductance matrix were defined already within previous sections. The model is completed by adding Newton equation and the torque expression $T_{em} = (3/2)(\Psi_{\alpha S}\, i_{\beta S} - \Psi_{\beta S}\, i_{\alpha S})$. This set of differential equations and algebraic expressions constitutes mathematical model of induction machine, based on previously adopted approximations. The model is summarized in (15.43), (15.44), (15.45), and (15.46). It can be used in its present form to predict dynamic behavior and steady-state properties of induction machines. For that to be achieved, it is sufficient to enter (15.43), (15.44), (15.45), and (15.46) into program for computer simulation of dynamic systems. Hence, developed model is the correct representation of behavior of induction machines. Yet, it has drawbacks that hinder further analytical considerations and introduce difficulties in drawing conclusions and deriving the steady-state characteristics.

There could be specific situations where the given model cannot serve as an accurate representation of the induction machine. In cases where the iron losses cannot be neglected, or the magnetic saturation is emphasized, as well as in cases where the remaining two approximations do not hold, the model may give erroneous results. In such cases, the model has to be modified and upgraded so as to include the effects that were neglected in the first place. The four approximations that were adopted in modeling electrical machines are listed and explained in introductory chapters.

The model contains α_S and β_S components of the stator variables, as well as α_R and β_R components of the rotor variables. Equations (15.43), (15.44), and (15.45) remain unaltered whatever the choice of the leading coefficient K of $3\Phi/2\Phi$ transform. In (15.46), it is assumed that $K = 2/3$ is used. This choice is used throughout the book, and it requires the αβ power and torque to be multiplied by 3/2 in order to obtain the power and torque of the original.

It is of interest to recall that the choice of K has to do with selecting the number of turns $N_{\alpha\beta}$ of the equivalent two-phase machine. As already shown, the choice $N_{\alpha\beta} = (3/2)^{0.5}\, N_{abc}$ and $K = (2/3)^{0.5}$ results in a two-phase equivalent which has power, impedance, and inductance invariance. The two-phase equivalent is a mathematical fiction, and it does not have to be actually made. Yet, envisaging the variables $i_{\alpha S}$, $i_{\beta S}$, $i_{\alpha R}$, and $i_{\beta R}$ as electrical currents of actual phase windings helps understanding the basic voltage, current, and flux vectors of the machine, and it helps using the model.

Components of the voltage, current, and flux in the model are projections of the relevant vectors of the voltage, current, and flux on axes of coordinate systems α_S–β_S and α_R–β_R. Stator vectors are projected on axes α_S and β_S of stationary coordinate system, while rotor vectors are projected on axes α_R and β_R of coordinate system that revolves with the rotor.

Complete model is summarized by (15.43), (15.44), (15.45), and (15.46). The symbol p in (15.46) represents the number of pairs of magnetic poles, discussed in Chap. 16. Preceding considerations assumed that $p = 1$, namely, that magnetic field has one north pole and one south pole:

$$u_{\alpha s} = R_s i_{\alpha s} + \frac{d\Psi_{\alpha s}}{dt}, \quad u_{\beta s} = R_s i_{\beta s} + \frac{d\Psi_{\beta s}}{dt}, \tag{15.43}$$

$$0 = R_R i_{\alpha R} + \frac{d\Psi_{\alpha R}}{dt}, \quad 0 = R_R i_{\beta R} + \frac{d\Psi_{\beta R}}{dt}, \tag{15.44}$$

$$\begin{bmatrix} \Psi_{\alpha s} \\ \Psi_{\beta s} \\ \Psi_{\alpha R} \\ \Psi_{\beta R} \end{bmatrix} = \begin{bmatrix} L_s & 0 & L_m\cos\theta_m & -L_m\sin\theta_m \\ 0 & L_s & L_m\sin\theta_m & L_m\cos\theta_m \\ L_m\cos\theta_m & L_m\sin\theta_m & L_R & 0 \\ -L_m\sin\theta_m & L_m\cos\theta_m & 0 & L_R \end{bmatrix} \cdot \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \\ i_{\alpha R} \\ i_{\beta R} \end{bmatrix}, \tag{15.45}$$

$$T_{em} = \frac{3}{2} p \left(\Psi_{\alpha S} i_{\beta S} - \Psi_{\beta S} i_{\alpha S} \right). \tag{15.46}$$

15.18 Drawbacks

The above model is an adequate representation of dynamic and steady-state behavior of induction machines, but it has drawbacks that make further uses more difficult. Such uses are the steady-state analysis, deriving equivalent circuits, and conceiving and designing control algorithms. The key problems with the model are:

1. The presence of trigonometric functions in differential equations
2. The state variables exhibiting sinusoidal change even in steady state

The consequences of the above issues on clarity and usability of the model can be seen by considering the voltage balance equation. Further on, discussion will close by proposing further steps that should be taken to obtain a more clear and more intuitive model.

By expressing the flux in terms of electrical currents and introducing the resulting expression in voltage balance equations for the stator phase windings, the following expressions are obtained:

$$\begin{aligned} u_{\alpha s} = R_s i_{\alpha s} + L_s \frac{\mathrm{d}i_{\alpha s}}{\mathrm{d}t} + L_m \cos\theta_m \frac{\mathrm{d}i_{\alpha R}}{\mathrm{d}t} - \omega_m L_m \sin\theta_m i_{\alpha R} \\ - L_m \sin\theta_m \frac{\mathrm{d}i_{\beta R}}{\mathrm{d}t} - \omega_m L_m \cos\theta_m i_{\beta R}, \end{aligned}$$

$$\begin{aligned} u_{\beta s} = R_s i_{\beta s} + L_s \frac{\mathrm{d}i_{\beta s}}{\mathrm{d}t} + L_m \sin\theta_m \frac{\mathrm{d}i_{\alpha R}}{\mathrm{d}t} + \omega_m L_m \cos\theta_m i_{\alpha R} \\ + L_m \cos\theta_m \frac{\mathrm{d}i_{\beta R}}{\mathrm{d}t} - \omega_m L_m \sin\theta_m i_{\beta R}. \end{aligned}$$

Presence of trigonometric functions in differential equations makes the steady-state analysis more difficult. An attempt to derive a steady-state equivalent circuit becomes more involved. Hypothetic removal of trigonometric functions from the voltage balance equations results in

$$u_{\alpha S} = R_S i_{\alpha S} + L_S \frac{\mathrm{d}i_{\alpha S}}{\mathrm{d}t} + L_m \frac{\mathrm{d}i_{\alpha R}}{\mathrm{d}t}$$

which simplifies greatly the steady-state relations and makes it more obvious. Applying Laplace transform to the previous equations, an algebraic expression is obtained which relates the complex images of voltages and currents,

$$U_{\alpha S}(s) = R_S I_{\alpha S}(s) + s L_S I_{\alpha S}(s) + s L_m I_{\alpha R}(s)$$

At steady state, electrical current $i_{\alpha s}$ exhibits sinusoidal change. Assuming that, for the sake of an example, all the electrical currents have the angular frequency ω, the steady-state analysis implies that the operator s becomes $\mathrm{j}\omega$, resulting in the following expression:

$$U_{\alpha S}(s) = R_S I_{\alpha S}(s) + \mathrm{j}\omega L_S I_{\alpha S}(s) + \mathrm{j}\omega L_m I_{\alpha R}(s)$$

which represents the voltage balance equation in a contour comprising the voltage source $U_{\alpha S}$, resistance R_S, an inductance $(L_S - L_m)$ carrying current $I_{\alpha S}$, and an inductance L_m carrying current $(I_{\alpha R} + I_{\alpha S})$. Considered example demonstrates that a set of differential equations with constant coefficients provides the grounds for deriving an equivalent circuit that represent the machine in the steady state. The presence of trigonometric functions in voltage balance equations makes this impossible.

Sinusoidal change of state variables even in steady-state conditions brings in more difficulties in steady-state analysis. At the same time, similar difficulties are encountered in designing control structures for the current, flux, and torque regulation. The models with state variables that remain unaltered in the steady state are simple to grasp. In their steady-state equations, all the time derivatives of the state variables disappear, and the remaining expressions are easy to understand and use. Yet, the present model does not have this possibility. The state variables of the model, such as $i_{\alpha s}$, exhibit sinusoidal changes even in the steady state.

Most mathematical models are usually formulated in such way that their state variables remain constant in the steady state. The time derivatives of these state variables are then equal to zero. In such cases, steady-state relations are obtained from differential equations by removing the time derivatives.

Perpetual changes of state variables even in steady state make the control problems more difficult to solve. An example to that is the applications of electrical motors in industrial robots or autonomous vehicles, where the electrical motors are used for controlling the motion. For that to be achieved, it is necessary to *regulate* some relevant motor variables, such as the flux, torque, speed, and current. The term *regulation* implies:

- Definition of a desired *reference* value for the controlled variable (such as the phase current) that should be reached and maintained
- Measurement of the controlled variable (current) and calculation of the *error*, the deviation of the controlled variable from the desired value
- Performing *control algorithm*, calculation procedure or formula which receives the *error* and calculates the *control*, the output of the control algorithm[5]
- Bringing the control variable to the system under control through an *executive organ* or *actuator*[6]

When the reference value does not change, the steady-state error can be reduced to zero by adding integral control action into the control algorithm. Corresponding control action is proportional to the integral of the error. Yet, this maintains the error at zero only for constant references. In cases when the reference value has perpetual changes, control actions would act upon the controlled variable to track these changes. This is achieved at the cost of an error which cannot be removed.

[5] By using devices called *actuators* or *amplifiers*, control output can affect the system and change the controlled variable. Control algorithm is suited to reduce error and bring the controlled variable toward the reference. Well-suited control algorithm ensures progressive reduction of the error which, after a while, reaches zero. Hence, the steady-state value of errors is expected to be equal to zero.

[6] Control of phase currents of most electrical machines often includes switching power converters employing power transistors and PWM techniques, as well as digital signal controllers. The control algorithm is usually implemented by programming digital signal controllers. Control variables are obtained in numerical form, as binary-coded digital words that reside within processor registers. On the other side, the actual control variable that has the potential to change the phase current is the voltage across the phase winding which may change within the range of $\pm$600 V. The executive organ is often a switching transistor which receives gating signals from the digital signal controller.

Namely, controlled variable would track the reference with a tracking error that depends on the rate of change of the reference. The presence of error can jeopardize the operation of the whole system.

Induction machines have alternating currents in their phase windings. Hence, even in the steady state, the variables $i_{\alpha s}$ and $i_{\beta s}$ exhibit sinusoidal changes. Whenever there is a need to regulate stator currents of induction machine, controlled variables are variable even at steady state. This creates the need to transform the mathematical model in such way that the steady-state values of the state variables are constant and to use this model to formulate the control algorithm.

Question (15.9): Is there an operating mode of an induction machine where the frequency of stator currents is zero?

Answer (15.9): Considering induction machines supplied from constant frequency voltage sources, the frequency of the stator electrical currents is determined by the frequency of voltages fed to the stator winding. If induction machine is connected to the mains, the frequency of the stator current will be 50 Hz, irrespective of the rotor speed, torque, or power. For that reason, the stator currents may have different frequency only in cases when the induction machine is supplied from a variable frequency, variable voltage source such as the static power converter which uses semiconductor power switches and operates on pulse-width modulation principles. In this way, the stator winding can be supplied with a symmetrical system of three-phase voltages of variable amplitude and variable frequency.

In further considerations, it is assumed that the supply voltage has variable frequency ω_e which can be adjusted to achieve desired operating mode. For two-pole machine, $\omega_e = \Omega_m + \omega_{slip}$, where Ω_m is the rotor speed while ω_{slip} is the slip frequency. Operating mode where DC current flows through the stator is the one where $\Omega_m = -\omega_{slip}$, resulting in $\omega_e = 0$. It should be noted that the slip frequency is proportional to the developed electromagnetic torque. Therefore, the operating mode with $\omega_e = 0$ is reached when the torque and speed of rotation are of different signs. The rotor rotates at the speed which has the same magnitude as the slip frequency, but it has opposite sign. In one of such cases, the motor is stopped and develops the torque $T_{em} = 0$. In another example, the rotor revolves at -300 rpm, it develops positive torque, and it has the rotor currents of 5 Hz.

15.19 Model in Synchronous Coordinate Frame

The problems of analysis and control based on mathematical model in α–β coordinate frame arise due to the fact that the state variables exhibit sinusoidal change even in the steady state. Namely, projections of the vectors of currents, voltages, and flux linkages on the axes of α–β coordinate frame change as sinusoidal functions even in steady state, with constant amplitude of magnetomotive force F_S, constant flux, and constant rotor speed. This deficiency of the model can be removed by applying another transform of the state coordinates and replacing the existing state variables by the new ones. The new variables should be selected so that their values at steady state do not vary. The new coordinate transform should maintain the invariability in terms of impedance, inductance, and power. For that to be achieved, the same transformation matrix should be applied to all the variables, whether the currents, voltages, or flux linkages. The question arises about how to devise this new coordinate transform.

15.20 Park Transform

In the model of the machine which is formulated in stationary α–β frame, the stator currents $i_{\alpha S}$ and $i_{\beta S}$ are projections of the vector of magnetomotive force $\boldsymbol{F_S} = N_S\, \boldsymbol{i_S}$ on axes of the α_S–β_S coordinate system. The problem arises due to the fact that $\boldsymbol{F_S}$ is a rotating vector. Its rotation with respect to α–β frame makes the projections $i_{\alpha S}$ and $i_{\beta S}$ variable. In steady state, they become sinusoidal function. In order to solve the problem, it is necessary to envisage a new coordinate system which is rotating at the same speed as the magnetomotive force $\boldsymbol{F_S}$. In this case, projections of this vector on new pair of axes do not change in the steady state, as the relative position between the revolving vector and the new frame does not change. The same conclusion applies for the voltage and flux vectors. Therefore, the new transform of the state coordinates should formulate the model of induction machine in a new coordinate system which revolves synchronously with the field.

By adopting a synchronously rotating coordinate system having axes d and q, projections i_d and i_q of the stator current vector $\boldsymbol{i_S} = \boldsymbol{F_S}/N_S$ on these new axes have constant steady-state values. Therefore, by transforming all the stator quantities from the stationary α_S–β_S coordinate system to the synchronously rotating d–q coordinate system, a model of the stator winding is obtained where the relevant quantities have constant steady-state values.

Transform implies a relation that expresses the variables of the d–q system (i_d and i_q) in terms of the variables of the α_S–β_S coordinate system ($i_{\alpha S}$ and $i_{\beta S}$). Applying a coordinate transform does not introduce any change to the considered induction machine nor to its flux or torque. The transform merely represents a *different point of view* and represents one and the same actual systems by means of another mathematical representation. Therefore, no matter whether the mathematical model is formulated in the stationary or in the synchronous coordinate frame, it must describe the same vectors of the magnetomotive force, flux, voltage, and current as well as the same torque, speed, and power of electromechanical conversion. Consequently, both the model in α–β frame and the model in d–q frame must have one and the same vector of the stator magnetomotive force F_S. In other words, the vector $\boldsymbol{F_S}$ created by currents $i_{\alpha S}$ and $i_{\beta S}$ must have the same amplitude and spatial orientation as the vector $\boldsymbol{F_S}$ created by i_d and i_q, the stator currents transformed into d–q coordinate frame.

Coordinate transform is essentially a mathematical operation, and it does not have to result in state variables that correspond to an actual, physical machine.

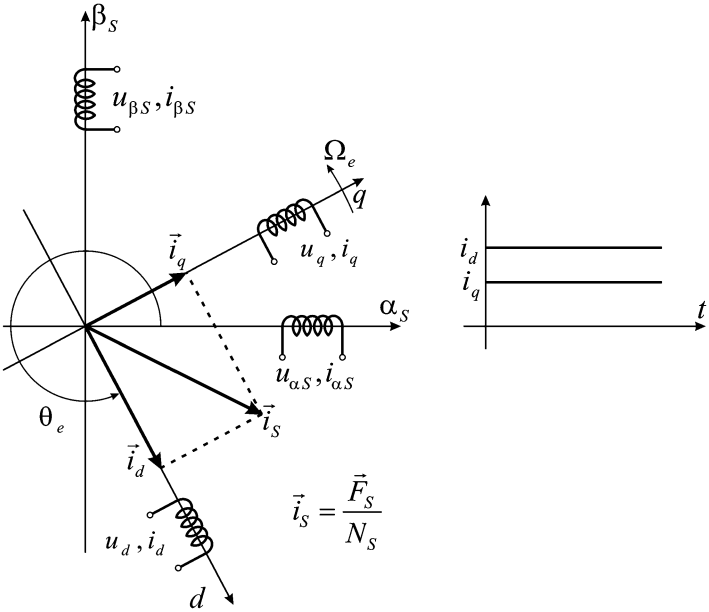

Fig. 15.8 Position of *d*–*q* coordinate frame and corresponding steady-state currents in virtual phases *d* and *q*

In many cases, it is not even possible to construct an induction machine that would have the voltages and currents that correspond to the ones obtained from coordinate transform. Yet, an attempt to represent the variables i_d and i_q as electrical currents of a new, virtual stator winding may help understanding and using the new model. The new transform can be represented by *removing* the stator phase windings α_S and β_S and *installing* new stator phases *d* and *q*, with their magnetic axes lying along the *d* and *q* of the new coordinate frame, as shown in Fig. 15.8. These *d*–*q* windings cannot be actually made, so it is correct to call them *virtual*. Electrical currents i_d and i_q of these new virtual windings must result in the same vector F_S as the previous, created by electrical currents in phases α_S and β_S.

In the following figures and illustrations that involve the new *d*–*q* coordinate frame, new notation is introduced for the phase windings and electrical currents in *d*–*q* frame. The *virtual* stator windings are denoted by *d* and *q*, while the *virtual* rotor windings are denoted by *D* and *Q*. The stator currents in new coordinate frame are denoted by i_d and i_q, while the rotor currents are i_D and i_Q. Components of electrical currents in *d*–*q* frame are equal to projections of current vectors on axes *d* and *q*. Should virtual windings *d* and *q* actually exist, the currents i_d and i_q would result into the same vector of the stator magnetomotive force F_S which actually exists in the original machine. Neither the stator nor the rotor could have their actual phase windings residing in the *d*–*q* frame. The only purpose of showing them in illustrations is to help visualizing the state variables which are obtained by rotational transform, such as i_d and i_q, and also to facilitate the model analysis and use.

15.21 Transform Matrix

Relations of the rotational transform can be derived from the condition of invariability of the vector F_S. In the preceding figure, the angle θ_e denotes the angular shift of the revolving axis d with respect to the still axis α_S. Projection of the current component $i_{\alpha S}$ on axis d is $i_{\alpha S}$ cosθ_e, while projection of current component $i_{\beta S}$ on the same axis is $i_{\beta S}$ sinθ_e. Hence, projection of the vector $\boldsymbol{F_S}/N_S$ on axis d is equal to $i_{\alpha S}$ cosθ_e + $i_{\beta S}$ sinθ_e. Considering d–q frame variables, the current i_q produces the component of $\boldsymbol{F_S}$ in direction of the axis q, and it cannot contribute to d component of $\boldsymbol{F_S}$. Hence, the current i_d has to be equal to $i_{\alpha S}$ cosθ_e + $i_{\beta S}$ sinθ_e in order to achieve invariability of vector $\boldsymbol{F_S}$ (Fig. 15.9). By summing projections of currents $i_{\alpha S}$ and $i_{\beta S}$ on axis q, one obtains that current i_q in virtual phase winding q must be equal to $-\ i_{\alpha S}$sinθ_e + $i_{\beta S}$cosθ_e. Park rotational transform is summarized by using the matrix given by (15.47). Determinant of the transform matrix is equal to 1. By transforming any vector from stationary frame to synchronously rotating frame, one obtains the vector of the same amplitude as the original. By applying the same transform matrix to voltages, currents, and fluxes, one obtains the relevant variables in d–q coordinate system (u_d, u_q, i_d, i_q, Ψ_d, Ψ_q). Proposed transform is invariable in terms of impedance, inductance, and power (det T = 1). Variables Ψ_d and Ψ_q denote total flux linkages of the virtual stator windings in d and q axes.

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{bmatrix} \cos\theta_s & \sin\theta_s \\ -\sin\theta_s & \cos\theta_s \end{bmatrix} \cdot \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \end{bmatrix} = \underline{T} \cdot \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \end{bmatrix} \tag{15.47}$$

Question (15.10): Starting from expression $u_{\alpha S} = R_S i_{\alpha S} + \mathrm{d}\Psi_{\alpha S}/d$t, is it possible to state that $u_d = R_S i_d + \mathrm{d}\Psi_d/\mathrm{d}t$?

Answer (15.10): The stator phase winding d is a virtual winding. Even though, it is possible to formulate the voltage balance equation which comprises the variables u_d, i_d

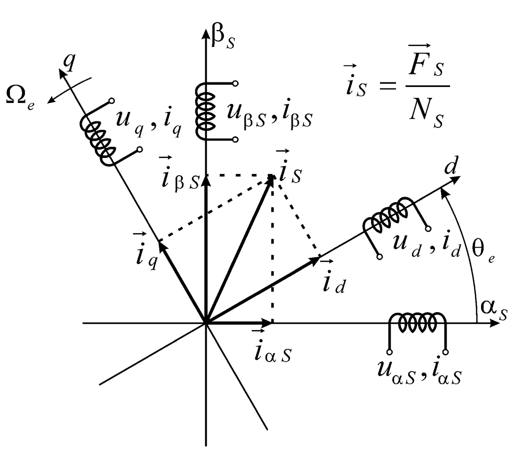

Fig. 15.9 Projections of $\boldsymbol{F_S}/N_S$ on stationary and rotating frame

and other variables of the d–q frame. The presence of unchanged stator resistance R_S is expected since the transform is invariant in terms of impedance. However, it cannot be stated that $u_d = R_S i_d + \mathrm{d}\Psi_d/\mathrm{d}t$. Namely, the voltage balance equation $u = Ri + \mathrm{d}\Psi/\mathrm{d}t$ applies to any winding that actually exists or the winding that could actually be made. Relation $u = Ri + \mathrm{d}\Psi/\mathrm{d}t$ was used to model the voltage balance in α_S–β_S coordinate frame. Although the original induction machine has three phases, the two-phase stator winding in α_S–β_S frame can be actually made. Therefore, the stator phase windings α_S and β_S can be considered as real windings. Thus, the voltage balance equation in α_S–β_S frame is $u = Ri + \mathrm{d}\Psi/\mathrm{d}t$. On the other hand, the phase windings in dq frame cannot be made. The analysis performed later on proves that the voltage u_d is not equal to $R_S i_d + \mathrm{d}\Psi_d/\mathrm{d}t$. The actual voltage balance equations in dq frame, describing the virtual voltages, will be derived by applying the transform matrix to equations $u_{\alpha S} = R_S i_{\alpha S} + \mathrm{d}\Psi_{\alpha S}/\mathrm{d}t$ and $u_{\beta S} = R_S i_{\beta S} + \mathrm{d}\Psi_{\beta S}/\mathrm{d}t$.

15.22 Transforming Rotor Variables

It is beneficial to have both the stator and the rotor variables residing in the same frame of coordinates. For that to be achieved, it is necessary to apply rotational transform to the rotor variables and find their DQ equivalents.

Park transform of the stator phase windings α_S and β_S results in two virtual stator phases in dq frame. In the same way, it is necessary to replace the short-circuited rotor cage by two virtual rotor phases residing in d–q coordinate system. For clarity, notation adopted hereafter implies that the stator variables have lower case subscripts dq, while the rotor variables have upper case subscripts DQ.

It is necessary to explain the need for transforming the stator and rotor variables to the same, synchronously rotating, coordinate system. First of all, it should be recalled that the voltage balance equations in stationary α–β coordinate frame are difficult to cope with due to variable coefficients of differential equations. These variable coefficients come from variable elements in the inductance matrix, such as $L_{13} = L_m \cos(\theta_m)$. Some mutual inductances between stator and rotor phases are variable (15.48) due to change in relative position of the two windings. In (15.45), this comes as a consequence of the α_R–β_R frame moving with respect to the α_S–β_S frame. The mutual inductance between the two windings does not change as long as they maintain the same relative position. Hence, any pair of windings residing in the same coordinate frame, whether revolving or stationary, does not change relative position, and therefore, they have constant mutual inductance. Transformation of rotor windings from $\alpha_R\beta_R$ frame to d–q coordinate frame can be depicted as removing the original $\alpha_R\beta_R$ windings and replacing them with a pair of virtual rotor phases denoted by DQ, residing in the same d–q coordinate frame as the virtual stator phases denoted by dq. Now the stator dq windings do not move with respect to the rotor DQ windings. Therefore, all the self-inductances and all the mutual inductances are constant. The inductance matrix that corresponds to virtual windings in dq frame has constant

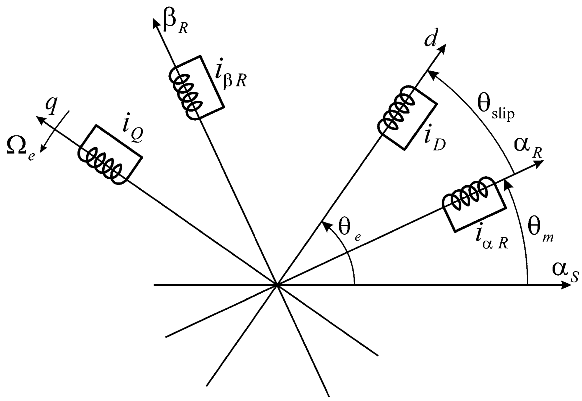

Fig. 15.10 Rotor coordinate system and *dq* system

elements. It will be shown later that, as a consequence, the voltage balance equations in *dq* frame do not have variable coefficients.

In addition to obtaining a constant inductance matrix, transformation of the rotor variables to *d–q* system is also required in order to obtain a mathematical model where the rotor variables have constant steady-state values. The actual rotor cage has AC currents of angular frequency $\omega_{slip} = \omega_e - \omega_m = \omega_e - p\Omega_m$. These currents create magnetomotive force and flux of the rotor which revolve at the speed Ω_{slip} with respect to the rotor. With $p = 1$, their speed with respect to the stator is $\Omega_{slip} + \Omega_m = \omega_{slip} + \omega_m = \omega_e$. In other words, the rotor magnetomotive force and flux vectors revolve at the same speed (in synchronism) with the stator vectors. Some phase and/or spatial shift among these variables may exist at steady state. Therefore, projections of the rotor variables on *d–q* do not vary at steady state. Hence, the actual rotor variables should be transformed to *d–q* coordinate frame.

Rotational transform of the rotor variables is illustrated in Fig. 15.10. It should be noted that the product of the rotor currents and the number of rotor turns $N_R\, i_{\alpha R}$ and $N_R\, i_{\beta R}$ are equal to projections of the rotor magnetomotive force vector $\boldsymbol{F_R}$ on axes α_R–β_R of the rotor coordinate system. The angular displacement between axes α_R and α_S is determined by the rotor position θ_m. Synchronously rotating *d–q* coordinate system leads by θ_{slip} with respect to the rotor; thus, its advance with respect to the stator is $\theta_e = \theta_{slip} + \theta_m$.

Transformation matrix for the rotor variables is derived starting from invariability of the rotor magnetomotive force. Projection of the rotor phase current $i_{\alpha R}$ on axis d is equal to $i_{\alpha R} \cos\theta_{slip}$, while projection of the rotor phase current $i_{\beta R}$ on the same axis is equal to $i_{\beta R} \sin\theta_{slip}$. Therefore, the current of the virtual rotor winding D is equal to $i_D = i_{\alpha R} \cos\theta_{slip} + i_{\beta R} \sin\theta_{slip}$. Similarly, $i_Q = -i_{\alpha R} \sin\theta_{slip} + i_{\beta R} \cos\theta_{slip}$. These relations can be written in matrix form, as shown in the (15.48). The same transformation matrix is applied to all rotor variables, resulting in voltages of virtual rotor windings $u_D = u_Q = 0$ and providing total flux linkages Ψ_D and Ψ_Q:

$$\begin{bmatrix} i_D \\ i_Q \end{bmatrix} = \begin{bmatrix} \cos\theta_{slip} & \sin\theta_{slip} \\ -\sin\theta_{slip} & \cos\theta_{slip} \end{bmatrix} \cdot \begin{bmatrix} i_{\alpha R} \\ i_{\beta R} \end{bmatrix}. \tag{15.48}$$

15.23 Vectors and Complex Numbers

Park transform relates coordinates of vectors in α–β coordinate system to coordinates of the same vectors in d–q coordinate system. All the voltages, currents, magnetomotive forces, and fluxes can be represented by vectors, either in $\alpha\beta$ frame or in dq frame. All the vectors have two components, being associated with two-phase windings. Projection of each voltage, current, or flux vector on one of the axes corresponds to voltage, current, or flux linkage in the phase winding residing on corresponding axis.

With all the vectors residing in plane, it is possible to use complex notation in representing individual vectors. Namely, a vector can be represented by a complex number, with real and imaginary parts representing the projections of the vector on the two orthogonal axes. In this way, Park transform notation can be simplified, and the transformation matrix reduced to a complex number. As an example, the vector of the stator current can be expressed in terms of its α_S component, directed along α_S axis, and its β_S component, directed along β_S axis. If α_S axis is given attributes of the real axis and axis β_S attributes of the imaginary axis, the α_S–β_S plane is interpreted as the complex plane, where the current vector is represented by complex number,

$$\vec{i}_{\alpha\beta S} = \vec{\alpha}_0 i_{\alpha S} + \vec{\beta}_0 i_{\beta S} \Rightarrow \underline{i}_{\alpha\beta S} = i_{\alpha S} + j\, i_{\beta S}.$$

Similarly, considering the current vector in d–q system, axis d can be given attributes of the real axis, while axis q can be treated as imaginary axis, converting in this way d–q space into a complex plane, where the current is represented by the complex number $\underline{i}_{dq} = i_d + ji_q$. Complex notation of current vectors is not unique. With axis d as real axis, the number $\underline{i}_{dq}$ is obtained, different than the complex number $\underline{i}_{\alpha\beta}$ obtained with axis α_S as real axis.

15.23.1 Simplified Record of the Rotational Transform

By using the complex notation, Park transform can be written as

$$\underline{i}_{dq} = i_d + ji_q = \underline{i}_{\alpha\beta S} e^{-\mathrm{j}\theta_e},$$

$$\begin{aligned} \underline{i}_{\alpha\beta S} e^{-j\theta_e} &= \left(i_{\alpha S} + j\, i_{\beta S}\right)(\cos(\theta_e) - j\,\sin(\theta_e)) \\ &= (i_{\alpha S}\cos(\theta_e) + i_{\beta S}\sin(\theta_e)) + j\left(-i_{\alpha S}\sin(\theta_e) + i_{\beta S}\cos(\theta_e)\right). \end{aligned}$$

The inverse transform can be written as

$$\underline{i}_{\alpha\beta S} = \underline{i}_{dq} e^{+j\theta_e}.$$

Complex number $\underline{i}_{dq} = i_d + ji_q$ is a compact way of representing the stator current vector by representing the two state variables, i_d and i_q, as a single complex number. The choice of real axis (d) and complex axis (q) is in accordance with the fact that axis q leads by π/2 and that imaginary unit $j = \exp(j\pi/2)$ represents the phase shift of π/2. Complex notation $\underline{i}_{dq}$ acquires another significance in steady state. With i_d and i_q remaining constant, the steady-state value of $\underline{i}_{dq}$ becomes a complex constant, *phasor*. The amplitude and argument of this phasor determine the amplitude and the initial phase of the stator current.

15.24 Inductance Matrix in *dq* Frame

By application of Park transform, the rotor and stator windings are transferred to synchronously rotating d–q system. The virtual stator windings are denoted by subscripts d and q, while virtual rotor windings are denoted by subscripts D and Q. Rotation of dq frame does not change relative position of stator and rotor virtual phases; thus, all coefficients of relevant inductance matrix remain constant. The flux of the stator winding d is determined by the first row of matrix, $\Psi_d = L_S i_d + L_m i_D$. The mutual inductance is constant since windings d and D do not change their relative positions.

Revolving frame has an angular speed Ω_e which is the same as the speed of the revolving field. The speed Ω_e is determined by the angular frequency of stator voltages and currents, ω_e. For two-pole machines with $p = 1$, $\Omega_e = \omega_e$. The angle between axes d and α_S is equal to

$$\theta_e = \theta_e(0) + \int_0^t \Omega_e d\tau. \tag{15.49}$$

Stator currents are

$$\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e \\ -\sin\theta_e & \cos\theta_e \end{bmatrix} \cdot \begin{bmatrix} i_{\alpha s} \\ i_{\beta s} \end{bmatrix}. \tag{15.50}$$

Complex notation of the stator current is

$$\underline{i}_{dq} = i_d + ji_q = e^{-j\theta_e}\underline{i}_{\alpha\beta}. \tag{15.51}$$

Rotor currents are

$$\begin{bmatrix} i_D \\ i_Q \end{bmatrix} = \begin{bmatrix} \cos\theta_{slip} & \sin\theta_{slip} \\ -\sin\theta_{slip} & \cos\theta_{slip} \end{bmatrix} \cdot \begin{bmatrix} i_{\alpha R} \\ i_{\beta R} \end{bmatrix}. \tag{15.52}$$

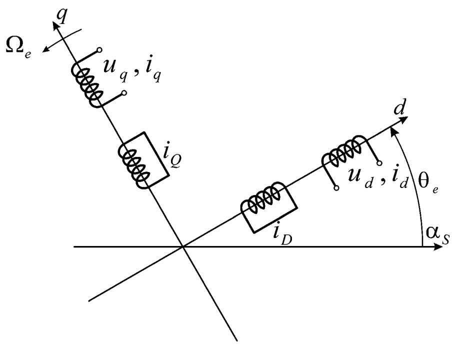

Fig. 15.11 Stator and rotor windings in *dq* coordinate frame

The stator and rotor windings are represented by virtual *dq* and *DQ* windings residing in *dq* frame. Their magnetic axes coincide, and they do not move relative to one another. Therefore, their mutual inductances do not change. The mutual inductance between virtual windings *d* and *D* is L_m, as well as the mutual inductance between windings *q* and *Q*. The mutual inductance between windings in orthogonal axes, such as *d* and *Q*, is equal to zero. The inductance matrix is

$$\begin{bmatrix} \Psi_d \\ \Psi_q \\ \Psi_D \\ \Psi_Q \end{bmatrix} = \begin{bmatrix} L_s & 0 & L_m & 0 \\ 0 & L_s & 0 & L_m \\ L_m & 0 & L_R & 0 \\ 0 & L_m & 0 & L_R \end{bmatrix} \cdot \begin{bmatrix} i_d \\ i_q \\ i_D \\ i_Q \end{bmatrix} = \underline{L} \cdot \begin{bmatrix} i_d \\ i_q \\ i_D \\ i_Q \end{bmatrix}. \tag{15.53}$$

Question (15.11): The electromagnetic torque acting on moving part of the system which comprises several magnetically coupled contours is determined from the expression $T_{em} = ½\, i^T(\mathrm{d}\boldsymbol{L}/\mathrm{d}\theta_m)i$, where $\boldsymbol{L}$ is the inductance matrix. Taking into account the matrix given in (15.53), it is concluded that $\mathrm{d}\boldsymbol{L}/\mathrm{d}\theta_m = 0$ which, introduced into the torque expression, gives $T_{em} = 0$. Is this consideration correct?

Answer (15.11): Expression $T_{em} = ½\, i^T(\mathrm{d}\boldsymbol{L}/\mathrm{d}\theta_m)i$ has been derived starting from the expression for the field energy and using the voltage equilibrium equations for actual physical windings. There is no proof that the same expression applies for the inductance matrix and electrical currents of virtual, inexistent windings. The torque expression $½\, i^T(\mathrm{d}\boldsymbol{L}/\mathrm{d}\theta_m)i$ can be used only with the inductance matrix $\boldsymbol{L}$ and electrical currents that correspond to actual physical windings. Hence, substitution of (15.53) into the torque expression is erroneous. Notice that the torque expression can be used in conjunction with $\alpha\beta$ variables. This is due to the fact that one can actually build a two-phase machine with $\alpha\beta$ windings which represents the original three-phase machine. Virtual windings such as *d, q, D,* and *Q* cannot exist in an actual machine. Yet, they are introduced as a means to understand and use Park rotational transform.

Question (15.12): The self-inductance L_S of the stator phase windings of a three-phase machine and coefficient of magnetic coupling between the stator and rotor *k* are known. Determine the coefficients of inductance matrix for the winding system in Fig. 15.11.

Answer (15.12): The inductance matrix in (15.53) is obtained by applying Clarke $3\Phi/2\Phi$ transform to the original three-phase machine and then applying Park rotational transform to the two-phase $\alpha\beta$ equivalent. The applied transforms are invariant in terms of inductance. Therefore, elements L_{11} and L_{22} of the matrix are equal to L_S. Considering the rotor model, it is usually assumed that the short-circuited rotor winding has the same number of turns as the stator and that magnetic resistance along paths of the two fluxes is approximately equal; thus, $L_S = L_R$. The coefficients of mutual inductance are equal to $L_m = k(L_S L_R)^{0.5} = kL_S$.

Successive application of Clarke and Park transforms results in the transform known as Blondel transform.

15.25 Voltage Balance Equations in *dq* Frame

Mathematical model of the electrical subsystem comprises the voltage balance equations, differential equations that express the equilibrium of the supply voltage, the voltage drops, and electromotive forces in actual phase windings. These equations can be transformed from $\alpha\beta$ frame to dq frame, synchronous with the revolving field. In $\alpha\beta$ stationary coordinate system, voltage equilibrium in the stator phase windings is given by

$$u_{\alpha S} = R_S i_{\alpha S} + \mathrm{d}\Psi_{\alpha S}/\mathrm{d}t, \qquad u_{\beta S} = R_S i_{\beta S} + \mathrm{d}\Psi_{\beta S}/\mathrm{d}t. \tag{15.54}$$

Multiplication of the second equation by j and summing the two equations result in a single voltage balance equation with voltages, fluxes, and currents expressed as complex numbers,

$$\begin{aligned} \underline{u}_{\alpha\beta S} &= (u_{\alpha S} + ju_{\beta S}) = R_S(i_{\alpha S} + ji_{\beta S}) + \mathrm{d}(\Psi_{\alpha S} + j\Psi_{\beta S})/\mathrm{d}t \\ &= R_S \underline{i}_{\alpha\beta S} + \mathrm{d}\underline{\Psi}_{\alpha\beta S}/\mathrm{d}t. \end{aligned} \tag{15.55}$$

Voltages of virtual d-phase and q-phase windings are obtained from $\alpha\beta$ voltages by applying Park transform,

$$\underline{u}_{dq} = u_d + ju_q = \underline{u}_{\alpha\beta S} e^{-j\theta_e} = \left(R_S \underline{i}_{\alpha\beta S} + \mathrm{d}\underline{\Psi}_{\alpha\beta S}/\mathrm{d}t\right) e^{-j\theta_e}. \tag{15.56}$$

Variables $\underline{i}_{\alpha\beta S}$ and $\Psi_{\alpha\beta S}$ of the stationary coordinate system can be represented in terms of dq variables by applying inverse Park transform, $\underline{i}_{\alpha\beta S} = \underline{i}_{dq}\exp(-j\theta_e)$, to obtain

$$\begin{aligned} \underline{u}_{dq} &= \left(R_S \underline{i}_{dq} e^{+\mathrm{j}\theta_e} + \mathrm{d}\left(\underline{\Psi}_{dq} e^{+j\theta_e}\right)/\mathrm{d}t\right) e^{-\mathrm{j}\theta_e} \\ &= R_S \underline{i}_{dq} + \mathrm{d}\underline{\Psi}_{dq}/\mathrm{d}t + j\omega_e \underline{\Psi}_{dq}. \end{aligned} \tag{15.57}$$

Therefore, voltage balance equations of virtual stator phases in dq frame do not have the form $u = Ri + \mathrm{d}\Psi/\mathrm{d}t$. They contain an additional member which comes as a consequence of Park transform. The above equation with complex numbers can be separated into real and imaginary parts, resulting in two scalar equations. The same procedure can be applied to the voltage balance equations of the rotor windings, but this time the angle θ_e is replaced by angle θ_{slip}. This is due to the fact that the angle between the original rotor coordinate system (α_R–β_R) and the target dq coordinate system is $\theta_{slip} = \theta_e - \theta_m$. For the virtual rotor windings (DQ), the following voltage balance equations are obtained:

$$\begin{aligned} \underline{u}_{DQ} &= \left(R_R \underline{i}_{DQ} e^{+\mathrm{j}\theta_{slip}} + \mathrm{d}\left(\underline{\Psi}_{DQ} e^{+\mathrm{j}\theta_{slip}}\right)/\mathrm{d}t\right) e^{-\mathrm{j}\theta_{slip}} \\ &= R_R \underline{i}_{DQ} + \mathrm{d}\underline{\Psi}_{DQ}/\mathrm{d}t + \mathrm{j}\omega_{slip} \underline{\Psi}_{DQ}. \end{aligned} \tag{15.58}$$

15.26 Electrical Subsystem

This section summarizes the model of the electrical subsystem of induction machine expressed in synchronous dq frame, where the state variables are constant in the steady state and where the inductance matrix of virtual windings has constant elements. This model is based on the four approximations adopted in modeling induction machines. They are detailed in introductory chapters, and they include:

1. Neglected effects of distributed parameters.
2. The energy of electrical field is neglected along with parasitic capacitances.
3. Neglected are the iron losses.
4. Magnetic saturation is neglected along with nonlinear $B(H)$ characteristic of ferromagnetic materials.

Moreover, the leading coefficient of Clarke transform, used to replace the three-phase original with the two-phase equivalent, is $K = 2/3$. For simplicity, further considerations assume that the induction machine under consideration is two-pole machine where $p = 1$ and that electrical angular frequency ω corresponds to mechanical angular speed Ω, resulting in $\omega = p\Omega$.

The following equations give complete mathematical model of the electrical subsystem of an induction machine in the synchronously rotating dq coordinate system. The two voltage balance equations with complex variables are split into four voltage balance equations with scalars:

$$u_{\mathrm{d}} = R_s i_{\mathrm{d}} + \frac{\mathrm{d}\Psi_{\mathrm{d}}}{\mathrm{d}t} - \omega_e \Psi_q, \tag{15.59}$$

$$u_q = R_s i_q + \frac{\mathrm{d}\Psi_q}{\mathrm{d}t} + \omega_e \Psi_d, \tag{15.60}$$

$$0 = R_R i_D + \frac{\mathrm{d}\Psi_D}{\mathrm{d}t} - \omega_{slip} \Psi_Q, \tag{15.61}$$

$$0 = R_R i_Q + \frac{\mathrm{d}\Psi_Q}{\mathrm{d}t} + \omega_{slip} \Psi_D. \tag{15.62}$$

The inductance matrix relates the flux linkages and currents of virtual windings:

$$\begin{bmatrix} \Psi_d \\ \Psi_q \\ \Psi_D \\ \Psi_Q \end{bmatrix} = \begin{bmatrix} L_s & 0 & L_m & 0 \\ 0 & L_s & 0 & L_m \\ L_m & 0 & L_R & 0 \\ 0 & L_m & 0 & L_R \end{bmatrix} \cdot \begin{bmatrix} i_d \\ i_q \\ i_D \\ i_Q \end{bmatrix}. \tag{15.63}$$

Question (15.13): Starting from expression for the torque $T_{em} = (3/2)(\Psi_{\alpha S}\, i_{\beta S} - \Psi_{\beta S}\, i_{\alpha S})$, express the torque T_{em} as function of the fluxes and currents in dq coordinate frame.

Answer (15.13): By using complex notation in vector representation, where $\Psi_{\alpha\beta S} = \Psi_{\alpha S} + \mathrm{j}\Psi_{\beta S}$ and $\underline{i}_{\alpha\beta S} = i_{\alpha S} + \mathrm{j}\; i_{\beta S}$, the torque can be written as $T_{em} = (3/2)$ Im $(\Psi^*{}_{\alpha\beta S}\; \underline{i}_{\alpha\beta S})$, where Im denotes the function taking imaginary part of the complex number, while $\Psi^*{}_{\alpha\beta S}$ is conjugate value of the complex number, the number with the imaginary part having the opposite sign. On the basis of Park transform expression $\Psi_{dq} = \Psi_d + \mathrm{j}\Psi_q = \exp(\text{-j}\theta_e)\; \Psi_{\alpha\beta S}$ and $\underline{i}_{dq} = i_d + \mathrm{j}\; i_q = \exp(-\mathrm{j}\theta_e)\; \underline{i}_{\alpha\beta S}$, one obtains

$$\begin{aligned} T_{em} &= \frac{3}{2}\mathrm{Im}\left[\left(\underline{\Psi}_{dq} e^{\mathrm{j}\theta_e}\right)^* \left(\underline{i}_{dq} e^{\mathrm{j}\theta_e}\right)\right] \\ &= \frac{3}{2}\mathrm{Im}\left[\underline{\Psi}^*_{dq} e^{-\mathrm{j}\theta_e} \underline{i}_{dq} e^{\mathrm{j}\theta_e}\right] \\ &= \frac{3}{2}\mathrm{Im}\left[\underline{\Psi}^*_{dq} \underline{i}_{dq}\right] = \frac{3}{2}\mathrm{Im}\left[\underline{\Psi_d i_q -}\; \Psi_q i_d\right]. \end{aligned}$$