Modeling Soil Moisture

28

Richards equation is a model of water content and soil water flux in response to energy gradients. It is incorporated into the *WINDS* model. Water, energy, and conductivity relationships are calculated with the van Genuchten equations. Determining the rate of water table movement in response to water gain or loss requires calculation of specific yield. The WINDS model utilizes integrated forms of the Brooks-Corey and van Genuchten equations to calculate water volume in the soil profile above a water table and to calculate the specific yield of the water table. This approach requires that some soil layers in the model are in hydraulic equilibrium with the water table while flow in upper layers is calculated based on energy differences and hydraulic conductivity. Lastly, this chapter discusses upward movement of water from the water table and compares Anat's equation to a discretized solution.

Combined Energy and Volume Balance Equation

As with Chap. 26, the vertical axis, z, is defined in the positive upward direction, the datum is placed at the bottom of the control volume, and layer numbers increase from the bottom to the top of the control volume (Fig. 28.1).

The Darcy velocity, v, is calculated with the Darcy equation with the energy gradient including both elevation and matric potential.

$$\mathrm{v} = -K_e \frac{H_j - H_{j+1}}{\Delta z} \tag{28.1}$$

where

$H = h_c + z$
Δz = height of layer, m,
H = total energy at center of layer, m,
h_c = matric potential, m,
z = elevation above datum, m.

The soil physics equation that is used to model water flow in soils is the Richard's equation.

$$\frac{\delta\theta}{\delta t} = \frac{\delta}{\delta z}\left[K(h)\frac{\delta h}{\delta z} + 1\right] \tag{28.2}$$

Thermal and osmotic energy gradients influence water movement in soils, but they are ignored in this chapter and in many soil-water models.

In the following derivation, j refers to the layer, j − 1 refers to the layer below, j + 1 refers to the layer above, and n refers to the number of layers. $Z_{L-1.}$ The depth of water in the layer is θ Δz. The change in depth of water is the product of the change in water content and the thickness of the layer, Δθ Δz.

Equation 28.3 is derived from Eq. 27.20. Although the cross-sectional area between layers remains constant in soils, the hydraulic conductivity changes dramatically with water content so the effective conductivity, K_e, calculated as the geometric or arithmetic mean of the conductivities in the adjacent layers is used in Eq. 28.3.

$$\Delta\theta_j \Delta z = \left(-K_{e(j\&\ j-1)}\frac{H_j - H_{j-1}}{\Delta z} + K_{e(j\&\ j+1)}\frac{H_{j+1} - H_j}{\Delta z}\right)\Delta t \tag{28.3}$$

Divide both sides by Δz and solve for change in water content.

$$\Delta\theta_j = \frac{1}{\Delta z}\left(-K_{e(j\&\ j-1)}\frac{H_j - H_{j-1}}{\Delta z} + K_{e(j\&\ j+1)}\frac{H_{j+1} - H_j}{\Delta z}\right)\Delta t$$
$$\Delta\theta_j = \frac{\Delta t}{\Delta z^2}\left(-K_{e(j\&\ j-1)}\left(H_j - H_{j-1}\right) + K_{e(j\&\ j+1)}\left(H_{j+1} - H_j\right)\right) \tag{28.4}$$

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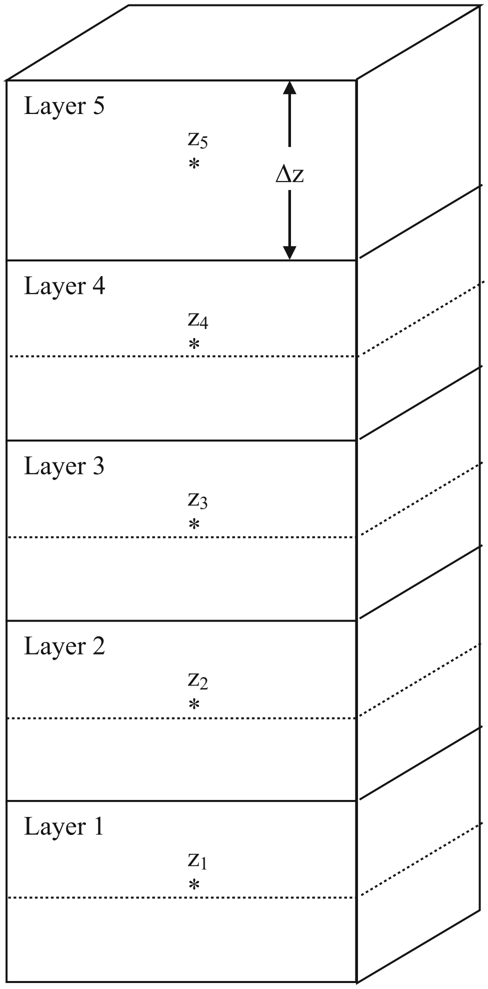


Fig. 28.1 Soil model with 5 layers beginning with layer 1 at the bottom of the control volume

Final water content is the initial water content + change in water content.

$$\theta_{j-\text{final}} = \theta_{j-\text{initial}} + \Delta\theta_j$$
$$\theta_{final} = \theta_{initial} + \frac{\Delta t}{\Delta z^2}\left(-K_{e(j\&\ j-1)}\left(H_j - H_{j-1}\right) + K_{e(j\&\ j+1)}\left(H_{j+1} - H_j\right)\right) \quad (28.5)$$

The total energy, H is the sum of matric potential energy for the layer and elevation of the midpoint of the layer above the datum. If the layers have the same thickness, Δz, then Δz can be substituted for the difference in elevation potential energy between two layers as follows.

$$H_j - H_{j-1} = \left(h_{c\,j} - h_{c\ \ j-1} + \Delta z\right)$$
$$\theta_{j-final} = \theta_{initial} + \frac{\Delta t}{\Delta z^2}\left(-K_{e(j\&\ j-1)}\left(h_j - h_{j-1} + \Delta z\right) + K_{e(j\&\ j+1)}\left(h_{j+1} - h_j + \Delta z\right)\right) \quad (28.6)$$

In the case of a confining layer at the lower end of the soil profile with no water movement below the root zone, the final water content in the lower layer (layer 1) is

$$\theta_{final} = \theta_{initial} + \frac{\Delta t}{\Delta z^2}\left(K_{e(1\,\&\,2)}(H_2 - H_1)\right) = \theta_{initial} + \frac{\Delta t}{\Delta z^2}\left(K_{e(1\,\&\,2)}(h_2 - h_1 + \Delta z)\right) \quad (28.7)$$

The final water content in the upper layer (layer n) is

$$\theta_{n-final} = \theta_{n-initial} + \frac{\Delta t}{\Delta z^2}\left(K_{e(n\,\&n-1)}(H_n - H_{n-1})\right) = \theta_{n-initial} + \frac{\Delta t}{\Delta z^2}\left(K_{e(n\,\&\,n-1)}(h_n - h_{n-1} + \Delta z)\right) + \frac{i}{\Delta z} \quad (28.8)$$

where

i = infiltration from the soil surface (positive downward), m.

The new matric potential energy (Eq. 28.9) is calculated at the beginning of each time step, and substituted into Eqs. 28.3, 28.4, 28.5, 28.6, 28.7, and 28.8 in order to calculate flux and change in water content.

Brooks-Corey van Genuchten

$$h_c = \frac{h_b}{\left(\frac{\theta-\theta r}{\theta_s-\theta_r}\right)^{1/\lambda}} \qquad h_c = \frac{\left(\theta_e^{-1/m} - 1\right)^{1/n}}{\alpha} \quad (28.9)$$

The effective hydraulic conductivity between two layers, j and j + 1, can be calculated as the simple average or calculated as the geometric mean.

$$K_e = \frac{\Delta z}{\frac{\Delta z/2}{K_j} + \frac{\Delta z/2}{K_{j+1}}} = \frac{2}{\frac{1}{K_j} + \frac{1}{K_{j+1}}} \quad (28.10)$$

Neither the simple nor the geometric average is correct since the changes in water content and conductivity between two points are nonlinear. In the WINDS model, a conditional statement is used: the average mean is used if the water content is greater than a threshold value and if the water content in the upper layer is greater than the water content in the lower layer. The hypergeometric mean is used in if the either of the two conditions are not true.

Hydraulic conductivity is dramatically reduced with a reduction in water content because pore area available for water flow increases with water content and because the water path through the soil becomes more tortuous as water content decreases. The ratio of unsaturated hydraulic conductivity to saturated hydraulic conductivity is called the relative permeability, K_r.

$$K_r = \frac{K(\theta)}{K_S} \tag{28.11}$$

where

$K(\theta)$ = conductivity at water content θ, cm/day
K_S = saturated hydraulic conductivity, cm/day
K_r = relative permeability

Permeability is a property of the soil matrix whereas conductivity is a function of both the soil matrix geometry and the water viscosity. The relative permeability, K_r, is the ratio between the permeabilities of the saturated and unsaturated soil conditions. Through experiments and mathematical derivation, Brooks-Corey found the following relationship between relative permeability and matric potential with the term η equal to $2 + 3\lambda$:

$$K_{rw} = \left(\frac{h_b}{h_c}\right)^{2+3\lambda} \quad K_{rw} = \left(\frac{h_b}{h_c}\right)^{\eta} \quad K(h) = K_s K_{rw} = K_s\left(\frac{h_b}{h_c}\right)^{\eta} \tag{28.12}$$

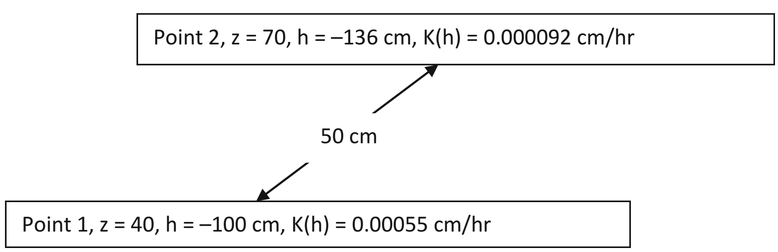
Example 28.1 Find the relative permeability and effective saturation of Wagram loamy sand ($h_b = -30$ cm $= -0.3$ m, $\lambda = 1.27$) at $h_c = -100$ cm.

$$K_{rw} = \left(\frac{-30}{-100}\right)^{2+3(1.27)} = 0.0009$$

The hydraulic conductivity at -100 cm capillary pressure is approximately 1,000 times less then the saturated hydraulic conductivity. The effective saturation at -100 cm capillary potential is 0.21. Thus, the effective saturation is approximately 5 times lower than for a fully saturated soil, but the conductivity is approximately 1,000 times lower.

$$S_e = \left(\frac{h_b}{h_c}\right)^{\lambda} = \left(\frac{-30}{-100}\right)^{1.27} = 0.21$$

Example 28.2 Calculate the Darcy velocity between two points in an unsaturated Wagram loamy sand with matric potential values of -100 cm and -136 cm and elevations of 40 and 70 cm, respectively. The saturated hydraulic conductivity of Wagram loamy sand is 0.6 cm/hr. The distance between the two points is 50 cm. Use the Brooks-Corey model for hydraulic conductivity and the hypergeometric mean to calculate the effective conductivity.



The relative conductivity, K_{rw}, at -100 cm was calculated in Example 28.1. Calculate the relative conductivity at -136 cm and the calculate actual hydraulic conductivity at each point.

$$K_{rw} = \left(\frac{-30}{-136}\right)^{2+3(1.27)} = 0.00015$$

$K(h_{100}) = K_S K_{rw\ 100} = 0.6\ \mathrm{cm/hr} * 0.00091 = 0.00055\ \mathrm{cm/hr}$
$K(h_{136}) = K_S K_{rw\ 136} = 0.6\ \mathrm{cm/hr} * 0.00015 = 0.000092\ \mathrm{cm/hr}$

Calculate the hypergeometric mean conductivity.

$$K_e = \frac{L_1 + L_2}{\frac{L_1}{K_1} + \frac{L_2}{K_2}} = \frac{25 + 25}{\frac{25}{0.000092} + \frac{25}{0.00055}} = 0.00016\ \ \mathrm{cm/hr}$$

Calculate energy difference, h_f, between the points.

$$h_f = \Delta H = h_2 + z_2 - h_1 - z_1 = -136 + 70 - (-100) - 40 = -6\ \ \mathrm{cm}$$

The fact that the energy gradient is negative means that the energy at point 1 is higher than the energy at point 2. Thus, flow is from point 1 to point 2. Calculate Darcy velocity, v.

$$v = -K_e \frac{\Delta H}{L} = -0.00016 \frac{-6}{50} = 0.000019\ \ cm/hr$$

Van Genuchten derived an expression for relative permeability, which is used in the *WINDS* model, in terms of effective saturation.

$$K(h) = K(\theta) = K_0 {\theta_e}^L \left[1 - \left(1 - {\theta_e}^{1/m}\right)^m\right]^2 \qquad (28.13)$$

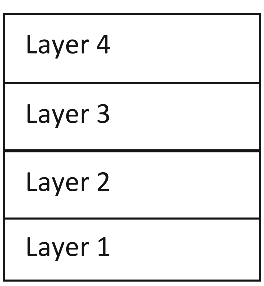
where

K(θ) = conductivity at water content θ, cm/day
K_0 = matching hydraulic conductivity at saturation, cm/day
L = curve fitting parameter, dimensionless number theoretically = 0.5 but not in actual practice (Table 7.2).

Equation 28.13 is written in terms of the matching hydraulic conductivity at saturation rather than the saturated hydraulic conductivity in order to match experimental curves more closely with the model. The matching and saturated hydraulic conductivity terms are similar but not the same (Table 27.2). The matching hydraulic conductivity is used instead of the hydraulic conductivity for the purpose of matching the mathematical curve to the experimental curve. The matching conductivity is in the *Soils* dialog box with the column heading Ko (Fig. 26.4).

Modeling Flux and Energy in Unsaturated Soils

The sections shows how *WINDS* models soil water content and flux in unsaturated soils with the Richards equation and the van Genuchten soil parameters. Example 28.3 shows that equilibrium is reached when total energy is the same in all layers.



Example 28.3 Four layers have 0.5 m depth. Saturated water content = 0.45. Residual water content = 0.067, n = 1.41, α = 0.02, L = 0.5, and K_0 = 10.8 cm/day. Initial water content in all layers is 30 %. There is no infiltration and no seepage of water below the control volume. Use daily time steps. Calculate effective hydraulic conductivity between layers with different water contents with the geometric mean. Make manual calculations for two time steps and compare with *WINDS* model output. Run the *WINDS* model for 200 days. The information for this simulation is in the #1 position in the *Crop_data* worksheet.

Calculate m.

$$m = 1 - 1/n = 1 - 1/1.41 = 0.29$$

Time step 1 (i = 1). Layer 1. (j = 1)

Calculate initial effective water content and matric potential.

$$\theta_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} = \frac{0.30 - 0.067}{0.45 - 0.067} = 0.608$$

Calculate initial matric potential.

$$h_c = -\frac{\left(\theta_e^{-1/m} - 1\right)^{1/n}}{\alpha} = -\frac{\left(0.608^{-1/0.29} - 1\right)^{1/1.41}}{0.02}$$
$$= -145.85 \ cm = -1.4585 \ m$$

Calculate hydraulic conductivity.

$$K(\theta) = K_0 {\theta_e}^{0.5}\left[1 - \left(1 - {\theta_e}^{1/m}\right)^m\right]^2$$
$$= 10.8*(0.608)^{0.5}\left[1 - \left(1 - 0.608^{1/0.29}\right)^{0.29}\right]^2$$
$$= 0.0268 \ cm/d = 0.000268 \ m/d$$

Calculate final water content in layer 1.

$$\theta_{final} = \theta_{initial} + \frac{\Delta t}{\Delta z^2}\left(K_{e(1\,\&\,2)}(h_2 - h_1 + \Delta z)\right)$$
$$\theta_{final} = 0.30 + \frac{1}{0.5^2}(0.000268(-1.4585 - (-1.4585) + 0.5))$$
$$= 0.30054$$

Layer 2. (j = 2) Fluxes from top and bottom are the same so there is no change in water content.

Day 2

Calculate new effective water content in layer 1 (layers 2 and 3 remain the same)

$$\theta_{e-1} = \frac{\theta - \theta_r}{\theta_s - \theta_r} = \frac{0.30054 - 0.067}{0.45 - 0.067} = 0.6098$$

Calculate new matric potential in layer 1.

$$h_{c-1} = -\frac{\left(\theta_e^{-1/m} - 1\right)^{1/n}}{\alpha} = -\frac{\left(0.6098^{-1/0.29} - 1\right)^{1/1.41}}{0.02}$$
$$= -144.85 \ cm = -1.4485 \ m$$

Calculate new hydraulic conductivity in layer 1 (layers 2 and 3 remain the same).

$$K(\theta)_1 = K_0\theta_e^{0.5}\left[1-\left(1-\theta_e{}^{1/m}\right)^m\right]^2$$
$$= 10.8*(0.608)^{0.5}\left[1-\left(1-0.608^{1/0.29}\right)^{0.29}\right]^2$$
$$= 0.0273\ \ cm/d = 0.000273\ \ m/d$$

Calculate effective hydraulic conductivity between layers 1 and 2 with the geometric mean.

$$K_{e-1-2} = \frac{2}{\frac{1}{K_j}+\frac{1}{K_{j+1}}} = \frac{2}{\frac{1}{0.000268}+\frac{1}{0.000273}}$$
$$= 0.00027049\ \ m/day$$

Calculate final water content in layer 1.

$$\theta_{1-final} = \theta_{2-initial} + \frac{\Delta t}{\Delta z^2}\left(K_{e(2\&\,1)}(h_2-h_1+\Delta z)\right)$$
$$\theta_{1-final}\ \ = 0.30054 + \frac{1}{0.5^2}(0.0002705*\,(-1.4585-(-1.4485)+0.5)) = 0.3011$$

Calculate the final water content in layer 2.

$$\theta_{2-final} = \theta_{2-initial} + \frac{\Delta t}{\Delta z^2}\left(-K_{e(2\&\,1)}(h_2-h_1+\Delta z)+K_{e(2\,\&\,3)}(h_3-h_2+\Delta z)\right)$$
$$\theta_{2-final} = 0.3 + \frac{1}{0.5^2}(-0.0002705,*(-1.4585,-(-1.4485)\ \ +0.5)+0.000268(0.5)) = 0.300006$$

The calculated values agree with the WINDS model calculations on the first two days in layers 1 and 2. The following text explains the calculations in written form.

$$\begin{array}{l}
\text{Water content in layers 2 and 3} = 0.3\ \ \text{m/m}\\
\text{Energy difference between layer 3 and layer } 2 = 0.5\\
\text{Average conductivity and flow in vertical direction} = 0.000268\ \ \text{m/day}\\
\text{Water content change in layer } 2 = 0.000268\ \ \text{m}/0.5\ \ \text{m} = +0.000536\ \ \text{m/m}\\
\text{Updated water content in layer } 2 = 0.300536 - \text{the same as the layer 1 water content}
\end{array}$$

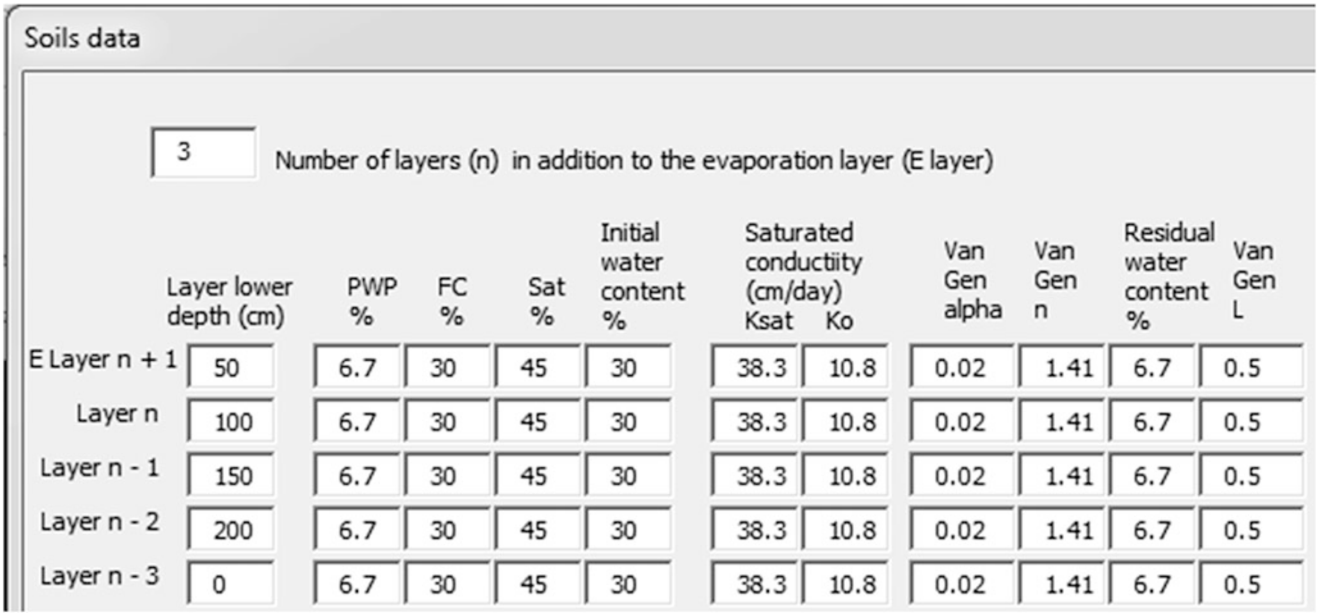
$$\begin{array}{l}
\text{Energy difference} = 0.5 + (-0.014585-(-0.014485)) = 0.49\\
\text{Average conductivity and flow in vertical direction} = (0.000268+0.000273)/2 = 0.0002705\\
\text{Water content change in layer } 2 = 0.49*0.0002705/0.5^2 = -0.0005302\ \ \text{m/m}\\
\text{Updated water content in layer } 2 = 0.300536 - 0.0005302 = 0.300006
\end{array}$$

	I	J	K	L	M
34	TRUE	No infiltration			
35	TRUE	No ET			
36	FALSE	No stress reduction			
37	TRUE	No ET frac adjustment			
38	FALSE	Continue drainage rate			
39	FALSE	No redistribution by Richards equation			
40	TRUE	Turn off field capacity restriction			
41	TRUE	Bypass upper layer in nitrogen application			
42	FALSE	No nitrogen frac adjustment			

Fig. 28.2 Main page true-false values for Example 28.3

In order to run the model in WINDS, the True-False values in column I must be set as shown in Fig. 28.2. Turn off salinity and nitrogen calculations in row 5. The van Genuchten parameters are set in the *Soils* dialog box (Fig. 28.3). With the sensitivity of energy-driven water flux calculations to discontinuities, the evaporation layer cannot be ignored as it was in Chap. 26. It becomes the upper layer in this example, but with no evaporation.

Water nearly reaches equilibrium during the 200 day simulation period (Fig. 28.4). Equilibrium is reached when the total energy (Fig. 28.5), which is the elevation energy + matric potential energy (Fig. 28.6), is the same in all layers.



Soils data

3 Number of layers (n) in addition to the evaporation layer (E layer)

	Layer lower depth (cm)	PWP %	FC %	Sat %	Initial water content %	Saturated conductiity (cm/day) Ksat	Ko	Van Gen alpha	Van Gen n	Residual water content %	Van Gen L
E Layer n + 1	50	6.7	30	45	30	38.3	10.8	0.02	1.41	6.7	0.5
Layer n	100	6.7	30	45	30	38.3	10.8	0.02	1.41	6.7	0.5
Layer n - 1	150	6.7	30	45	30	38.3	10.8	0.02	1.41	6.7	0.5
Layer n - 2	200	6.7	30	45	30	38.3	10.8	0.02	1.41	6.7	0.5
Layer n - 3	0	6.7	30	45	30	38.3	10.8	0.02	1.41	6.7	0.5

Fig. 28.3 Soil parameters for Example 28.3

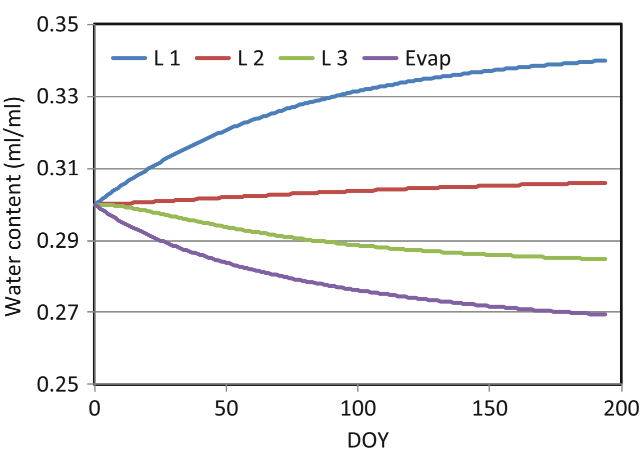


Fig. 28.4 Water content versus time from *WINDS* model for Example 28.3

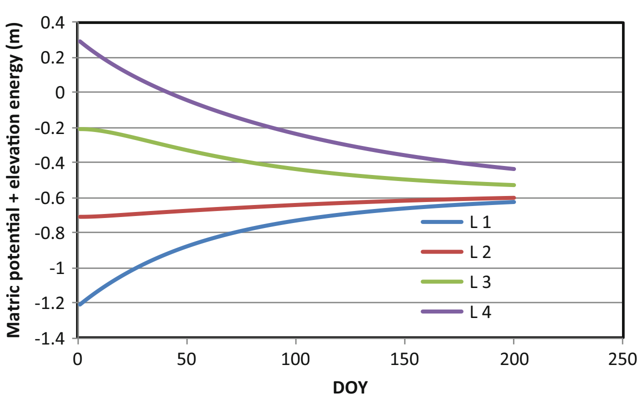


Fig. 28.5 Total energy versus time from *WINDS* model for Example 28.3

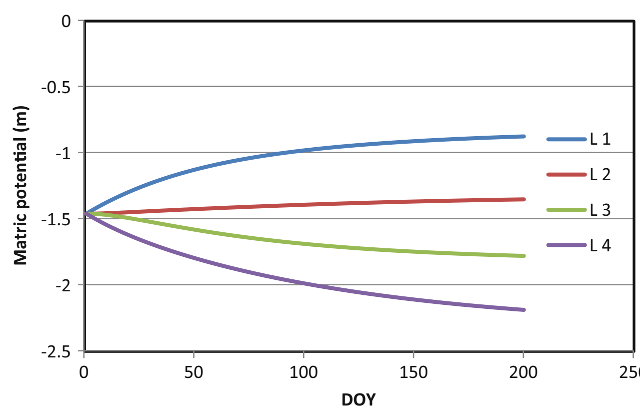


Fig. 28.6 Matric potential versus time from *WINDS* model for Example 28.3

Example 28.4 Leaving other parameters the same as in 28.3, change the van Genuchten parameters to $n = 1.75$ and $\alpha = 0.078$. Run the WINDS model for 300 days. Results are shown in Fig. 28.7. The information for this simulation is in the #2 position in the *Crop_data* worksheet.

Example 28.4 demonstrates the dramatically different results that can be obtained with two soils with the same textural classification but with different properties. The soils in Examples 28.3 and 28.4 are silt loams. Layer 1 becomes saturated after 50 days (there is no leaching below layer 1) in Example 28.4.

Normally, there is not an impermeable layer directly below the root zone (as in Example 28.4), and water either pools in a water table or seeps to a lower soil layer. For soils without a water table, the same matric potential can be assigned to the region below the lower layer in order to allow the lower layer to drain by gravity (difference in

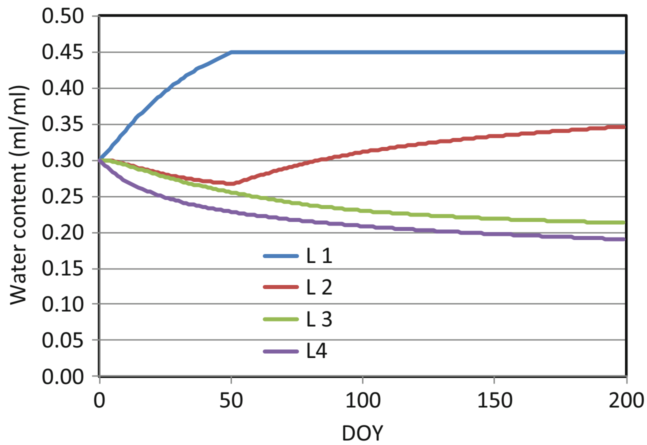


Fig. 28.7 Water content versus time from WINDS model for Example 28.4

head equal to Δz). This is accomplished by rewriting the water balance equation for layer 1 (Eq. 28.6) as

$$\theta_{final} = \theta_{initial} + \frac{\Delta t}{\Delta z^2}\left(K_{e(1\,\&\,2)}(h_2 - h_1 + \Delta z) - K_1 \Delta z\right) \quad (28.14)$$

where K_1 Δt is the flux below layer 1. Equation 28.14 is used at the lower boundary in Example 28.5. For saturated soils with a water table, seepage rate is generally a function of water table elevation, which is described in the following sections.

Water and Energy Distribution with a Water Table

When a lower confining layer such as in Example 28.4 prevents the rapid downward movement of water, the lower soil layers reach saturation. In soils with a water table, the soil can be divided into the saturated zone below the water table and the vadose zone above the water table. The saturated zone may lose water slowly to a drainage system or to a low permeability aquitard below the soil. The water just above the water table tends to remain in hydraulic equilibrium with the water table. However, at some distance above the water table, matric potentials become more negative due to water extraction by evapotranspiration, and the layers disconnect hydraulically from the water table. Thus, *WINDS* models soils with a water table in two parts: the upper section in which water moves from layer to layer by matric potential gradients caused by ET, and the lower section, which remains in equilibrium with the water table.

Specific yield is the depth of water drained from the soil per change in depth of the water table. The concept is used in both aquifer and soil drainage calculations to estimate the amount of water that is removed from an aquifer for a given change in water table elevation. The United States Bureau of Reclamation calculates specific yield of a soil as the total water drained when the water table drops from the soil surface to a given depth.

$$SY = \frac{d_d}{DTWT} \quad (28.15)$$

where

SY = Specific yield, dimensionless,
d_d = Depth drained, m,
DTWT = Depth to the water table from the soil surface, m.

Example 28.5 20 cm depth of water is drained from the soil as the water table is lowered from the soil surface to 2 m. Calculate the specific yield.

$$SY = \frac{d_d}{DTWT} = \frac{0.2\ m}{2\ m} = 0.1 = 10\%$$

Specific yield can also be defined as the incremental change in volume drained per incremental change in water table depth.

$$SY = \frac{\Delta d_d}{\Delta DTWT} = -\frac{\Delta d_d}{\Delta z_{WT}} \quad (28.16)$$

where

Δz_{WT} = change in water table elevation above a datum (z is positive upward), m.

Example 28.6 The water table drops from 1.5 to 1.6 m below the ground surface, and the incremental specific yield at 1.5 m depth is 20 %. Calculate the depth of water drained. Calculate in terms of the elevation above the datum instead of the depth to the water table to maintain consistency with later equation development.

Assume that the datum is at 2 m depth. Initial water table elevation is 0.5 m above the datum and final water table elevation is 0.4 m.

$$\Delta z_{WT} = z_{WT-2} - z_{WT-1} = 0.4\ \text{m} - 0.5\ \text{m} = -0.1\ \text{m}$$
$$d_d = -SY^* \Delta z_{WT} = -0.2^*(-0.1) = 0.02\ m$$

Specific yield can be calculated with the van Genuchten or Brooks-Corey equations. For the Brooks-Corey equation, if the water table is lowered from the soil surface elevation, z_t, to an elevation, z_{WT}, then the total depth drained, d_d, is the area of the hatched region in Fig. 28.8. The total depth of water in the soil above the water table is the integral of the equations from the water table to the soil surface. Depth drained from the soil when the water table drops from the soil surface to elevation, $z_{WT,}$ is calculating by subtracting the drained depth

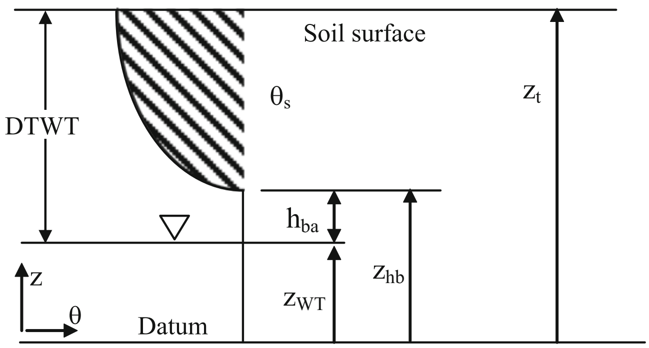


Fig. 28.8 Schematic for calculation of volume drained

from the total depth in the control volume. The drained depth is the integral of the hatched area in Fig. 28.8 with limits of integration as the absolute value of the matric potential at the lower (h_{ba}) and upper ($z_t - z_{hb} + h_{ba}$) boundaries.

z_{WT} = elevation of water table
z_{hb} = elevation of bubbling pressure
z_t = elevation of surface above datum.
h_{ba} = Abs. value of bubbling pressure

All units in m.

The area of the hatched region in Fig. 28.7 is calculated as follows:

$$d_d = \theta_s(z_t - z_{hb}) - \int_{hba}^{Zt-ZWT} \theta(z)dz \qquad (28.17)$$

where

z_t = elevation of soil surface above the datum, m,
z_{hb} = elevation of bubbling pressure point above datum (z_{WT} + h_b), m,
d_d = total depth drained from profile, m,
θ(z) = water content at elevation z above the datum, m/m,

The Brooks-Corey equation is substituted for θ(z)

$$\theta(z) = \theta_r + (\theta_s - \theta_r)\left(\frac{h_b}{h_c}\right)^{\lambda} = \theta_r + (\theta_s - \theta_r)\left(\frac{h_{ba}}{z}\right)^{\lambda}$$

$$d_d = \theta_s(z_t - z_{hb}) - \left(\int_{hba}^{Zt-ZWT}\left(\theta_r + (\theta_s - \theta_r)\left(\frac{h_{ba}}{z}\right)^{\lambda}\right)dz\right)$$

$$d_d = \theta_s(z_t - z_{hb}) - (z_t - z_{hb})\theta_r - \frac{(\theta_s - \theta_r)h_{ba}^{\lambda}}{-\lambda + 1}\left((z_t - z_{WT})^{-\lambda+1} - (h_{ba})^{-\lambda+1}\right)$$

$$d_d = (\theta_s - \theta_r)(z_t - z_{hb}) - \frac{(\theta_s - \theta_r)h_{ba}^{\lambda}}{-\lambda + 1}\left((z_t - z_{WT})^{-\lambda+1} - (h_{ba})^{-\lambda+1}\right)$$

$$d_d = (\theta_s - \theta_r)\left(z_t - z_{hb} - \frac{h_{ba}^{\lambda}}{-\lambda + 1}\left((z_t - z_{WT})^{-\lambda+1} - (h_{ba})^{-\lambda+1}\right)\right) \qquad (28.18)$$

Example 28.7 Calculate the depth of water drained from the soil profile and specific yield if the water table drops from the soil surface to a depth of 1 m in Wagram loamy sand. Use Eq. 28.18. Assume that the datum is 2 m below the soil surface.

Water table depth = 1.0 m, bubbling pressure, h_{ba} = 0.3 m, Saturated water content, θ_s = 0.305,

Residual water content, θ_r = 0.044, Pore size distribution index, λ = 1.27

$$z_{hb} = z_{WT} + h_{ba} = 1 + 0.3 = 1.3 \text{ m}$$

$$d_{d(1.0\ meter)} = 0.261\left(2 - 1.3 - \frac{0.3^{1.27}}{-1.27 + 1}\left((2-1)^{-1.27+1} - 0.3^{-1.27+1}\right)\right) = 0.1022\ m$$

Thus, 10 cm of water is drained from the soil profile when the water table drops from the soil surface to 1 m depth. The specific yield is calculated as follows:

$$SY = \frac{d_d}{DTWT} = \frac{0.10}{1} = 0.1 = 10\%$$

The incremental specific yield is the depth drained for a given change in water table depth from one position to the next. This is found by taking the derivative of d_d with respect to z_{hb}. Fortunately, terms cancel in the derivative so Eq. 28.19 is relatively short.

$$SY = \frac{d(d_d)}{d(z_{WT})} = -(\theta_s - \theta_r)D_{zhb}\left(z_t - z_{hb}, -\frac{h_{ba}^{\lambda}}{-\lambda + 1}\left((z_t - z_{WT})^{-\lambda+1} - (h_{ba})^{-\lambda+1}\right)\right)$$

$$SY = \frac{d(d_d)}{d(z_{WT})} = -(\theta_s - \theta_r)\left[-1 + \left(\frac{h_{ba}}{z_t - z_{WT}}\right)^{\lambda}\right] = -(\theta_s - \theta_r)\left[-1 + \left(\frac{h_{ba}}{DTWT}\right)^{\lambda}\right] \qquad (28.19)$$

Example 28.8 For Wagram loamy sand, find the incremental specific yield if the water table is at 1 m depth.

$$\frac{d(d_d)}{d(z_{WT})} = -(0.305 - 0.044)\left[-1 + \left(\frac{0.3}{1}\right)^{1.27}\right]100\% = S = 20.5\%$$

Thus, for every 1 cm change in water table depth, 0.205 cm is drained from the soil. For example, if the water table depth increased from 1 to 1.02 m (z_{wt} decreased from 1 to 0.98 m), then 0.02 m * 20.5 = 0.0041 m would be drained from the soil. Depth drained would be 0.1022 + 0.0041 = 0.1063. Check the results by calculating the total volume drained at 1.02 m with Eq. 28.18.

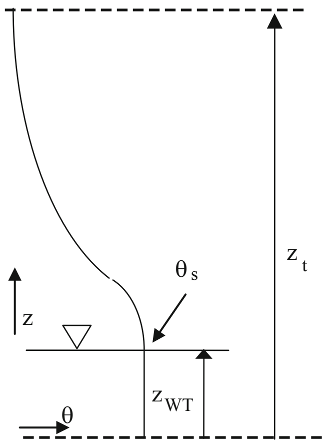


Fig. 28.9 Water profile above datum for soil with a water table for Van Genuchten model

$$d_{d(1.02\ \ meter)} = 0.261\left(2-, 1.28, -\frac{0.3^{1.27}}{-1.27+1}\left((1.02)^{-1.27+1} - 0.3^{-1.27+1}\right)\right) = 0.1063\ \ m$$

Equation 28.17 can be modified to find the total depth of water in the soil profile: this technique is used in the *WINDS* model for the lower layers that are in equilibrium with the water table.

$$d_{total} = \theta_s(z_{hb}) + \int_{hba}^{Zt-ZWT} \theta(z)dz \tag{28.20}$$

where

d_{total} = total depth of water in the soil profile, m,

Integrate Eq. 28.20 to find the total depth of water in the soil profile.

$$d_{total} = \theta_s(z_{hb}) + \theta_r(z_t - z_{hb}) + \frac{(\theta_s - \theta_r)h_{ba}^{\lambda}}{-\lambda+1}\left((z_t - z_{WT})^{-\lambda+1} - (h_{ba})^{-\lambda+1}\right) \tag{28.21}$$

The depth of water in a soil profile with a water table can also be found with the van Genuchten equation (Fig. 28.9). Write Eq. 28.20 in terms of van Genuchten parameters as a function of elevation z (matric potential). Substitute z, elevation above the water table, for – h_c. This is the water table algorithm in the *WINDS* model.

$$d_{total} = \theta_s(z_{WT}) + \int_0^{Zt-Zwt} \theta(z)dz \qquad \theta = \theta_r + (\theta_s - \theta_r)[1 + (-\alpha\ \ {*h_c})^n]^{-m} \tag{28.22}$$

$$d_{total} = \theta_s(z_{WT}) + \int_0^{Zt-Zwt} \theta_r + (\theta_s - \theta_r)[1 + (\alpha\ \ {*z})^n]^{-m}dz \tag{28.23}$$

The right side of the integrand in Eq. 28.23 is the Gauss hypergeometric function, $_2F_1(a, b, c, w)$ where $a = 1/n$, $b = 1 - 1/n$, $c = 1 + 1/n$, and $w = -((\alpha\ (z_t - z_{WT}))^n$. Note that w is substituted for z because z represents elevation.

$$\int_0^{Zt-Zwt} (1 + (\alpha\ \ z)^n)^{(1/n-1)} = z\ \ *\ {}_2F_1\left(\frac{1}{n},\ 1-\frac{1}{n};\ 1+\frac{1}{n};\ -((\alpha\ \ (z_t - z_{WT}))^n)\right) \tag{28.24}$$

The hypergeometric function (Equation 28.24) does not converge if the fourth term is greater than 1. In these water table calculations, the fourth term is greater than 1 if $(z_t - z_{WT})$ is large and/or α is large. A transformation can be made such that w in the hypergeometric function is always less than 1.

$$_2F_1(a, b, c, w) = (1-w)^{-b}{}_2F_1(c-a,\ \ b;\ \ c, w/(w-1))$$
$$\text{where } w = -((\alpha(z_t - z_{WT}))^n)$$

With the transformation, the integral of Eq. 28.23 is

$$d_{total} = \theta_s(z_{WT}) + \theta_r(z_t - z_{WT}) + (\theta_s - \theta_r)(z_t - z_{WT})(1-w)^{1/n-1}\ {}_2F_1\left(1,\ \ 1-\frac{1}{n};\ \ 1+\frac{1}{n},\ \ \frac{w}{w-1}\right) \tag{28.25}$$

It is not possible to find the derivative of Eq. 28.25 in order to find the incremental specific yield. Thus, the incremental specific yield must be found by solving for d_{total} at two different water table elevations, and then calculating the incremental specific yield with the two points as shown in Example 28.9.

Example 28.9 Calculate the incremental specific yield at a water table depth below the soil surface of 0.45. Van Genuchten parameters are $n = 1.75$, $\alpha = 0.078$, $z_t = 0.55$ m, $z_{WT} = 0.1$ m, $\theta_s = 0.45$, and $\theta_r = 0.08$ (Figure 28.10).

Because the van Genuchten parameters are listed in terms of cm, w must be calculated in units of cm so multiply by 100.

$$w = -((\alpha(z_t - z_{WT}))^n)$$
$$w = -\left((0.078{*}100{*}(0.55 - 0.1))^{1.75}\right) = -9.35$$

The hypergeometric function is passed the following parameters

$$\text{hg}(1, 1 - 1/1.75, 1 + 1/1.75,\ -9.35/(-9.35-1))$$

The hypergeometric function returns a value of 1.67

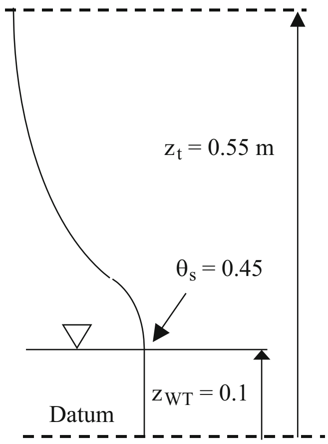


Fig. 28.10 Parameters for solution of Example 28.9

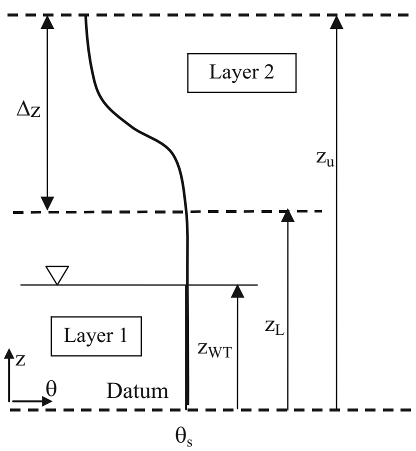


Fig. 28.11 Parameters for calculation of depth of water in a soil layer

The total depth of water in the soil is calculated as follows.

$$d_{total} = \theta_s(z_{WT}) + \theta_r(z_t - z_{WT}) + (\theta_s - \theta_r)(z_t - z_{WT}) (1-w)^{1/n-1} {}_2F_1\left(1,\ 1-\frac{1}{n};\ 1+\frac{1}{n},\ \frac{w}{w-1}\right)$$

$$d_{total} = 0.45(0.1) + 0.08(0.55 - 0.1) + (0.45 - 0.08)(0.55 - 0.1)(1 - (-9.35))^{(1/1.75-1)} * 1.67 = 0.18\ m$$

Calculate the incremental specific yield by also calculating the total depth of water in the soil profile at $z_{WT} = 0.09$ m → 0.1821 m. SY = (0.1844–0.1821)/(0.1–0.09) = 0.23 → 23 %.

The equations can be modified to find the depth of water within an individual layer that contains the water table (Layer 1 in Fig. 28.11).

The water depth below the water table is added to the water depth above the water table, where z_L and z_u are not the same as the z_L and z_u shown in Fig. 28.11 but refer to the lower and upper limits of Layer 1.

$$d_{layer} = \theta_s(z_{WT} - z_L) + \int_0^{Zu-Zwt} \theta(z)dz \tag{28.26}$$

where

z_u = elevation of upper boundary of layer, m,
z_L = elevation of lower boundary of layer, m.
d_{layer} = depth of water within the layer, m.

Integrate Eq. 28.26.

$$d_{layer} = \theta_s(z_{WT} - z_L) + \theta_r(z_u - z_{WT}) + (\theta_s - \theta_r) (z_u - z_{WT})(1-w)^{1/n=1} {}_2F_1\left(1,\ 1-\frac{1}{n};\ 1+\frac{1}{n},\ \frac{w}{w-1}\right) \tag{28.27}$$

where

$$w = -\left(\left(\alpha\left(z_u - z_{WT}\right)\right)^n\right)$$

If a layer is completely above the water table such as Layer 2 in Fig. 28.11, and the layer is in hydraulic equilibrium with the water table, then the depth of water in the layer is found as follows.

$$d_{layer} = \int_{Z_L - Z_{WT}}^{Zu - Z_{WT}} \theta(z)dz$$

$$d_{layer} = \theta_r(z_u - z_L) + (\theta_s - \theta_r)(z_u - z_{WT})(1-w_u)^{1/n-1} {}_2F_1\left(1,\ 1-\frac{1}{n};\ 1+\frac{1}{n},\ \frac{w_u}{w_u-1}\right) - (\theta_s - \theta_r)(z_L - z_{WT})(1-w_L)^{1/n-1} {}_2F_1\left(1,\ 1-\frac{1}{n};\ 1+\frac{1}{n},\ \frac{w_L}{w_L-1}\right) \tag{28.28}$$

where

$$w_u = -\left(\left(\alpha\left(z_u - z_{WT}\right)\right)^n\right)$$
$$w_L = -\left(\left(\alpha\left(z_L - z_{WT}\right)\right)^n\right)$$

The downward rate of seepage from the soil profile (from Layer 1 to below Layer 1) is controlled by the permeability of the confining layer below or by the resistance to flow to a subsurface drain system and by the depth of the water table. The drainage rate is a function of this permeability, or resistance, and the elevation of the water table.

$$d_{drainage} = f(K_d, z_{WT}, \Delta t) \tag{28.29}$$

where

K_d = conductivity of the drainage system, m/day,
Δt = time, days,
$d_{drainage}$ = depth of drainage below the control volume, m.

The equation for downward flux may be as simple as the following equation or it may be very complex and depend on the geometry of the subsurface flow to drains.

$$d_{drainage} = K_d * z_{WT} * \Delta t \tag{28.30}$$

One of the challenges with this drainage model is that the model crashes if the theoretical framework is inaccurate. This mass transfer between layers must be calculated with algorithms that have some level of theoretical validity. If only a single layer was connected to the water table, then the specific yield values would be unrealistic. This is complicated by the fact that the water table rises and falls; thus, the boundary between the hydraulically connected lower layers and hydraulically disconnected upper layers also rises and falls. The following example demonstrates how the *WINDS* model transfers water from the upper soil region to a lower region that is in equilibrium with the water table.

Water content in those layers that are the in equilibrium with the water table are all adjusted according to Eqs. 28.26 and 28.27 after each time step. Those layers that are not connected are only adjusted based on the energy difference and conductivity between layers. This technique prevents the upward movement of water from the water table connected layers into layers that are dried by evapotranspiration. The user can specify the approximate number layers in equilibrium with the water table by adjusting the fraction of saturation that causes connection of layers to the water table.

Example 28.10 A soil has four layers, 0.5 m thickness (Fig. 28.12). The soil parameters are the same as in Example 28.4 except that saturated water content is 40 % instead of 45 %. The rate of subsurface drainage below layer 1 is a function of the water table elevation, z_{WT}: $d_{drainage} = z_{WT} *$ 0.01 m/day * 1 day. The initial elevation of the water table is 1.5 m, and all layers are in equilibrium with the water table at the beginning of the first day (Fig. 28.13). Afterward, the saturation fraction required for equilibrium is 0.68. There is no irrigation or evapotranspiration, Run the *WINDS* model for 100 days. The information for this simulation is in the #3 position in the *Crop_data* worksheet. The drainage parameters are in Fig. 28.13. They are also in cells B231: B248 in the *Active_data* sheet.

The elevation of the water table vs. time is shown with the red line in Fig. 28.14. The equilibrium layer (blue line) descends with the water table. The change in water content vs. time is shown in Fig. 28.15. A comparison of Figs. 28.14 and 28.15 reveals that once layers are not in equilibrium with the water table, the rate of water content change decreases dramatically. The number of layers in equilibrium with the water table would be changed by changing the fraction of saturation number (increase from 0.68) in the *Drainage* dialog box (Fig. 28.13).

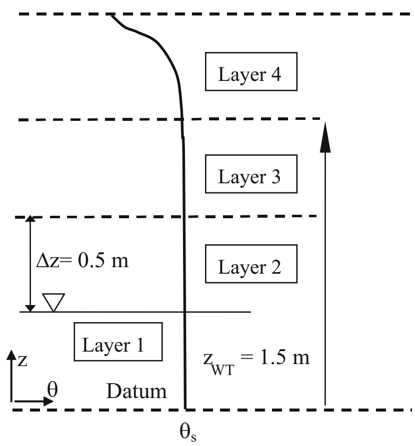


Fig. 28.12 Geometry for Example 28.10

The mass balance (Fig. 28.16) shows that mass is conserved. The change in water depth in the soil profile corresponds with the amount of water lost to drainage. In addition, the drainage rate is as expected (15 mm/day at 1.5 m water table elevation, etc...). The smoothness of curve is the result of including several layers in the water table change calculation. Otherwise, specific yield fluctuates with proximity to layer boundaries. Seepage (infiltration) below each layer vs. time is shown in Fig. 28.17. All layers with saturated lower boundaries have the same seepage rate. Once the upper end of a layer begins to become unsaturated, the amount of water entering the upper end of the layer does not match the amount of water exiting the lower end of the layer.

Example 28.11 A soil has four layers, 0.5 m thickness (Fig. 28.18). Calculate the initial and final water contents, and the change in water table depth over one day for the soil parameters in Example 28.4; however, change the saturated water content is 40 % instead of 45 %. The rate of subsurface drainage below layer 1 is a function of the water table elevation, z_{WT}: $d_{drainage} = z_{WT}$ * 0.01 m/day * days). The initial elevation of the water table is 0.5 m. Let all layers be in equilibrium with the water table for the initial distribution of water, but only the lower two layers are in equilibrium with the water table afterwards. Evapotranspiration removes 0.5 cm/day from each of layers 3 and 4. Compare calculations with the WINDS model. The information for this simulation is in the #4 position in the *Crop_data* worksheet.

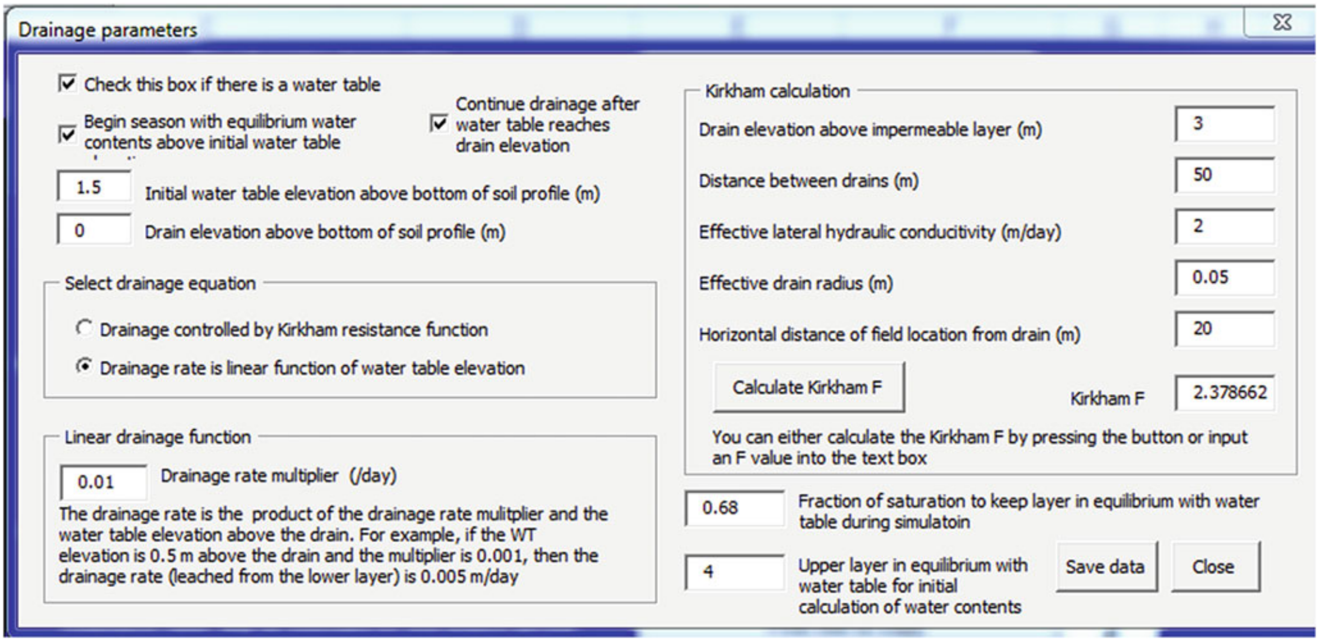


Fig. 28.13 *Drainage* dialog box for Example 28.10

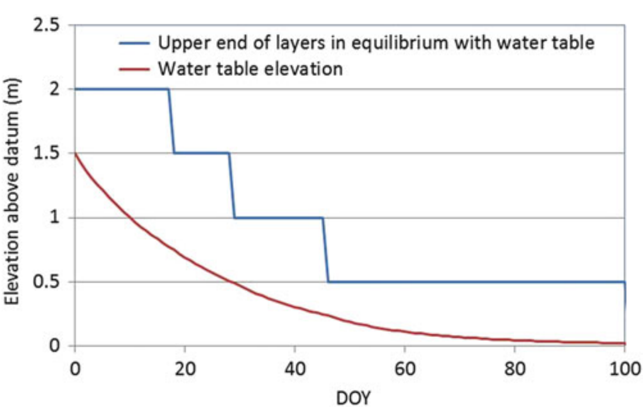


Fig. 28.14 Position of water table and cells in equilibrium with the water table versus time for Example 28.10

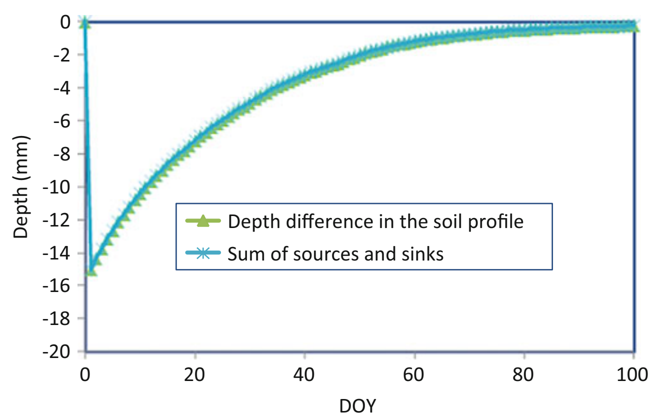


Fig. 28.16 Volume balance for Example 28.10

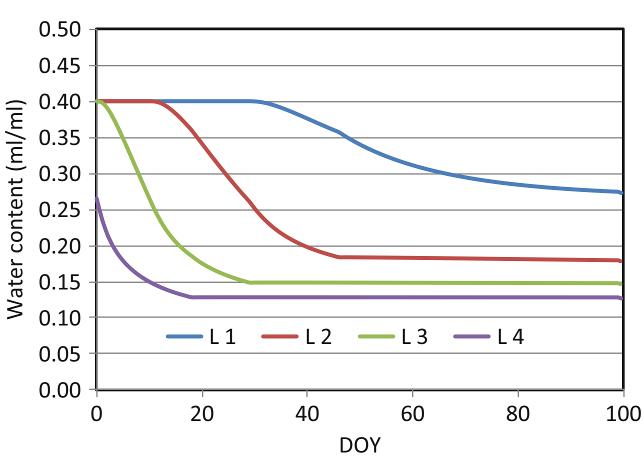


Fig. 28.15 Change in water content versus time for Example 28.10

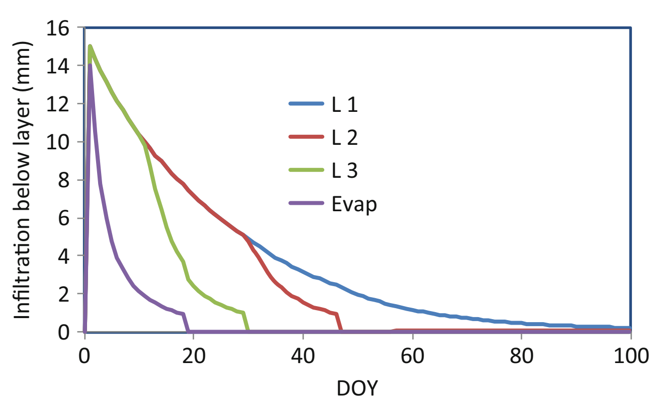


Fig. 28.17 Seepage below each layer for Example 28.10

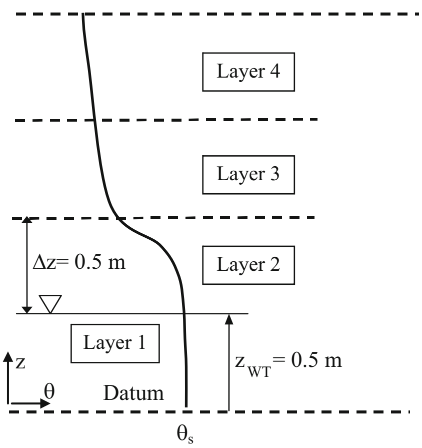


Fig. 28.18 Geometry for Example 28.11

In order to run this example, the Kcb values are set to 1.0 in cells B13:B15 in the *Active_data* worksheet, and the No_ET value in cell I35 in the *Main* worksheet must be set to False, and cells C2:C3 in the *ET_fractions* worksheet must be set to 0.5 to extract water from the upper two layers in which the root zone occupies 2 layers. Set the root zone depth = 0.99 m so that WINDS does not extract water from layer 2, as it would if the roots touched layer 2 (1.0 m).

The water table is at the upper boundary of layer 1 so layer 1 is saturated (water content = 40 %).

$$\mathrm{d_{layer}} = 0.4\ \mathrm{m/m}(0.5\ \mathrm{m}) = 0.2\ \mathrm{m}$$

The average water content in layer 2 can be found with Eq. 28.28 with the water table elevation $z_{WT} = z_L$ for layer 2 (layer 2 is in equilibrium with the water table).

$$\begin{aligned} d_{layer} &= \theta_s(z_{WT} - z_L) + \theta_r(z_u - z_{WT}) \\ &\quad + (\theta_s - \theta_r)(z_u - z_{WT})(1-w)^{1/n=1} \\ &\quad {}_2F_1\left(1,\ 1-\frac{1}{n};\ 1+\frac{1}{n},\ \frac{w}{w-1}\right) \\ \mathrm{w} &= -((\alpha(z_u - z_{WT}))^n) = -\left((0.078*100(0.5))^{1.75}\right) \\ &= -10.823 \\ d_{layer} &= 0.4(0) + 0.067(0.5) + (0.33)(0.5) \\ &\quad (1-(-10.823))^{1/1.75=1} \\ &\quad {}_2F_1\left(1,\ 1-\frac{1}{1.75};\ 1+\frac{1}{1.75},\ \frac{-10.823}{-10.823-1}\right) \\ d_{layer} &= 0.4(0) + 0.067(0.5) + (0.333)(0.5)(0.347)(1.715) \\ &= 0.1325\ \mathrm{m} \\ \theta_2 &= \mathrm{d_{layer}}/\Delta z = 0.1325\ \mathrm{m}/0.5\ \mathrm{m} = 0.265 = 26.5\%. \end{aligned}$$

The average water content in layer 3 is found with Eq. 28.29.

$$\mathrm{w_u} = -((\alpha(z_u - z_{WT}))^n) = -\left((0.078*100(1.5-0.5))^{1.75}\right) = -36.405$$

$$\mathrm{w_L} = -((\alpha(z_L - z_{WT}))^n) = -\left((0.078*100(1.0-0.5))^{1.75}\right) = -10.823$$

$$\begin{aligned} d_{layer} &= \theta_r(z_u - z_L) + (\theta_s - \theta_r)(z_u - z_{WT})(1-w_u)^{1/n-1} \\ &\quad {}_2F_1\left(1,\ 1-\frac{1}{n};\ 1+\frac{1}{n},\ \frac{w_u}{w_u-1}\right) \\ &\quad - (\theta_s - \theta_r)(z_L - z_{WT})(1-w_L)^{1/n-1} \\ &\quad {}_2F_1\left(1,\ 1-\frac{1}{n};\ 1+\frac{1}{n},\ \frac{w_L}{w_L-1}\right) \end{aligned}$$

$$\begin{aligned} d_{layer} &= 0.067(0.5) + (0.333)(1.0)(1-(-36.405))^{1/1.75-1}\ (1.99) \\ &\quad -(0.333)(0.5)(1-(-10.823))^{1/1.75-1}\ (1.715) = 0.0747\ m \end{aligned}$$

$$\theta_3 = \mathrm{d_{layer}}/\Delta z = 0.07477\ \mathrm{m}/0.5\ \mathrm{m} = 0.1495 = 15.0\%.$$

The average water content in layer 4 is also found with Eq. 28.29 and is equal to 11.5 %.

The total depth of water in the soil profile is (0.4 + 0.265 + 0.150 + 0.115) * 0.5 = 0.465 m.

Equation 28.23 can be used to verify that the total depth of water in the entire soil profile is equal to the sum of the depths in each soil layer.

$$\begin{aligned} d_{total} &= \theta_s(z_{WT}) + \theta_r(z_t - z_{WT}) + (\theta_s - \theta_r) \\ &\quad (z_t - z_{WT})(1-w)^{1/n=1} \\ &\quad {}_2F_1\left(1,\ 1-\frac{1}{n};\ 1+\frac{1}{n},\ \frac{w}{w-1}\right) \\ \mathrm{w} &= -((a(z_t - z_{WT}))^n) = -\left((0.078*100(1.5))^{1.75}\right) \\ &= -74.016 \\ d_{total} &= 0.4(0.5) + 0.067(1.5) + (0.333)(1.5) \\ &\quad (1-(-74.016))^{1/1.75=1}\ (2.096) = 0.465\ \mathrm{m} \end{aligned}$$

Next, find the water contents after the first day. Calculate the water lost to drainage.

$$\begin{aligned} \mathrm{d_{drainage}} &= z_{WT}*0.02\ \mathrm{m/day}*1\ \mathrm{days} = 0.5*0.02 \\ &= 0.01\ \mathrm{m/day} \end{aligned}$$

The water table movement is based on the specific yield. The specific yield at any water table elevation can be found by calculating the total depth of water in the soil profile at water table elevations 0.05 m above and 0.05 m below the original water table elevation. The depths must be calculated for only the lower 2 layers since only those layers are in equilibrium with the water table in this example.

$$\begin{aligned} d_{profile}(z_{wt} &= 0.495) = 0.33148\ m \\ \mathrm{w} &= -\left((\alpha(z_t - z_{WT}))^n\right) = -\left((0.078*100(1-0.495))^{1.75}\right) \\ &= -11.0135 \\ d_{total} &= 0.4(0.495) + 0.067(0.505) + (0.333)(0.505) \\ &\quad (1-(-11.0135))^{1/1.75-1}\ (1.72) = 0.33148\ m \\ \mathrm{d_{profile}}(z_{wt} &= 0.505) = 0.33367\ \mathrm{m} \end{aligned}$$

The specific yield is then calculated with low and high water contents

$$SY = (0.33367 - 0.33148)/(0.505 - 0.495) = 0.2188$$

The new water table elevation is calculated as follows (j represents the previous time step).

$$z_{wt}(j) = z_{wt}(j-1) - \text{drainage}/SY = 0.5 - 0.01/0.2188 = 0.454\ \text{m}$$

The *WINDS* model uses the *Find_wt* function to find the height of the water table. The calculated value is 0.455 m. *Find_wt* uses an iterative procedure that moves the water table until the total depth of water in the equilibrium layers is equal to the depth of water in the cells.

The new water content in layer 1 is calculated with the d_layer_WT function in the *WINDS* model.

$$\theta_1 = \text{d\_layer\_WT}(ts, tr, zu, zl, zwt(j), alpha, n)/dz$$
$$\theta_1 = \text{d\_layer\_WT}(0.4, 0.067, 0.5, 0, 0.455, 0.078, 1.75)/0.5 = 0.3999\ \text{ml/ml}$$

The new water content in layer 2 is calculated with the d_layer_Eq function in the *WINDS* model

$$\theta_2 = \text{d\_layer\_Eq}(ts, tr, zu, zl, zwt(j), alpha, n)/dz$$
$$\theta_2 = \text{d\_layer\_Eq}(0.4, 0.067, 1, 0.5, 0.455, 0.078, 1.75)/0.5 = 0.246\ \text{ml/ml}$$

The new water contents in layers 3 and 4 are calculated based on the amount of water removed by evapotranspiration since they are no longer in equilibrium with the water table. There is no flux from layer to layer because all layers are in equilibrium at the beginning of the time step. Thus, the only water loss is due to evapotranspiration.

$$\theta_3 = 0.1498 - 0.005\ \text{m}/0.5\ \text{m} = 0.1398\ \text{ml/ml}$$
$$\theta_4 = 0.115 - 0.005\ \text{m}/0.5\ \text{m} = 0.105\ \text{ml/ml}$$

The flux between the upper layers is negligible (order 10^{-6} m/d) and does not significantly change the solution. Likewise, the flux from layer 3 into layer 2 is negligible.

The next example demonstrates the fluctuation of the water table with evapotranspiration and weekly irrigations. The *WINDS* model move the equilibrium layer to the top of the soil profile or to the maximum specified layer when water content increases in the upper layers beyond field capacity.

Example 28.12 A soil has four layers, 0.5 m thickness (Fig. 28.19). Soil parameters are the same as the previous two examples; however, change the permanent wilting point to 14 %. The rate of subsurface drainage below layer 1 is a function of the water table elevation, z_{WT}: $d_{drainage} = z_{WT} *$ 0.01 m/day *days). The initial elevation of the water table is 0.5 m. The fraction of saturation for the equilibrium max placement is 0.75. Evapotranspiration is 1 cm/day, with the nominal fraction removed from each of layers 3 and 4 equal 50 %. However, the fraction can be adjusted based on water content, and the total evapotranspiration can be reduced based on overall soil water content in layers 3 and 4. Set the root zone depth = 0.9 m for the entire season. The information for this simulation is in the #5 position in the *Crop_data* worksheet. Apply 70 mm/week with 15 % extra as a leaching fraction. Run *WINDS* for 100 days.

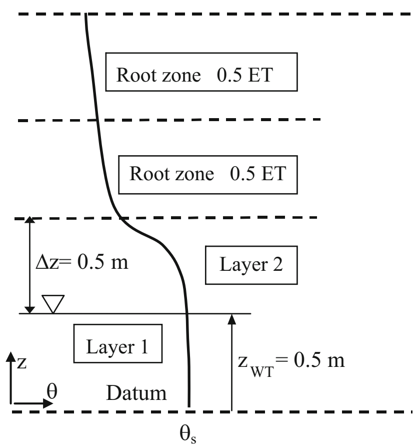


Fig. 28.19 Geometry for Example 28.12

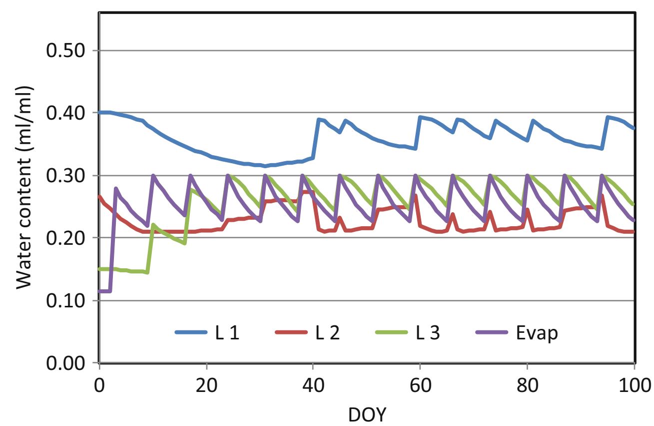


Fig. 28.20 Water content versus time for Example 28.12

Cells I43:I44 on the Main worksheet are set to True and 2, respectively, limiting the equilibrium zone to the lower two layers. Water content changes are shown in Fig. 28.20. The water table (red line) and upper limit of the equilibrium zone (blue line) are shown in Fig. 28.21.

The *WINDS* model has an algorithm, described in Chap. 29, to limit evapotranspiration if there is not sufficient water in the root zone. For example, both layers 3 and 4 begin the season at less than permanent wilting point (Fig. 28.20); thus, there is no evapotranspiration on the first two days (Fig. 28.22). The potential ET in this example

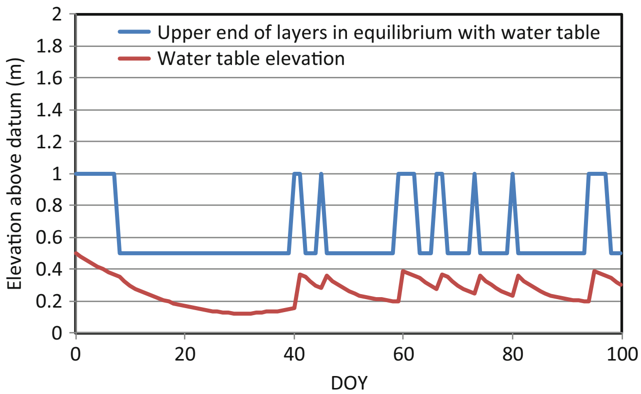


Fig. 28.21 Water table and equilibrium zone versus time for Example 28.12

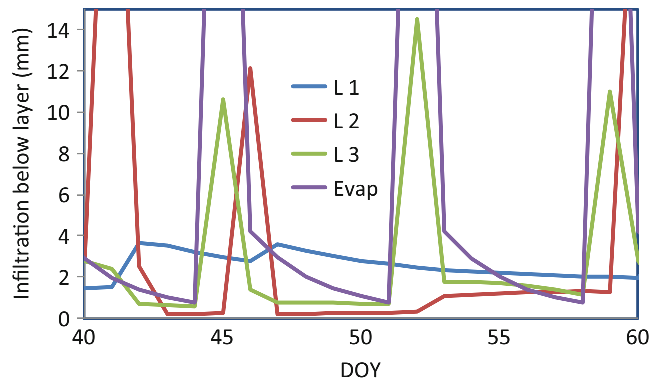


Fig. 28.24 Downward seepage between layers from DOY 40–60 for Example 28.12

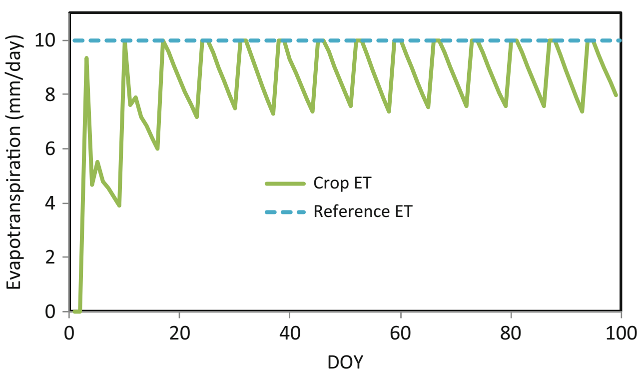


Fig. 28.22 Evapotranspiration versus time for Example 28.12

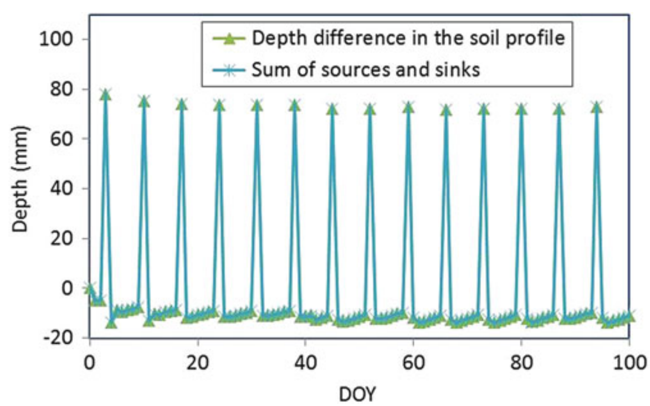


Fig. 28.25 Volume balance for Example 28.12

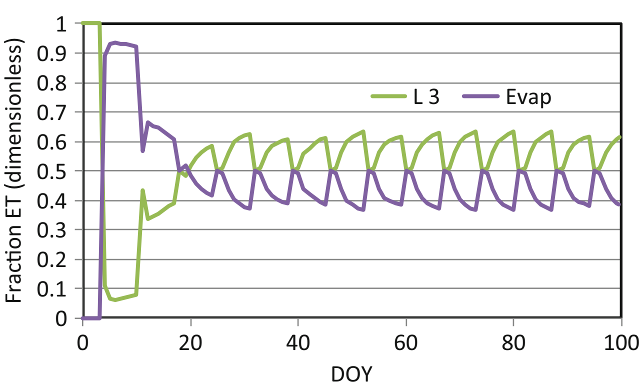


Fig. 28.23 Fractions of total ET removed from each layer for Example 28.12

is a constant 10 mm/day; however, the actual evapotranspiration (ET) is less than 10 mm/day on days (Fig. 28.22) when water content is below the MAD value.

The WINDS model shifts the majority of the ET to the root zone layers with the highest water content. Although the specified fraction of ET is 50 % each from layers 3 and 4, the actual fractions change with water content as shown in Fig. 28.23.

The reason that more water is extracted from layer 3 is that there is a slow downward seepage of water from layer 4 to layer 3 after the irrigation events. In Fig. 28.24, the top of the irrigation infiltration peaks are cut off in order to focus in the infiltration between layers during the interval between irrigation events. The volume balance shows agreement between change in storage and sum of sources and sinks (Fig. 28.25).

Upward Flux

A high water table can supply water to the plant root zone if the water table is close to the bottom of the root zone (Fig. 28.26). The phenomenon of upward flux from the water table to the bottom of the root zone can be modeled with an analytic solution derived by Anat.

$$v = \frac{K_s\left[h_b + \frac{1.89}{\eta^2+1}h_b\right]^\eta}{y^\eta} \tag{28.31}$$

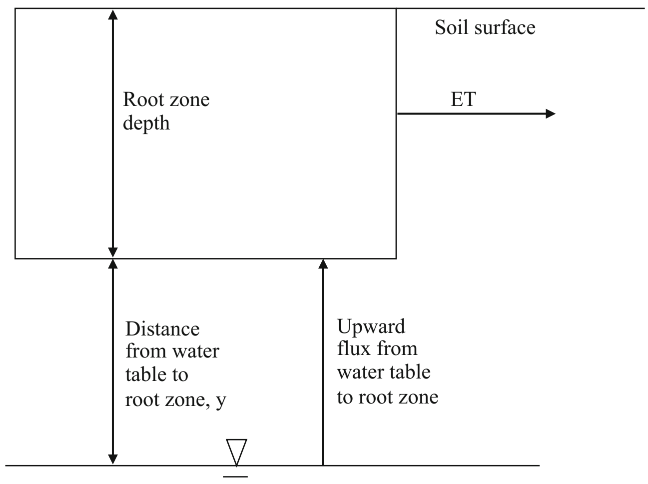


Fig. 28.26 Upward flux from water table to root zone and evapotranspiration

where

y = distance from the water table to the bottom of the root zone, m.
v = Darcy velocity, m/day.

Upward flux from the water table to the root zone or to the soil surface can also be calculated with a discretized solution of Darcy's law, which is rearranged to solve for change in matric potential (Eq. 28.32) from layer to layer (Fig. 28.27). The solution starts at the top layer with an assumed matric potential. The procedure continues until the matric potential equals zero, which would be the depth of the water table The final solution is not sensitive to the magnitude of the assumed matric potential in the upper layer because hydraulic conductivity becomes very low at very negative matric potentials.

$$\begin{aligned}
v &= -K_{e(j\&j-1)}\frac{H_j - H_{j-1}}{dz} = -K_{e(j\&j-1)}\frac{h_j + z_j - h_{j-1} - z_{j-1}}{dz}\\
&= -K_{e(j\&j-1)}\frac{h_j - h_{j-1} + \Delta z}{\Delta z} = K_{e(j\&j-1)}\frac{h_{j-1} - h_j}{\Delta z} - K_{e(j\&j-1)}\\
h_{j-1} &= h_j + \frac{v\ \Delta z}{K_{e(j\&j-1)}} + \Delta z
\end{aligned} \tag{28.32}$$

Example 28.13 The potential evapotranspiration is 0.01 m/day. Calculate the maximum distance between the water table and the bottom of the root zone for subirrigation in Wagram loamy sand. Compare the discretized solution (Eq. 28.32) to Anat's Eq. (28.31). Assume that saturated hydraulic conductivity of Wagram loamy sand is 0.144 m/day and that matric potential at the bottom of the soil root zone is −3.3 m (field capacity). Let the layers have a height (dz) of 0.01 m.

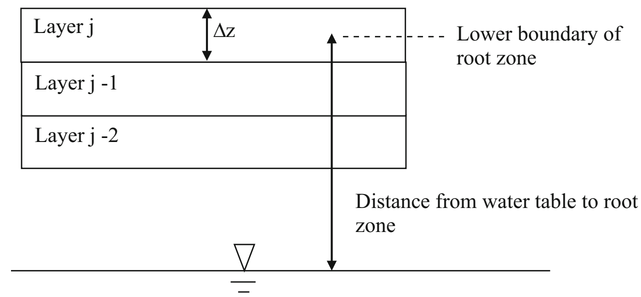


Fig. 28.27 Layer numbering for upward flux calculation

Start by finding the conductivity of the upper layer, j, assuming that matric potential is −3.3 m. Next calculate the matric potential of layer j − 1. Then, based on the new calculated j − 1 matric potential, calculate a new conductivity of layer j − 1, and so on.

$$\begin{aligned}
K_j &= K_s K_{rw} = K_S\left(\frac{h_b}{h_j}\right)^{2+3\lambda} = 0.144\left(\frac{-0.3}{-3.3}\right)^{2+3(1.27)}\\
&= 1.28{*}10^{-7}\ m/day\\
h_{j-1} &= h_j + \frac{v\ dz}{K_{e(j\&j-1)}} + dz\\
&= -3.3\ m + \frac{(0.01\ m/day)(0.01\ m)}{1.28{*}10^{-7}\ m/day} + 0.01\ m\\
&= 776\ m
\end{aligned}$$

The low hydraulic conductivity caused to solution to blow up. Try a less negative matric potential ($h_j = -0.7$ m) in the upper layer so that the solution does not blow up.

$$\begin{aligned}
K_j &= K_s K_{rw} = K_S\left(\frac{h_b}{h_j}\right)^{2+3\lambda} = 0.144\left(\frac{-0.3}{-0.7}\right)^{2+3(1.27)}\\
&= 0.001\ m/day\\
h_{j-1} &= -0.7\ m + \frac{(0.01\ m/day)(0.01\ m)}{0.001\ m/day} + 0.01\ m\\
&= -0.594\ m
\end{aligned}$$

Now, solve for matric potential in layer j-2.

$$\begin{aligned}
K_j &= K_s K_{rw} = K_S\left(\frac{h_b}{h_j}\right)^{2+3\lambda} = 0.144\left(\frac{-0.3}{-0.594}\right)^{2+3(1.27)}\\
&= 0.0027\ m/day\\
h_{j-1} &= -0.594\ m + \frac{(0.01\ m/day)(0.01\ m)}{0.0027\ m/day} + 0.01\ m\\
&= -0.548\ m
\end{aligned}$$

Continue to solve until $h_c = 0$ (Fig. 28.28). The matric potential reaches zero at a depth of 0.47 m below the bottom of the root zone. Thus, if the root zone is less than 0.47 m above the water table, then the crop will receive adequate water from the water table.

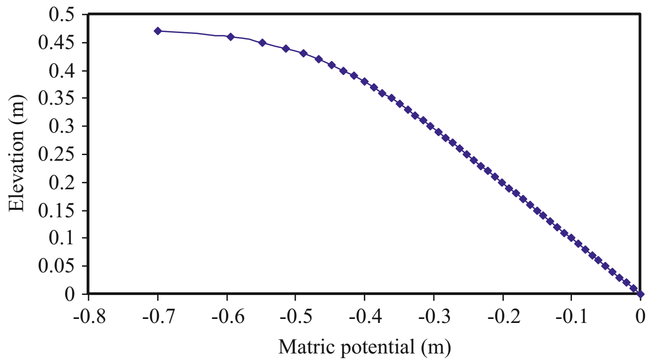


Fig. 28.28 Matric potential versus elevation above the water table

Find the upward flux with Anat's equation with a water table at a depth of 0.47 m below the root zone.

$$v = \frac{K_s\left[h_b + \frac{1.89\ \ h_b}{\eta^2+1}\right]^{\eta}}{y^{\eta}} = \frac{0.144\left[0.3 + \frac{1.89\ \ *0.3}{5.81^2+1}\right]^{5.81}}{0.47^{5.81}}$$

$$= 0.0144\ \ m/day$$

Anat's equation calculates an upward flux rate that is 44 % greater than the finite difference solution. However, at a slightly greater distance from the water table to the root zone, 0.52 m, the flux predicted by Anat's equation is 0.008 m/day, which is 20 % lower than the finite difference calculation.

Questions

1. Derive Eq. 28.3 from Eq. 27.20
2. Derive a finite difference equation from Eq. 28.3 that has the same structure as Eq. 28.2.
3. Include osmotic potential as a function of soil water EC in Eq. 28.5. Leave only the change in water content ($\Delta\theta$) on the left side of the equation. Use Eq. 5.2 to calculate osmotic potential based on soil water EC.
4. For the three cells listed below, write the equation derived in question 3 in terms of cell 4. Calculate the change in water content in cell 4 with and without the influence of salinity. Use a one day time step.

   $z_3 = 1.0$ m $\quad h_3 = -2$ m $\quad EC_{w-3} = 4$ dS/m
   $z_4 = 1.4$ m $\quad h_4 = -3$ m $\quad EC_{w-4} = 3$ dS/m
   $z_5 = 1.8$ m $\quad h_5 = -5$ m $\quad EC_{w-5} = 6$ dS/m

   The effective hydraulic conductivity between cells 3 and 4 is 0.01 m/day
   The effective hydraulic conductivity between cells 4 and 5 is 0.001 m/day
5. Calculate the water content, effective water content, effective saturation and hydraulic conductivity of Wagram loamy sand at −1 bar matric potential.
6. Calculate the Darcy velocity between two points in unsaturated Wagram loamy sand with matric potential values of −1,000 cm (point 1) and −2,000 cm (point 2) and elevations of 10 cm (point 1) and 20 cm (point 2), respectively. Assume that the saturated hydraulic conductivity of Wagram loamy sand is 0.6 cm/hr. The distance between the two points is 50 cm. Use the Brooks Corey model for hydraulic conductivity. Use the geometric mean to calculate effective hydraulic conductivity.
7. Three layers numbered 1 through 3 from bottom to top have 0.5 m depth. Use the van Genuchten equations to calculate water contents after 1 day and 2 days. Initial water content in all layers is 37 %. $n = 1.31$, $\theta_r = 0.095$, $\theta_S = 0.41$, $\alpha = 0.019$, $L = 0.5$, and $K_0 = 6.24$ cm/day. There is no infiltration and no seepage of water below the control volume. Calculate effective conductivity between two layers with the geometric mean. Calculate water contents after the first day and after the second day. It is impossible for any layer to have greater than the saturated water content.
8. Repeat question 7 but allow water to drain below the lower layer. Assume that there is no water table and use Eq. 28.27 for layer 1.
9. Repeat question 7 but restrict the downward movement of water to 0.1 cm/day from layer 1 (as with subsurface drainage and a water table).
10. Thirty cm is removed from the soil as the water table is lowered from 2.5 m to 0.5 m. Calculate the incremental specific yield between 0.5 m and 2.5 m.
11. The water table drops from 1.0 to 1.5 m below the ground surface, and the specific yield of the soil is 10 %. Calculate the depth of water removed from the soil profile.
12. What do the right and left terms on the right side of Eq. 28.17 represent? What is the hatched area in Fig. 28.8? What does $\theta(z)$ represent in the Eq. 28.17? Why is the Brooks-Corey calculation of $\theta(z)$, which is based on matric potential substituted into the equation derivation? Explain why the upper limit of integration is $z_t - z_{WT}$ and the lower limit is h_{ba}.
13. Calculate the depth of water drained and specific yield for a water table that drops from the soil surface to 0.8 m above the datum in Wagram loamy sand. Place the datum 1.5 m below the soil surface.
14. If the question 13 water table dropped from 0.7 m below the surface to 0.8 m below the surface, then how much water would you expect would be drained from the soil during this 0.1 m drop in water table elevation?

Calculate the depth drained in two ways: use the specific yield that you calculated in question 13 and also calculate the actual depth drained at 0.8 m depth with Eq. 28.18. Then use the two drained depths to calculate the incremental specific yield between 0.7 and 0.8 m depth below the soil surface.

15. Use Eq. 28.19 to find the incremental specific yield at 0.75 m depth below the soil surface for Wagram loamy sand and compare to the incremental specific yield that was calculated in question 14.
16. Find the depth drained if the water table drops from the soil surface to 50 cm depth, from the soil surface to 100 cm depth, and from the soil surface to 150 cm depth. Use the Brooks-Corey Goldsboro sandy loam parameters that you calculated in question 27.22. Set the datum at 3 m. Find the specific yield associated with the water table drop from the surface to each depth
17. Derive Eq. 28.21 from Eq. 28.20
18. Use the integrator at http://integrals.wolfram.com/index.jsp to integrate the left hand side of Eq. 28.24.
19. Find the hypergeometric function page at the Wolfram website listed in question 18 and find the transformation shown between Eqs. 28.24 and 28.25. List the line number of the transform on the Wolfram page to show that you found the transform. Find the series solution to the hypergeometric function on the same page and list the line number. Explain how you would implement the series solution in a computer code or spreadsheet.
20. A soil has the following parameters: $n = 1.5$, $\alpha = 0.06$, $z_t = 2$ m, $z_{WT} = 1$ m, $\theta_s = 0.45$, and $\theta_r = 0.08$. Calculate the a, b, c, and w terms for the transformed Gauss hypergeomtric function as shown in Eq. 28.25. A function is included in the Van Genuchten Excel/VBA program called hg that calculates the hypergeometric series solution. Calculate the series solution with the function hg by calling it from a worksheet with the following: "=hg(a,b,c,w/(w-1))". Make sure to write z in units of cm in the calculation of w since α has units of 1/cm. Finally, calculate the total depth of water in the soil profile from the datum to the soil surface.
21. Redo question 20, but place the water table at 0.1 m above the datum. Calculate the specific yield if the water table drops from 1 to 0.1 m above the datum.
22. Find the average water content in two layers. The upper layer has lower and upper boundaries that are 1 and 1.2 m above the datum. The lower layer has lower and upper boundaries that are 0.8 and 1.0 m above the datum. Both layers are in hydraulic equilibrium with the water table. The matric potential, h_c, at the center of the upper layer is – 0.5 m. Use the Brooks-Corey parameters for Wagram loamy sand.
23. Calculate the average water content in a layer that has upper and lower limits that are 1.4 m and 1.2 m above the datum. Use the parameters listed in question 20.
24. A crop requires 0.006 m/day evapotranspiration. Calculate the maximum distance between the water table and the bottom of the root zone for sub irrigation without water stress. Use Wagram loamy sand parameters. Compare the discretized solution (Eq. 28.32) to Anat's Eq. (28.31). The saturated hydraulic conductivity of Wagram loamy sand is 0.144 m/day.
25. Repeat Example 28.3; however, use the parameters for sandy loam in the *WINDS* model. Make sure to change the initial water content in the Active Data page to field capacity (0.21) before copying the data to Crop_data. Make sure that the final DOY is set to 200 days (cell E3) in order to run the simulation for the entire period. Show the water content, matric potential, and matric potential + elevation graphs. The Matric potential graph is in the *Matric potential* worksheet. Discuss results.
26. Repeat Example 28.3; however, use the parameters for sand in the *WINDS* model. Make sure to change the initial water content in the Active Data page to field capacity (0.1) before copying the data to Crop_data. Show the water content, matric potential, and matric potential + elevation graphs. The Matric potential graph is in the *Matric potential* worksheet. Discuss results. Is there reason to believe that the field capacity estimate may be too high?
27. Repeat Example 28.10; however, use the parameters for sand in the *WINDS* model. Show the water table graph and the water content graph. Evaluate drainage rate multipliers 0.01 and 0.02. The multiplier can be changed in the *Drainage* form, which is accessed from the *Active Data* worksheet.
28. Repeat Example 28.10; however, use the parameters for clay in the *WINDS* model. Show the water table graph and the water content graph. Evaluate drainage rate multiplier 0.02. The multiplier can be changed in the *Drainage* form, which is accessed from the *Active Data* worksheet. Explain the results.
29. Repeat Example 28.12; however, use the parameters for sandy loam in the *WINDS* model. Increase the drainage rate multiplier to 0.03. Set the initial elevation of the water table at 0.2 m. Show the water content graph and the water table elevation graph.