Appendix B
Matrix Results

We begin with two properties of determinants:

$$\det(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}) = \det(\boldsymbol{A}^{\mathrm{T}})\det(\boldsymbol{A}) = \det(\boldsymbol{A})^2 \tag{B.1}$$

and

$$\begin{vmatrix} \boldsymbol{T} & \boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{W} \end{vmatrix} = \mid \boldsymbol{T} \mid\mid \boldsymbol{W} - \boldsymbol{V}\boldsymbol{T}^{-1}\boldsymbol{U} \mid . \tag{B.2}$$

Let $\boldsymbol{A}$ be an $n \times n$ non-singular matrix, which we express as

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix},$$

where $\boldsymbol{A}_{11}$ is $k \times k$ and $\boldsymbol{A}_{12}$ is $k \times (n-k)$. The inverse $\boldsymbol{B} = \boldsymbol{A}^{-1}$ has elements

$$\begin{aligned}
\boldsymbol{B}_{11} &= (\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21})^{-1} \\
\boldsymbol{B}_{22} &= (\boldsymbol{A}_{22} - \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12})^{-1} \\
\boldsymbol{B}_{12} &= -\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\boldsymbol{B}_{22} \\
\boldsymbol{B}_{21} &= -\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\boldsymbol{B}_{11}.
\end{aligned}$$

For matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ of the appropriate dimensions:

$$(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^{\mathrm{T}})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{B} + \boldsymbol{C}^{-1})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{-1}. \tag{B.3}$$

We now describe how the expectation, variance and covariance operators deal with vectors of random variables.

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Suppose $\boldsymbol{U}$ is an $n \times 1$ vector of random variables, and $\boldsymbol{A}$ is an $m \times n$ matrix. Then

$$\mathrm{E}[\boldsymbol{AU}] = \boldsymbol{A}\,\mathrm{E}[\boldsymbol{U}]$$
$$\mathrm{var}(\boldsymbol{AU}) = \boldsymbol{A}\,\mathrm{var}(\boldsymbol{U})\boldsymbol{A}^{\mathrm{T}}.$$

Suppose $\boldsymbol{V}$ is an $m \times 1$ vector of random variables. Then $\mathrm{cov}(\boldsymbol{U}, \boldsymbol{V}) = \boldsymbol{C}$ is an $n \times m$ matrix with (i, j)th element $\mathrm{cov}(U_i, V_j)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Hence, $\mathrm{cov}(\boldsymbol{V}, \boldsymbol{U}) = \boldsymbol{C}^{\mathrm{T}}$. In addition,

$$\mathrm{cov}(\boldsymbol{U}, \boldsymbol{AU}) = \mathrm{cov}(\boldsymbol{U})\boldsymbol{A}^{\mathrm{T}}$$
$$\mathrm{cov}(\boldsymbol{AU}, \boldsymbol{U}) = \boldsymbol{A}\,\mathrm{cov}(\boldsymbol{U}).$$

The iterated expectation and covariance formulas are given by:

$$\mathrm{E}[\boldsymbol{Y}] = \mathrm{E}_{\boldsymbol{X}}\left[\mathrm{E}_{\boldsymbol{Y}\,|\,\boldsymbol{X}}(\boldsymbol{Y} \mid \boldsymbol{X})\right]$$
$$\mathrm{cov}(\boldsymbol{Y}, \boldsymbol{Z}) = \mathrm{E}_{\boldsymbol{X}}\left[\mathrm{cov}_{\boldsymbol{Y},\boldsymbol{Z}\,|\,\boldsymbol{X}}(\boldsymbol{Y}, \boldsymbol{Z} \mid \boldsymbol{X})\right] + \mathrm{cov}_{\boldsymbol{X}}\left[\mathrm{E}_{\boldsymbol{Y}\,|\,\boldsymbol{X}}(\boldsymbol{Y} \mid \boldsymbol{X}), \mathrm{E}_{\boldsymbol{Z}\,|\,\boldsymbol{X}}(\boldsymbol{Z} \mid \boldsymbol{X})\right].$$

Suppose $\boldsymbol{Z}$ is an $n \times 1$ random variable with $\mathrm{E}[\boldsymbol{Z}] = \boldsymbol{\mu}$, $\mathrm{var}(\boldsymbol{Z}) = \boldsymbol{\Sigma}$ and $\boldsymbol{A}$ is a symmetric $n \times n$ matrix. Then

$$\mathrm{E}[\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{AZ}] = \mathrm{tr}(\boldsymbol{A\Sigma}) + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{A\mu}. \tag{B.4}$$

See Schott (1997, p. 391) for a proof.