Chapter 2
Rocket Flight

We now want to tackle the problem of solving the equation of rocket motion Eq. (1.1.7). As will be seen in Sect. 2.1, even for many simple cases it can be solved only by numerical methods. Since this is not the objective of this book, we will treat only those important cases that can be analyzed analytically. This will give rise to an important characteristic quantity the so-called "delta-v budget". Its relevance will be explored in Sect. 2.4.1.

Before turning to this, we will introduce some essential rocket mass definitions we will use in this chapter

m instantaneous total mass of the rocket
m_0 total launch mass of the rocket
m_f mass of the rocket at burnout(final mass)
m_p propellant mass of the rocket before launch or before a maneuver
m_s structural mass of the rocket
m_L payload mass

From this it is obvious that

$$\begin{aligned} m_0 &= m_p + m_s + m_L \\ m_f &= m_0 - m_p = m_s + m_L \end{aligned} \tag{2.0.1}$$

In the following calculations, the variable parameter m can often be interpreted as the instantaneous mass or the mass at burnout, so in most cases $m = m_f$ is valid.

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2.1 General Considerations

Before we come to the few very important cases that can be examined analytically, let us have a look on the general solution of a flight in an external field. This may be any relevant external field, but it is almost always the gravitational field. To do this, we separate the variables on the left side of the motion Eq. (1.1.7), and we get

$$d\boldsymbol{v} = \frac{\boldsymbol{F}_* + \boldsymbol{F}_{ext}}{m} dt \tag{2.1.1}$$

$\boldsymbol{F}_*$ and $\boldsymbol{F}_{ext}$ are generally dependent on time. For example, during ascent in the atmosphere, the pressure thrust continually varies according to Eq. (1.2.14) because of the changing atmospheric pressure and/or the mass flow rate, which in particular holds for solid propellant rockets. In addition, the thrust direction changes because of the so-called gimbaling, i.e., the steering of the nozzle to change flight direction. In all these cases, with a given $\boldsymbol{F}_*(t)$, $\boldsymbol{F}_{ext}(t)$, $\dot{m}_p(t)$, or

$$m(t) = m_0 - \int_0^t \dot{m}_p(t')dt' \tag{2.1.2}$$

respectively, one can calculate the resultant velocity change by explicit integration:

$$\Delta\boldsymbol{v}(t) = \int_0^t \frac{\boldsymbol{F}_* + \boldsymbol{F}_{ext}}{m} dt' = \int_{m_0}^m \frac{\boldsymbol{v}_*}{m} dm + \int_0^t \frac{\boldsymbol{F}_{ext}}{m} dt' \tag{2.1.3}$$

where we have assumed $t_0 = 0$ for the sake of simplicity. We have written the left side of the equation in terms of the new and characteristic quantity "delta-v budget"

$$\Delta\boldsymbol{v}(t) := \int_{\boldsymbol{v}_0}^{\boldsymbol{v}} d\boldsymbol{v} = \boldsymbol{v}(t) - \boldsymbol{v}_0 \quad \textbf{delta-}\boldsymbol{v}\textbf{ budget} \tag{2.1.4}$$

or just "delta-v" for short, which will turn out to be quite handy to describe spacecraft (S/C) maneuvers in space (see Sect. 2.4.1). It describes the total change of the rocket's velocity vector due to all forces acting on the S/C over the time t.

In order to determine the position of the S/C as a function of time, Eq. (2.1.3) needs to be integrated once more. For nearly every practical case these integrations need to be done by numerical methods. There is only one important case where both integrations can be performed fully analytically for an external force: the continuous tangential thrust maneuver under a gravitational force. This important case will be covered separately in Sect. 8.4.5. We now consider some other important specific limiting cases.

2.2 Rocket in Free Space

A limiting case occurs in free space when there are no external forces, $\boldsymbol{F}_{ext} = 0$. In addition, in free space thrust maneuvers typically take place with a $\boldsymbol{v}_*$, which is constant both in absolute value and in direction. This special but most common situation simplifies Eq. (2.1.3) considerably to

$$\Delta \boldsymbol{v} = \boldsymbol{v}_* \int_{m_0}^{m} \frac{dm}{m}$$

so that it can be integrated straight away

$$\Delta \boldsymbol{v} = -\boldsymbol{v}_* \cdot \ln \frac{m_0}{m} \quad @ \quad \boldsymbol{v}_* = const \tag{2.2.1}$$

with $\Delta \boldsymbol{v} = \boldsymbol{v} - \boldsymbol{v}_0$, (v_0, m_0) the initial, and (v, m) the final state of the S/C. Note that the velocity change is independent of thrust level, of the duration, or any time dependence of the thrust, and hence via Eq. (1.1.5) on any variation of the mass flow rate. So, in free space, for a delta-v any specific engine characteristics does not matter, or whether a velocity-change boost is carried out over a short or a long time period. All what matters is initial and final mass and a constant effective exhaust velocity. Note that during ascent through an atmosphere v_* varies somewhat along with the pressure thrust due to the decreasing ambient atmospheric pressure. Therefore, the condition $v_* = const$ is not strictly fulfilled in this case.

Keeping in mind that $\Delta \boldsymbol{v}$ is always strictly antiparallel to $\boldsymbol{v}_*$ (see Fig. 2.1), Eq. (2.2.1) can be rewritten as an absolute value equation

$$\boxed{\Delta v = v_* \ln \frac{m_0}{m}} \quad @ \quad v_* = const \quad \textbf{rocket equation} \text{ (single stage)} \tag{2.2.2}$$

which is also known as **Tsiolkovsky rocket equation** or *ideal rocket equation*. Observe the close relationship to the relativistic rocket equation Eq. (1.4.10). Next

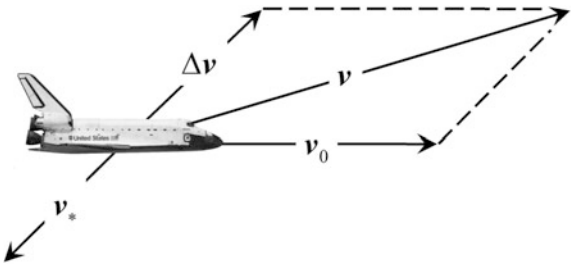

Fig. 2.1 Direction of delta-v for a maneuver in free space

to Eq. (1.1.5) this rocket equation is the most important equation in rocket flight. It can be written alternatively as

$$\frac{m}{m_0} = e^{-\Delta v/v_*} \tag{2.2.3}$$

or applying $m = m_f = m_0 - m_p$ at burnout

$$\frac{m_p}{m_0} = 1 - e^{-\Delta v/v_*} \quad @ \quad v_* = const \tag{2.2.4}$$

2.3 Rocket in a Gravitational Field

In this section we apply the equation of rocket motion (1.1.7) to evaluate the impact of an external field on the flight of a powered spacecraft. In outer space the external field is nearly always a gravitational field.

Only during ascent will a rocket also be subject to drag and sometimes also lift. The consideration of drag and lift complicates the treatment which is why we treat ascent motion only qualitatively in Chap. 6. Upon reentry there is also drag and lift, while the spacecraft is powerless. So, while the considerations in this section do not apply for reentry, the absence of thrust permits treating the motion in more detail, as done in Chap. 10.

Because the equation of rocket motion (1.1.7) is universal it also applies for a spacecraft without propulsion in a gravitational field as for powerless planet orbiting satellites or deep space probes. Although this is a frequent and hence extremely important case, it is somewhat difficult to treat in full detail. This is why we explore it separately in Chap. 7.

2.3.1 Impulsive Maneuvers

We first study the case when a spacecraft undergoes an impulsive maneuver in the presence of a gravitational field exerting the force F_G. Formally, "impulsive maneuver" means that for $t \to 0$ the total impulse $\lim_{t\to 0} \int_0^t F_* \, dt' = \Delta p > 0$ of the maneuver takes on a finite value. But since $\lim_{t\to 0} \int_0^t F_G \, dt' = 0$, this indirectly implies

$$F_* \gg F_G,$$

i.e., a minute but powerful thrust maneuver. We therefore find

For an impulsive maneuver external fields can be neglected and the so-called "delta-v" is determined solely by the thrust characteristics.

Note that, if the propellant mass expelled in an impulsive maneuver is not negligibly small compared to the total rocket mass, the delta-v of the maneuver is not

$$\Delta v = \int_0^t \frac{F_*}{m} dt' \neq \frac{F_*}{m} \int_0^t dt' = \frac{F_* t}{m} = \frac{\dot{m} v_* t}{m} = \frac{m_p v_* t}{tm} = v_* \frac{m_p}{m} = v_* \left(\frac{m_0}{m} - 1 \right)$$

as assumed quite frequently. Rather for a constant thrust (equals constant mass flow rate at $v_* = const$)

$$\Delta v = \int_0^t \frac{F_*}{m} dt' = \int_{m_0}^{m_f} \frac{v_*}{m} dm = v_* \int_{m_0}^{m_f} \frac{dm}{m} = v_* \cdot \ln \frac{m_0}{m}$$

i.e., the rocket Eq. (2.2.2). Impulsive maneuvers are of high relevance for orbit transfers. This is why we will investigate their effects on orbits in more detail in Chap. 8.

2.3.2 Brief Thrust

If the thrust $\boldsymbol{F}_* = -\dot{m}_p \boldsymbol{v}_*$ is constant, i.e., $\dot{m}_p = const$ and $\boldsymbol{v}_* = const$, but if $F_* \gg F_G$ does not apply, we have to take into account the effect of the gravitational field during the maneuver. From Eq. (2.1.3) we find

$$\Delta \boldsymbol{v}(t) = \boldsymbol{v}_* \int_{m_0}^{m} \frac{dm}{m} + \int_{m_0}^{m} \frac{\boldsymbol{F}_G}{\dot{m}} \frac{dm}{m}$$

Often the thrust maneuver is short compared to any variation of the gravitational field, which implies $\boldsymbol{F}_G = const$. In this case

$$\Delta \boldsymbol{v} = -\left(\boldsymbol{v}_* - \frac{\boldsymbol{F}_G}{\dot{m}_p} \right) \cdot \ln \frac{m_0}{m_f} = \frac{\boldsymbol{F}_* + \boldsymbol{F}_G}{\dot{m}_p} \cdot \ln \frac{m_0}{m_f} \quad @\ \boldsymbol{v}_*, \boldsymbol{F}_G = const \qquad (2.3.1)$$

This is quite an interesting result. It claims that for a brief thrust maneuver we only need to substitute $\boldsymbol{v}_* \to \boldsymbol{v}_* - \boldsymbol{F}_G / \dot{m}_p$ in the familiar rocket Eq. (2.2.1). In closing, we mention that the impulsive maneuver considered above is just a limiting case of Eq. (2.3.1) for $\boldsymbol{F}_* = -\dot{m}_p \boldsymbol{v}_* \gg \boldsymbol{F}_G$.

2.3.3 Gravitational Loss

We now consider the ascent of a rocket from a celestial body (cf. Sect. 6.4) with the gravitational force F_G as the only external force acting on the rocket. From Eq. (2.1.1), we find for the instantaneous speed gain

$$d\boldsymbol{v} = \frac{\boldsymbol{v}_*}{m} dm + \frac{\boldsymbol{F}_G}{m} dt$$

We assume that in course of the ascent the gravitational field does not change significantly and therefore $\boldsymbol{F}_G = m\boldsymbol{g}_0 \approx const$. To find the absolute value of the instantaneous speed gain we multiply this equation with the instantaneous unit velocity vector $\hat{\boldsymbol{v}}(t)$, finding

$$dv = \frac{\hat{\boldsymbol{v}}\boldsymbol{v}_*}{m} dm + g_0 \cos(\gamma + 90^\circ) \cdot dt = \frac{\hat{\boldsymbol{v}}\boldsymbol{v}_*}{m} dm - g_0 \sin\gamma \cdot dt$$

where $\gamma = \angle(\boldsymbol{v}, \boldsymbol{F}_G) - 90^\circ$ is the so-called *flight path angle*, which is the angle between the flight path and the local horizon (see Figs. 6.8 and 7.9). For a steering-free ascent $\hat{\boldsymbol{v}}\boldsymbol{v}_* = -v_* \approx const$ holds. In general $\hat{\boldsymbol{v}}\boldsymbol{v}_*/m$ is the mass-specific speed gain in a gravitational-free environment, the integral of which is not of specific interest for us here, so we will denote it quite generally as Δv_0. Therefore, we find for the velocity after time t

$$\Delta v = \Delta v_0 - \underbrace{g_0 \int_0^t \sin\gamma \cdot dt'}_{gravitational\ loss} \tag{2.3.2}$$

So, in contrast to a rocket in free space, the achieved velocity for ascent depends on the flight direction relative to the local horizon and the time t to engine shutdown. For ascent $\gamma > 0^\circ$ and therefore the integral term in Eq. (2.3.2) is positive. (For a reentering S/C with $\gamma < 0^\circ$ it would be negative.) Therefore, an ascending rocket does not achieve the same velocity increase as for a propulsion maneuver in free space, which is why the entire integral term is called **gravitational loss** term. Gravitational loss is particularly striking for vertical ascent, when $\gamma = 90^\circ$. In this case

$$\Delta v = \Delta v_0 - g_0 t \quad @ \ \text{vertical ascent}$$

In the extreme case when the thrust at ascent just balances the gravitational force, then $g_0 t = \Delta v_0$ and therefore $\Delta v = v_0 = 0$: the rocket barely hovers above the

launch pad until the fuel is used up. We, therefore, derive the following general rule:

> The longer the ascent time and the larger the angle between the flight path and the gravitational force, the smaller is the final speed at engine shutdown.

Therefore, for vertical ascent, the specific impulse $I_{sp} \propto v_*$ is not the only figure of merit of an engine but also a high thrust, which reduces ascent time and thus, gravitational losses. Consequently, powerful but admittedly low-efficient chemical boosters are regularly used during vertical ascent; while for the upper stages when the rocket turns horizontally higher-efficient but lower-power liquid hydrogen/oxygen thrusters take over.

Though we have found a clue how to get into orbit efficiently we are still far from answering the question: What is the optimal ascent trajectory? We will investigate this problem in more detail in Sect. 6.4.

2.4 Delta-v Budget and Fuel Demand

2.4.1 Delta-v Budget

The figure $\Delta\boldsymbol{v}$ appeared in the above equations for the first time. It has a special, double relevance in astronautics. On the one hand, it represents the mass-specific momentum change of a rocket: $\Delta\boldsymbol{v} = \Delta\boldsymbol{p}/m$. Momentum changes are necessary to change from a given Keplerian orbit to another Keplerian orbit, or from a Hohmann transfer orbit into a planetary orbit, or vice versa. We know from conservation laws that momentum is a basic physical parameter. Another important basic parameter is energy. To track energy changes is very important, as the initial increase in kinetic energy $\Delta E = m\boldsymbol{v} \cdot \Delta\boldsymbol{v}$ generated by a small $\Delta\boldsymbol{v}$ may transfer into different forms of energy by means of the energy conservation law, e.g., into potential energy, and in lower Earth orbits unfortunately also into frictional energy. This is why it should come as no surprise that a rocket, which formally gains velocity through a kick-burn Δv, may actually decrease its total velocity v when for instance a rocket fires in a gravitational potential. Then more kinetic energy is transferred into potential energy than kinetic energy is produced by the kick-burn. Overall, due to a higher final orbit and in accordance with $v = \sqrt{\mu(2/r - 1/a)}$ (see Eq. (7.3.14)) the orbital velocity is paradoxically reduced, even though the spacecraft initially received a velocity increase Δv.

The nice feature with $\Delta\boldsymbol{v}$ is that it measures all the possible energy demands of a mission. Since it also measures momentum and angular momentum demand, it is a perfect measure for the total thrust demand for an entire mission.

Remark *Changes of angular momentum caused by thrust maneuvers are related to the Δv demand in a quite complicated way, as because of $h = |\boldsymbol{r} \times \boldsymbol{p}| = mrv \cdot \sin(\angle(\boldsymbol{r}, \boldsymbol{v})) = mrv \cdot \cos\gamma$, every change of angular momentum depends strongly on the thrust direction and flight path angle γ. For a circular orbit and for a small tangential thrust $\Delta\boldsymbol{v}_{||}$, from the above it is easy to show that $\Delta h = mr \cdot \Delta v_{||}$.*

Even better, according to Eq. (2.2.4), the all important propellant demand m_p has a one-to-one correspondence to Δv. This is a very handy relationship: Although the required propellant is actually the determining factor in space flight, later on, it does not show up any more in orbit calculations. Rather it is the Δv measure that comes from astrodynamical considerations, such as from the vis-viva Eq. (7.2.15). Equation (2.2.4) now links the two parameters in a very convenient way. Even more conveniently, the propellant demand of two successive Δv maneuvers corresponds to the sum $\Delta v_1 + \Delta v_2$ of the individual maneuvers according to

$$\Delta v = v_* \ln \frac{m_0}{m_2} = v_* \ln \frac{m_0 m_1}{m_1 m_2} = v_* \ln \frac{m_0}{m_1} + v_* \ln \frac{m_1}{m_2} = \Delta v_1 + \Delta v_2$$

Since the propulsion effort is independent on the sign of Δv, we derive for the effort

$$\boxed{\Delta v = \sum_{all\ maneuvers} |\Delta v_i|} \tag{2.4.1}$$

This is called *delta-v* budget (a.k.a. *propulsion demand*) or just *delta-v* for short. So we can make the following comment:

> Δv (delta-v budget) is a perfect quantity to determine mission effort, because it relates astrodynamical properties (momentum, angular momentum, and total energy changes), which may be derived from mission considerations, in a convenient additive way to the propellant demand of a mission.

2.4.2 Fuel Demand—Star Trek Plugged

Keep in mind that, according to Eq. (2.2.4), the propellant demand and the propulsion demand have a non-linear relationship: propellant demand strongly grows with increasing propulsion demand. For practical purposes, the relative change of the launching mass of a rocket $\Delta m_0/m_0$ for a given change in propulsion demand $\Delta(\Delta v)$ is a very interesting relation. This relation of relative change can be shown (exercise, Problem 1.2) to be

$$\frac{\Delta m_0}{m_0} = \exp\left[\frac{\Delta(\Delta v)}{v_*}\right] - 1 \approx \frac{\Delta(\Delta v)}{v_*} \quad @ \quad \Delta(\Delta v) \ll v_* \tag{2.4.2}$$

Example

An interplanetary probe is to be accelerated to escape velocity to leave the solar system. A flyby maneuver via Jupiter requires $\Delta v_2 = 6.33$ km/s. Direct escape from the Earth orbit requires $\Delta v_1 = 8.82$ km/s. Question: How much more

launching mass do you need for direct escape, if the chemical propulsions have an I_{sp} of 306 s? Answer:

$$v_* = g_0 \cdot I_{sp} = 9.80 \cdot 306 \text{ m/s} = 3.00 \text{ km/s}$$

$$\frac{\Delta m_0}{m_0} = \exp\left[\frac{2.49}{3.00}\right] - 1 = 1.29$$

The additional propulsion demand for direct escape is only 39%, the increase of the launch mass due to the additional propellant demand, however, is 129%!

Let us have a closer look at Eq. (2.2.4). It refutes what many science fiction fans believe: that good classical propulsion just needs a lot of energy.

Remark *With "good classical propulsion" we refer to classical recoil propulsion not to exotic propulsion like warp propulsion. When you see on a cinema screen a spacecraft accelerating with a thundering roar (which of course does not make sense at all in a vacuum as outer space) during a spacecraft battle, this obviously is recoil propulsion.*

The truth, however, is this: What a flight maneuver needs more than anything else is propellant mass. A lot of it. As for large maneuvers, such as an inversion of the flight direction, Δv gets very large, the exponent tends to zero, and the used propellant mass tends to 100% of the spacecraft mass, which is an extremely uncomfortable perspective for the passengers. You could object arguing that "Star Trek" et al. have engines providing unlimited exhaust velocity v_*, which would reduce the propellant demand in line with Eq. (2.2.4). But that is not possible. Because, from Einstein, we know that the maximum possible exhaust velocity is the velocity of light c. Assuming that Star Trek's relativistic rocket engines (of course) have $v_* = c$, one can prove (cf. Sect. 1.4) that, for big maneuvers, Eq. (2.2.4) takes on the form

$$\frac{m_p}{m_0} = 1 - \frac{1}{2}\sqrt{1 - \left(\frac{\Delta v}{c}\right)^2} \quad @ \quad v_* = c, \quad \Delta v \to c$$

If Capt. Kirk now wants to carry out a reversion maneuver, then he has to decelerate first, i.e., $\Delta v = v \approx c$, then he has to accelerate again in the opposite direction, i.e., again $\Delta v = v \approx c$. This relativistic equation has to be used for both maneuvers one after the other, leading to

$$\frac{m_p}{m_0} = 1 - \frac{1}{4}\left[1 - \left(\frac{v}{c}\right)^2\right] \quad @ \quad v_* = c, \quad \Delta v = 2v \to c$$

In other words, if, in a galactic fight with an enemy, Kirk only flew with 90% velocity of light (which would indeed be below his dignity), he would need 95.25% of the spacecraft's mass as propellant for one single reversion maneuver. If he flew with 99% velocity of light, he would already need 99.5% of the spacecraft's mass.

It is quite strange that you never see any of the necessary huge propellant tanks in the movies.

But propellant shortage would be Kirk's smallest problem. The energy required for the reversion maneuver would be more of a problem. A relativistic calculation for the total energy demand of a full slam on the brakes would be

$$E = \frac{2m_0c^2}{\sqrt{1 - v^2/c^2}}$$

If the spacecraft does a reversion maneuver using up the double amount of energy, and let us assume the ideal case that Kirk gets his energy from a 100% efficient annihilation of matter and antimatter—the very best that could be imagined—then he would need the energy mass-equivalent

$$m = \frac{4m_0}{\sqrt{1 - v^2/c^2}}$$

of matter and antimatter, half the amount of each. So, if Kirk flies with 90% velocity of light, he would need the mass equivalent of 9.18, and with 99% velocity of light he would need 28.4 spacecraft masses. But from a logical point of view, this is not possible at all, as the spacecraft only has one spacecraft mass.

2.5 Problems

Problem 2.1 *Launch Mass Changes*
Prove Eq. (2.4.2) by deriving the relative change of the launch mass of a rocket $\Delta m_0/m_0$ for a relative change of the propulsion demand $\Delta(\Delta v)/(\Delta v)$. Find the approximation for small $\Delta(\Delta v)$.