Chapter 3
Rocket Staging

In Sect. 1.3.1, we found out that there are limits to the obtainable payload mass because of the finite structural mass. This limit is crucial for the construction of a rocket. If a rocket is to reach the low Earth orbit, a propulsion demand of about 9 km s^{-1} has to be taken into account (see end of Sect. 6.4.7), which is at the limit of feasibility for today's chemical propulsions. If a higher payload ratio beyond 3% is required, or the S/C needs to leave the gravitational field of the Earth requiring a higher propulsion demand, one has to take measures to increase the rocket efficiency.

The best method to do this is rocket staging. "Staging" means to construct a rocket such that some tanks and/or engines are integrated into one stage, which can be jettisoned after use thereby reducing the mass to be further accelerated. Figure 3.1 depicts four types of rocket staging, where parallel staging and serial staging are the most common.

3.1 Serial Staging

3.1.1 Definitions

Serial staging (a.k.a. multistaging, multistepping, tandem staging) means that several rockets (n stages) sit on top of each other. One stage after the other is fired during operation, and the burnt-out stages are jettisoned. The advantages are:

1. The engines can be adapted to the changing environment upon ascent: the lowest stage can be chosen for a high thrust to quickly escape the Earth's gravitational potential, whereas the upper stage(s) in (almost) free space can be dimensioned for best efficiency (I_{sp}).
2. Jettisoning the lower engines, which are no longer necessary, and the tanks, decreases structural mass and hence increases payload mass.

U. Walter, *Astronautics*, https://doi.org/10.1007/978-3-319-74373-8_3

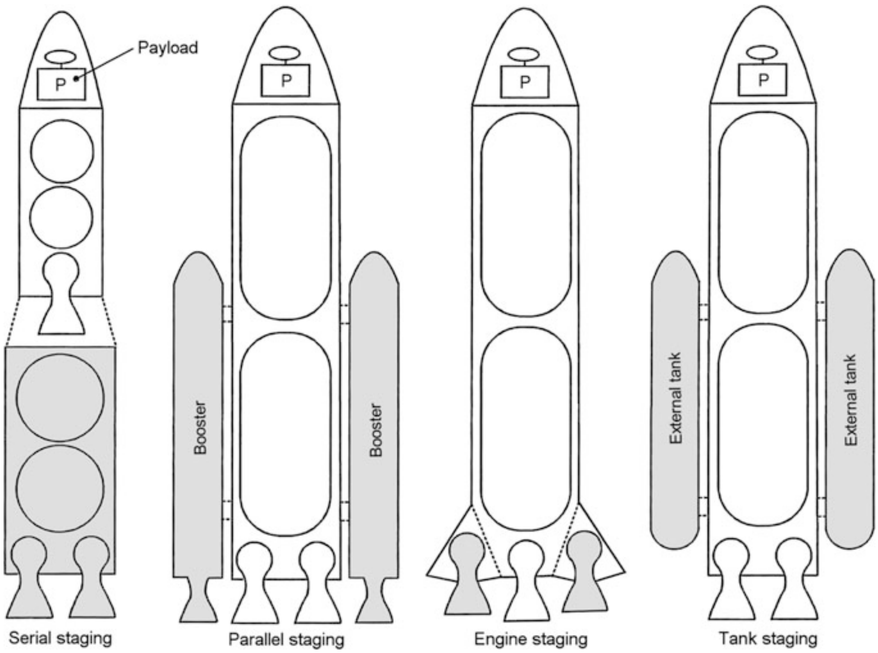

Fig. 3.1 Four types of rocket staging. Stages that are jettisoned during ascent are marked in gray

The concept of serial staging dates back to the first military applications in 1529. Figure 3.2 depicts the sketch of the first staged rocket as introduced by the Austrian military technician Conrad Haas in his *Kunstbuch* (German, meaning *Art Book*). The concept of staging then passed via a publication of the German rocket pioneer Johannes Schmidlap to the Polish military engineer Kazimierz Siemienowicz, whose book *Artis Magnae Artilleriae pars prima* (*Great Art of Artillery, the First Part*) published in 1650 was translated into many languages in Europe, and became for two centuries the basic artillery manual. So rocket staging became a well-known method in artillery rocketry and passed from there also to the rocket-hype decades 1920–1940.

Already in those days the question was raised, whether ignition of the next stage should wait until the rocket's speed had reduced to zero, i.e., make use of the full impetus or to ignite it immediately. The answer is given by Eq. (2.3.2), in saying: Any delay of ignition will lift the fuel of the next stage to higher altitudes thus converting kinetic energy of the rocket into potential energy of the fuel which is a waste of effort (see Problem 3.1). So, the gravitational loss demands for a preferably short ascent and hence a preferably short ignition sequence.

For a mathematical analysis of serial staging, we introduce the concept of a **partial rocket** *i*. Let us assume we have four separate propulsion units (see Fig. 3.3), which are set up in four stages on top of each other. The first partial

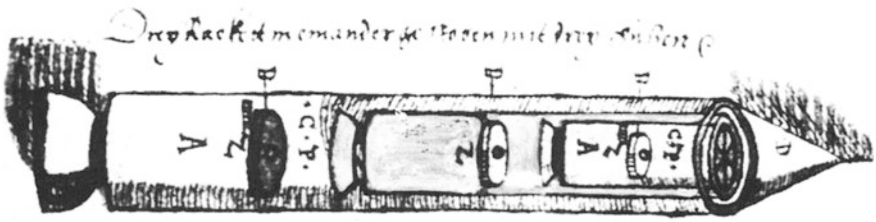

Fig. 3.2 A three-stage rocket of Conrad Haas from 1529. The picture inscription (German) reads: “Three nested rockets with three shots”. Note that each stage has already a bell-shaped nozzle. By courtesy of Barth (2005)

rocket refers to the sum of all the four propulsion units plus the true payload. The second partial rocket refers to the sum of the three stages 2–4 arranged on top of that, plus the true payload. The third partial rocket to the sum of the third and fourth stage plus true payload. Finally, last the fourth partial rocket to the upper unit plus the true payload.

We define the payload of a partial rocket as the mass of the next partial rocket, i.e.,

$$m_{L,i} := m_{0,i+1} \tag{3.1.1}$$

The **true payload** is defined as the payload of the last partial rocket. In line with the single-stage rocket $m_{0,i}$ and $m_{f,i}$ will be the initial and the final mass of the ith partial stage; therefore

$$\begin{aligned} m_{0,i} &= m_{p,i} + m_{s,i} + m_{0,i+1} \\ m_{f,i} &= m_{0,i} - m_{p,i} = m_{s,i} + m_{0,i+1} \end{aligned} \tag{3.1.2}$$

We also define the following ratios (cf. Eqs. (1.3.1)–(1.3.3)) for each partial rocket i:

$$\mu_i := \frac{m_{f,i}}{m_{0,i}} = \frac{m_{0,i+1} + m_{s,i}}{m_{0,i}} = \frac{m_{L,i} + m_{s,i}}{m_{0,i}} < 1 \quad \textbf{mass ratio} \tag{3.1.3}$$

$$\varepsilon_i := \frac{m_{s,i}}{m_{0,i} - m_{0,i+1}} = \frac{m_{s,i}}{m_{s,i} + m_{p,i}} < 1 \quad \textbf{structural ratio} \tag{3.1.4}$$

$$\lambda_i := \frac{m_{L,i}}{m_{0,i} - m_{0,i+1}} = \frac{m_{L,i}}{m_{s,i} + m_{p,i}} = \frac{m_{0,i+1}}{m_{s,i} + m_{p,i}} < \frac{m_{L,i}}{m_{s,i}} \quad \textbf{payload ratio} \tag{3.1.5}$$

The inverse value of the mass ratio is sometimes called the *growth factor of the stage*. Observe that the structural mass and the payload mass of a partial rocket are

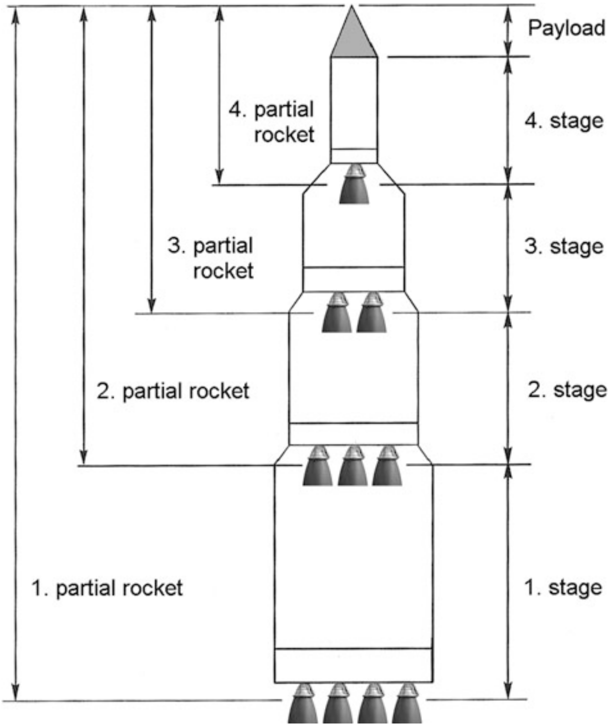

Fig. 3.3 Definitions for a staged rocket

taken relative to the mass of its lower stage and not its total mass. This is reasonable because it will facilitate the optimization of partial rockets, which will be the main objective of this chapter. Because

$$1 + \lambda_i = \frac{m_{s,i} + m_{p,i} + m_{0,i+1}}{m_{s,i} + m_{p,i}} = \frac{m_{0,i}}{m_{s,i} + m_{p,i}}$$

it follows that

$$\frac{\lambda_i}{1 + \lambda_i} = \frac{m_{0,i+1}}{m_{s,i} + m_{p,i}} \frac{m_{p,i} + m_{s,i}}{m_{0,i}} = \frac{m_{0,i+1}}{m_{0,i}}$$

and

$$\frac{\varepsilon_i}{1 + \lambda_i} = \frac{m_{s,i}}{m_{p,i} + m_{s,i}} \frac{m_{p,i} + m_{s,i}}{m_{0,i}} = \frac{m_{s,i}}{m_{0,i}}.$$

For the so-called total *payload ratio* (Ratio of the true payload to the total launch mass. Its inverse value is sometimes called *total growth factor*):

$$\lambda_* := \frac{m_{L,n}}{m_{0,1}} = \frac{m_{L,n}}{m_{0,i}} \frac{m_{0,i}}{m_{0,i-1}} \cdots \frac{m_{0,3}}{m_{0,2}} \frac{m_{0,2}}{m_{0,1}} \quad \textbf{total payload ratio}$$

we therefore obtain

$$\lambda_* = \prod_{i=1}^{n} \frac{\lambda_i}{1+\lambda_i} \tag{3.1.6}$$

and

$$\mu_i = \frac{m_{L,i} + m_{s,i}}{m_{0,i}} = \frac{\varepsilon_i + \lambda_i}{1+\lambda_i} \tag{3.1.7}$$

Remark *In order to be consistent to Eq. (3.1.5) λ_* should be defined as $\lambda_* := m_{L,n} / (m_{0,1} - m_{L,n})$. In this case in all the following equations $\lambda_* \rightarrow \lambda_* / (1 + \lambda_*)$ should be replaced. In Eq. (3.2.2) the replacement should read $\lambda_* \rightarrow \lambda_* (1 + \lambda_*)^2$. Because the inconsistent definition is used throughout the literature and because it is insignificantly different to the consistent form we will adopt it here also.*

3.1.2 Rocket Equation

We are now looking for a serial-staged rocket equation equivalent to Eq. (2.2.2) for the single-staged rocket. According to Eq. (2.2.2) for each partial rocket i,

$$\Delta v_i = v_{*,i} \ln \frac{m_{0,i}}{m_{f,i}} = -v_{*,i} \ln \mu_i$$

holds. Serial staging with instant firing of the following stage means that the terminal velocity of one partial rocket is the initial velocity of the following partial rocket, i.e., $v_{f,i} = v_{0,i+1}$. So the following is valid:

$$\Delta v = v_{f,n} - v_{0,1} = \left(v_{f,n} - v_{0,n}\right) + \left(v_{f,n-1} - v_{0,n-1}\right) + \cdots + \left(v_{f,1} - v_{0,1}\right) = \sum_{i=1}^{n} \Delta v_i$$

Therefore we get for the total velocity increase (propulsion demand):

$$\Delta v = -\sum_{i=1}^{n} v_{*,i} \ln \mu_i = \sum_{i=1}^{n} v_{*,i} \ln \frac{1+\lambda_i}{\varepsilon_i + \lambda_i} \quad \textbf{serial-stage rocket equation} \tag{3.1.8}$$

3.2 Serial-Stage Optimization

3.2.1 Road to Stage Optimization

Our goal now is to optimize the serial stages such that we get a maximum payload into orbit or to achieve a maximum velocity gain. In order to know how to perform the optimization let us state the problem explicitly. For the optimal construction of a serial-staged rocket, the following quantities must be considered:

- **Technically given quantities**
  - exhaust velocities $v_{*,i}$
  - structural ratios ε_i
  - total launch mass $m_{0,1}$ of the rocket
- **Technically variable parameters**
  - number of stages n
  - payload ratios λ_i
- **Target quantities**
  - total payload ratio λ_*
  - propulsion demand Δv

The latter are determined by the technical parameters through Eqs. (3.1.6) and (3.1.8).

The objective of a stage optimization now is to first specify one target quantity and then to maximize the other by variation of n and λ_i. However, n is not a true variable: first because n is an integer; and, second, because, with every additional stage, the rocket becomes more efficient (see Fig. 3.6) and therefore an optimal rocket would have infinitive many stages. So, optimizing n cannot be the true objective. Rather the following holds. Because every stage adds to the propulsion demand by a summand (see Eq. (3.1.8)), in the following optimization procedure n is the smallest stage number, for which at a given propulsion demand the optimized payload ratios can be just determined. We will see at the end of Sect. 3.2.2 precisely what this means.

Therefore, the payload ratios λ_i remain as the only variables, which have to be optimized by the following two different optimization approaches:

1. max λ_*: maximize the total payload ratio λ_* at a given Δv
2. max Δv: maximize the obtainable Δv at a given λ_*

The first approach is taken for instance by Ruppe (1966) and the second for instance by Griffin and French (2004). So, with max λ_* Eq. (3.1.6) is the target function to maximize, and Eq. (3.1.8) is the secondary condition; and for max Δv it is the other way round. As both proceedings are described in literature, and also used in practice, we also want to explore both, and will see that in principle they are equivalent.

3.2.2 General Optimization

Now λ_* and Δv are to be maximized with respect to λ_i under secondary conditions. This can be achieved with the so-called Lagrangian Multiplier Method, whereby a secondary condition can be taken into account by adding it to the partial derivatives (to be set to zero) via a so-called Lagrangian Multiplier γ.

Remark *For a comprehensible description of the Lagrangian multiplier method and why it works see for instance: Reif (1965, Appendix A10), or* http://mathworld.wolfram.com/LagrangeMultiplier.html.

By this method we find for both max cases and for each partial rocket $j = 1, \ldots, n$ the condition equations

$$\frac{\partial \lambda_*}{\partial \lambda_j} + \gamma \frac{\partial(\Delta v)}{\partial \lambda_j} = 0 \quad @ \ \ \max \lambda_* \tag{3.2.1a}$$

and

$$\frac{\partial(\Delta v)}{\partial \lambda_j} + \gamma' \frac{\partial \lambda_*}{\partial \lambda_j} = 0 \quad @ \ \ \max \Delta v \tag{3.2.1b}$$

respectively. Obviously these two equations actually set the same mathematical problem if one identifies $\gamma = 1/\gamma'$. Applying the partial derivations

$$\frac{\partial(\Delta v)}{\partial \lambda_j} = v_{*,j} \frac{\varepsilon_j + \lambda_j}{1 + \lambda_j} \frac{\partial}{\partial \lambda_j} \left(\frac{1 + \lambda_j}{\varepsilon_j + \lambda_j} \right) = -v_{*,j} \frac{1 - \varepsilon_j}{1 + \lambda_j} \frac{1}{\varepsilon_j + \lambda_j}$$
$$\frac{\partial \lambda_*}{\partial \lambda_j} = \frac{\lambda_*}{\lambda_j / (1 + \lambda_j)} \frac{\partial}{\partial \lambda_j} \left(\frac{\lambda_j}{1 + \lambda_j} \right) = \frac{\lambda_*}{\lambda_j (1 + \lambda_j)}$$

to Eq. (3.2.1a) yields

$$\frac{\lambda_*}{\lambda_j (1 + \lambda_j)} = \gamma v_{*,j} \frac{1 - \varepsilon_j}{1 + \lambda_j} \frac{1}{\varepsilon_j + \lambda_j}$$

From this it follows that

$$v_{*,j} \lambda_j \frac{1 - \varepsilon_j}{\varepsilon_j + \lambda_j} = \frac{\lambda_*}{\gamma} =: \alpha = const \quad @ \ \ j = 1, \ldots, n \tag{3.2.2}$$

This is a key equation: first, because it claims that if $v_{*,i} = v_* = const$ and $\varepsilon_i = const$ then all payload ratios are equal; and, second, because, once the constant α is known, all optimized payload ratios follow from it immediately by solving for λ_j:

$$\lambda_{j,opt} = \frac{\alpha\varepsilon_j}{(1-\varepsilon_j)v_{*,j} - \alpha} \quad @ \ j = 1, \ldots, n \tag{3.2.3}$$

Inserting this equation into Eqs. (3.1.8) and (3.1.6) one arrives at

$$\Delta v = \sum_{i=1}^{n} v_{*,i} \ln \frac{v_{*,i} - \alpha}{\varepsilon_i v_{*,i}} \tag{3.2.4}$$

and

$$\lambda_* = \prod_{i=1}^{n} \frac{\alpha\varepsilon_i}{(1-\varepsilon_i)(v_{*,i} - \alpha)} \tag{3.2.5}$$

The two optimization methods now differ merely in that in the

- $\max\lambda_*$ case, at a given Δv the constant α is numerically determined from secondary condition Eq. (3.2.4), which inserted into Eq. (3.2.5) yields the maximized λ_*,
- $\max\Delta v$ case, at a given λ_* the constant α is numerically determined from secondary condition Eq. (3.2.5), which inserted into Eq. (3.2.4) yields the maximized Δv.

Quite generally α can be determined from the above equations only numerically. If, for instance, Newton's method is applied, the equation for the recursive iteration for solving Eq. (3.2.4) reads

$$\alpha_{i+1} = \alpha_i - \frac{\Delta v - \sum_{i=1}^{n} v_{*,i} \ln \dfrac{v_{*,i} - \alpha_i}{\varepsilon_i v_{*,i}}}{\sum_{i=1}^{n} \dfrac{v_{*,i}}{v_{*,i} - \alpha_i}} \tag{3.2.6}$$

and that for Eq. (3.2.5)

$$\alpha_{i+1} = \alpha_i \cdot \left\{ 1 + \frac{\ln\left[\lambda_* \cdot \prod_{i=1}^{n} \dfrac{1-\varepsilon_i}{\varepsilon_i}\left(\dfrac{v_{*,i}}{\alpha_i} - 1\right)\right]}{\sum_{i=1}^{n} \dfrac{v_{*,i}}{v_{*,i} - \alpha_i}} \right\} \tag{3.2.7}$$

Note that due to either of the constraints $\lambda_{i,opt} > 0$, $\Delta v_i > 0$, or $\lambda_i/(1+\lambda_i) < 1$ we have from Eqs. (3.2.3), (3.2.4), or (3.2.5) that α is restricted to

$$0 < \alpha < \min_{i=1,n}\left[v_{*,i}(1-\varepsilon_i)\right] \tag{3.2.8}$$

Newton's iteration as given by Eqs. (3.2.6) and (3.2.7) does not take this restriction into account, so it must be addressed as a constraint in the iteration process.

If we insert α so derived into Eq. (3.2.3), we also obtain the optimal payload ratios. With this we have finally achieved the stage optimization goal.

Remark *One might argue that the procedure laid out here will not work in the max* λ_* *case because according to Eq. (3.2.2), the constant* α *depends on* λ_**, which in itself is a variable to be determined. Rather one would have to set up a system of* $n+1$ *equations from the corresponding secondary condition (3.2.4) or (3.2.5) plus the n Eqs. (3.2.3) to find from this the* $n+1$ *quantities* (λ_i, α) *self-consistently. The point is that* λ_* *is not a variable to be optimized (which are the* λ_i*) but a target quantity, which is mathematically a (yet to be determined) constant. With the introduction of* α *in Eq. (3.2.2) we merely redefine the constant Lagrange multiplier* γ *with the help of the quasi-constant* λ_*.

Example

Let us consider the Saturn V rocket from the Apollo era. Its characteristic partial rocket parameters are given for Apollo 11 in Fig. 3.4.

If we optimize the staging with the above procedure, we arrive at the dependencies as given in Fig. 3.5. Saturn V provided a $\Delta v = 12.4$ km s^{-1} to the Moon. Given this, we find as the optimal result $\lambda_* = 0.0175$, which is only marginally better than the actual Saturn V total payload ratio of $\lambda_* = 0.0162$, corresponding to 47 tons of payload (Command Module, Service Module, and Lunar Module) into translunar trajectory. Note that a two-stage version of Saturn V with the same staging parameters would have either permitted to transfer only half of the payload mass to the Moon or would have been twice as heavy on the launch pad at the given payload mass.

Choosing the Number of Stages

When α needs to be determined from Eq. (3.2.4) or Eq. (3.2.5) the number of stages (partial rockets) n, which occurs in both equations, deserves a treatment: What should n be chosen? In general, it can be said that, as long as Eq. (3.2.8) is obeyed, any additional stage adds to λ_* or Δv, respectively. So the answer is: "The more stages the better!" (cf. Fig. 3.6 for a rocket with uniform stages). However, since every stage adds to the complexity and hence to structural weight and to the cost of a rocket, the smallest number of stages might be pursued. Is there a lowest n that can be attained? In Eq. (3.2.5) $n \geq 1$ can be chosen arbitrarily, since we see from Eq. (3.1.6) that for every given $\lambda_* < 0.5$ and stage number n one can find a λ_i and

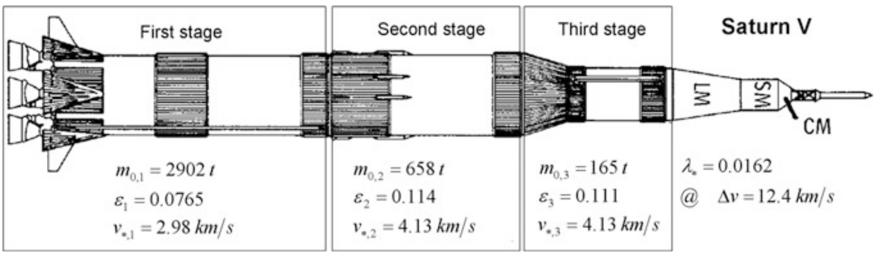

Fig. 3.4 Characteristic stage parameters of Saturn V (Apollo 11)

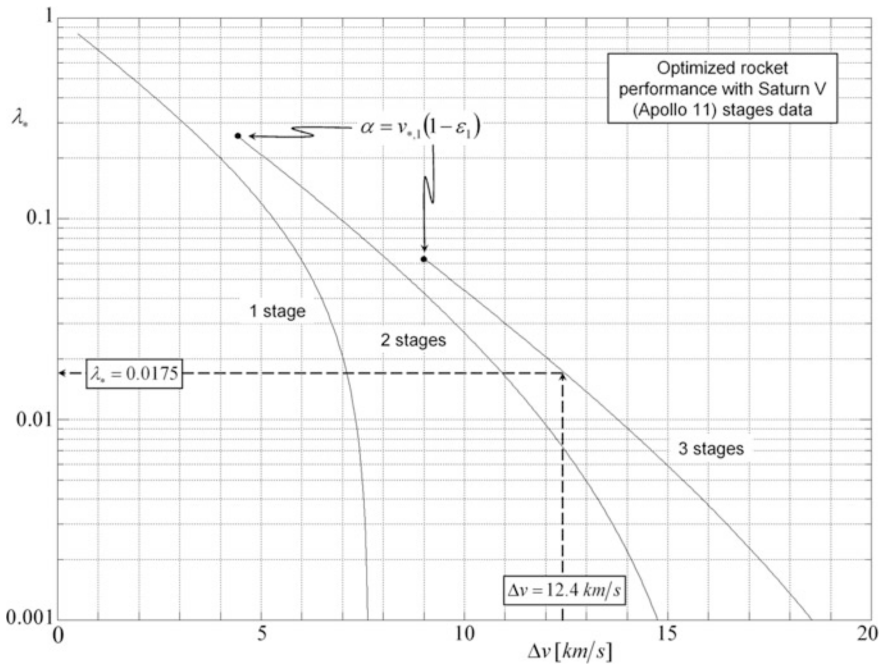

Fig. 3.5 Optimized rocket performance with Saturn V (Apollo 11) stage parameters as given in Fig. 3.4

hence an α. However, in Eq. (3.2.4) there may be a lower bound $n_{\min} > 1$. In the max λ_* case this happens because n has to be raised such that for a given Δv there exists an α. By the same token, in the max Δv case n has to be raised such that for an α derived from Eq. (3.2.5) a required propulsion demand is achieved.

But most importantly staging is governed by mission design rather than by optimization considerations. Take Saturn V as an example. The staging was chosen essentially such that the first two stages carried the third partial rocket into a low earth parking orbit (LEO), while the third stage injected the payload cluster into a translunar trajectory. Since the same will be true for future missions to Mars, the staging will have to be adapted to the mission sequence: launch pad → LEO parking and assembly orbit → low Mars parking orbit → descent to Mars surface → ascent to low Mars parking orbit → Earth reentry. For that reason the number of stages is determined by the sequence steps, while the partial rocket mass for each step is determined from the given delta-v budget for that step from the one stage rocket equation and working in the sequence backward beginning at the final payload mass (reentry capsule). The above stage optimization procedure has to be

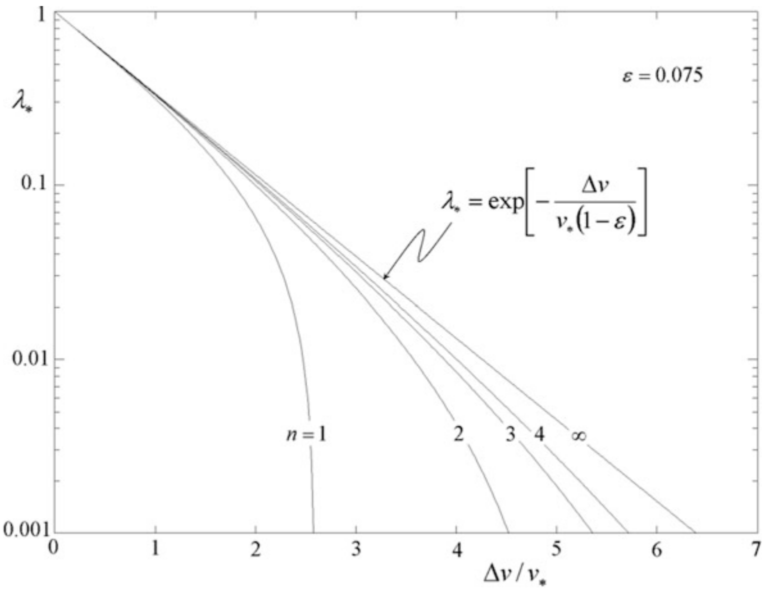

Fig. 3.6 Total payload ratio λ_* as a function of the normalized terminal velocity $\Delta v / v_*$

considered only to tune the overall mission. For the Apollo missions to the Moon, for instance, the LEO apogee kick-burn was not performed by the second stage but already by the third stage, which two and a half hours later reignited to inject the payload into a translunar trajectory. This trade-off shifted the payload ratios between the two partial rockets.

In summary we have the following guidelines for choosing the number of stages:

Guidelines for choosing *n*

1. Adapt the stages and hence also their number n to the propulsion needs of a mission sequence.
2. There is a minimum number of stages $n_{\min}$ such that only for $n \geq n_{\min}$ there exists an α and hence a self-consistent solution $\Delta v(\lambda_*)$ or $\lambda_*(\Delta v)$ from Eqs. (3.2.4) and (3.2.5).
3. The performance of a rocket increases with its number of stages—provided that $\alpha \leq v_{*,i}(1 - \varepsilon_i)$ holds for all of them—at the expense of rocket complexity.

3.3 Analytical Solutions

We are now seeking for analytical solutions to Eqs. (3.2.4) and (3.2.5). We start out with the most simple case of uniform staging and move toward more general cases. Whatsoever, the solution equation will be identical for both maximization cases, because mathematically we just merge the two equations by eliminating the constant α. We always obtain *one* equation, which relates Δv and λ_* via ε_i and $v_{*,i}$. This is why we will disregard in the following the index *max* of the variable to be maximized.

3.3.1 Uniform Staging

The most simple case is the uniform staging where the structural ratios and effective exhaust velocities are all the same:

$$v_{*,i} = v_* = const$$

and

$$\varepsilon_i = \varepsilon = const.$$

Via Eq. (3.2.3) this implies that all payload ratios must be identical, i.e., $\lambda_i = \lambda = const$. The single optimal payload ratio for all stages can derived from Eq. (3.1.6) as

$$\frac{1}{\lambda_{opt}} = \frac{1}{\lambda_*^{1/n}} - 1 \tag{3.3.1}$$

The maximized velocity gain of an n-staged rocket is finally obtained by inserting Eq. (3.3.1) into the serial-stage rocket Eq. (3.1.8)

$$\Delta v = -v_* \sum_{i=1}^{n} \ln\left[\lambda_*^{1/n}(1-\varepsilon)+\varepsilon\right] = -nv_* \ln\left[\lambda_*^{1/n}(1-\varepsilon)+\varepsilon\right] \tag{3.3.2}$$

From this follows

$$\lambda_*^{1/n} = \frac{e^{-\Delta v/nv_*} - \varepsilon}{1-\varepsilon} \tag{3.3.3}$$

This dependence is depictured in Fig. 3.6 for different stage numbers. Obviously the rocket performance increases steadily with increasing stage number.

In the max Δv case the maximized Δv is calculated with a given λ_* from Eq. (3.3.2), and with Eq. (3.3.3) it is just the other way around. In any case, the smaller the total payload ratio λ_*, the bigger is Δv. The absolute maximum Δv is achieved when λ_* vanishes, in which case we find

$$\Delta v_{\max} = -n v_* \ln \varepsilon \quad @\, \lambda_* = 0 \tag{3.3.4}$$

This sets an upper limit to what can be achieved. If one inserts Eq. (3.3.3) into Eq. (3.3.1) we derive for the optimized uniform payload ratio

$$\lambda_{opt} = \frac{\lambda_*^{1/n}}{1 - \lambda_*^{1/n}} = \frac{e^{-\Delta v / n v_*} - \varepsilon}{1 - e^{-\Delta v / n v_*}} \tag{3.3.5}$$

Having found an optimal solution to the serial-staged rocket we might ask: How does it compare with the single-stage rocket? In terms of achievable total payload ratio, Eq. (3.3.3) compares with Eq. (1.3.4) for the single stage. For a single stage rocket (SSTO, $\Delta v = 9\ \mathrm{km\,s^{-1}}$) with 7.5% structural ratio and $v_* = 4\ \mathrm{km\,s^{-1}}$ we are able to lift only 3.4% total payload ratio into LEO. With an optimized two-stage rocket we lift 7.3%, and hence twice as much under the same conditions; and with a three-stage rocket the total payload ratio we obtain is only a little more, viz. 7.9%. Obviously a two-stage rocket is a perfect vehicle to LEO. This is why nearly all rockets into LEO have two stages.

This simple case of uniform staging enables us to calculate the ultimate rocket with an infinite number of stages. The transition to infinite stages can be performed in Eq. (3.3.2). The result is (exercise, Problem 3.2):

$$\Delta v = -v_* (1 - \varepsilon) \ln \lambda_* \quad @\, n \to \infty$$

From this follows

$$\lambda_* = \exp\left[-\frac{\Delta v}{v_* (1 - \varepsilon)}\right] \quad @\, n \to \infty \tag{3.3.6}$$

So, even in this ideal case the total payload ratio of an optimized rocket decreases exponentially with Δv budget. Figure 3.6 displays the functional dependencies of Eqs. (3.3.3) and (3.3.6). So, with a given structural ratio Δv and stage number n the achievable total payload ratio λ_* can be easily derived and vice versa. The result shows that already with a few stages it is possible to considerably increase the fraction of the payload mass. However, it also demonstrates that with more than three or four stages, the benefit hardly outweighs the additional complexity of the rocket. Obviously the straight line Eq. (3.3.6) is an asymptote for the achievable λ_* and Δv, respectively. And, whatever the number of stages, the total payload ratio always decreases exponentially with an increasing terminal velocity.

3.3.2 Uniform Exhaust Velocities

We now relax the conditions to

$$v_{*,i} = v_* = const$$
$$\varepsilon_i \quad \text{arbitrary}$$

From Eq. (3.2.4) it follows that

$$e^{\Delta v/v_*} = \prod_{i=1}^{n} \frac{v_* - \alpha}{\varepsilon_i v_*} = \prod_{i=1}^{n} \frac{1}{\varepsilon_i} \prod_{i=1}^{n} \left(1 - \frac{\alpha}{v_*}\right) = \frac{1}{\bar{\varepsilon}^n} \left(1 - \frac{\alpha}{v_*}\right)^n$$

with $\bar{x} = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$ geometric mean. Solving for α/v_* yields

$$\frac{\alpha}{v_*} = 1 - \bar{\varepsilon} \cdot e^{\Delta v/nv_*} \qquad (3.3.7)$$

Inserting this into Eq. (3.2.5)

$$\lambda_* = \prod_{i=1}^{n} \frac{\alpha \varepsilon_i}{(1 - \varepsilon_i)(v_* - \alpha)} = \frac{\bar{\varepsilon}^n}{\left(\overline{1 - \varepsilon}\right)^n} \cdot \frac{1}{(v_*/\alpha - 1)^n}$$

delivers

$$\lambda_*^{1/n} = \frac{e^{-\Delta v/nv_*} - \bar{\varepsilon}}{\overline{1 - \varepsilon}} \qquad (3.3.8)$$

From this follows

$$\Delta v = -nv_* \ln\left[\lambda_*^{1/n}\left(\overline{1 - \varepsilon}\right) + \bar{\varepsilon}\right] \qquad (3.3.9)$$

depending on which is the target quantity to be maximized. Inserting Eq. (3.3.7) into Eq. (3.2.3) one obtains for the optimized payload ratios

$$\lambda_{i,opt} = \frac{e^{-\Delta v/nv_*} - \bar{\varepsilon}}{\frac{\bar{\varepsilon}}{\varepsilon_i} - e^{-\Delta v/nv_*}} = \frac{\lambda_*^{1/n}}{\frac{\bar{\varepsilon}}{\varepsilon_i} \frac{1-\varepsilon_i}{\overline{1-\varepsilon}} - \lambda_*^{1/n}} \qquad (3.3.10)$$

These are the analytical results for a rocket with uniform exhaust velocities $v_{*,i} = v_*$.

3.3.3 Uneven Staging

We finally examine the most general case where both the exhaust velocities and the structural ratios are uneven

$$\begin{array}{ll} v_{*,i} & \text{arbitrary} \\ \varepsilon_i & \text{arbitrary} \end{array}$$

Due to the complexity of Eqs. (3.2.4) and (3.2.5) no exact analytical solutions can be given. But in the following we will provide approximate solutions, which for a draft rocket design are sufficiently exact. Without loss of generality we first define

$$v_{*,i} = v_* + \Delta v_{*,i} \text{ and } \varepsilon_i = \varepsilon + \Delta\varepsilon_i$$

where

$$v_* = \frac{1}{n}\sum_{i=1}^{n} v_{*,i} \text{ and } \varepsilon = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i$$

are the arithmetic means of the exhaust velocities and structural ratios of all n stages. Assuming $\Delta v_{*,i} \ll v_*$, $\Delta\varepsilon_i \ll \varepsilon$, and according to Appendix B it is possible to derive the following approximate solutions

$$\lambda_*^{1/n} = \frac{e^{-\Delta v/nv_* - C} - \bar{\varepsilon}}{\overline{1-\varepsilon}} \quad \text{@} \ \exp\left(-\frac{\Delta v}{nv_*}\right) \ll \min_{i=1,n}\left(\frac{\bar{\varepsilon}}{\delta_i}\right) \tag{3.3.11}$$

with

$$C = \frac{1}{n}\sum_i \frac{\Delta v_{*,i}}{v_*}\left\{\frac{\Delta\varepsilon_i}{\varepsilon_i} + \frac{1}{2}\left(1 - \frac{1}{\bar{\varepsilon}}e^{-\frac{\Delta v}{nv_*}}\right)^2 \frac{\Delta v_{*,i}}{v_*} + O\left[\left(\frac{\Delta\varepsilon_i}{\varepsilon}\right)^2\right] + O\left[\left(\frac{\Delta v_{*,i}}{v_*}\right)^2\right]\right\}$$

From this follows

$$\Delta v = -nv_*\left\{\ln\left[\lambda_*^{1/n}\left(\overline{1-\varepsilon}\right) + \bar{\varepsilon}\right] + C'\right\} \tag{3.3.12}$$

with

$$C' = \frac{1}{n}\sum_i \frac{\Delta v_{*,i}}{v_*}\left\{\frac{\Delta\varepsilon_i}{\varepsilon_i} + \frac{\lambda_*^{2/n}}{2}\left(\frac{\overline{1-\varepsilon}}{\bar{\varepsilon}}\right)^2 \frac{\Delta v_{*,i}}{v_*} + O\left[\left(\frac{\Delta\varepsilon_i}{\varepsilon}\right)^2\right] + O\left[\left(\frac{\Delta v_{*,i}}{v_*}\right)^2\right]\right\}$$

and

$$\lambda_{i,opt} = \frac{e^{-\Delta v/nv_* - C'} - \bar{\varepsilon}}{\frac{\bar{\varepsilon}}{\varepsilon_i} - e^{-\Delta v/nv_* - C'}} = \frac{\lambda_*^{1/n} + K}{\frac{\bar{\varepsilon}}{\varepsilon_i}\frac{1-\varepsilon_i}{\overline{1-\varepsilon}} - \lambda_*^{1/n} - K} \tag{3.3.13}$$

with $K = \left(\lambda_*^{1/n} + \frac{\bar{\varepsilon}}{\overline{1-\varepsilon}}\right)C'$

Note that even for quite uneven exhaust velocities the solutions are acceptable.

Example

Saturn V for instance had three stages with $v_{*,1} = 304$ s and $v_{*,2} = v_{*,3} = 421$ s at vacuum, therefore $v_* = 382$ s. A calculation of C via the above equations introduces a relative error of only

$$\frac{\Delta C}{C} \approx \sum_{i=1}^{3} \left(\frac{\Delta v_{*,i}}{v_*}\right)^3 = 0.64\%$$

Nevertheless, in specific cases Eqs. (3.2.4) and (3.2.5) should rather be solved numerically.

3.4 Parallel Staging

Parallel staging means that several stages (here: k stages) are mounted, and also activated in parallel (see Fig. 3.1). Let us determine the corresponding rocket equation. The total thrust of a parallel-staged rocket is the sum of the generally different thrusts of all stage engines

$$F_* = \sum_{i=1}^{k} F_{*,i} = \sum_{i=1}^{k} \dot{m}_{p,i} v_{*,i} \tag{3.4.1}$$

Also the total mass flow rate is the sum of all stage engines

$$\dot{m}_p = \sum_{i=1}^{k} \dot{m}_{p,i} \tag{3.4.2}$$

Analogous to Eq. (1.1.5) we can set up a thrust equation

$$F_* = \dot{m}_p v_* \tag{3.4.3}$$

whereby we have indirectly defined a effective exhaust velocity

$$v_* := \frac{F_*}{\dot{m}_p} = \frac{\sum_{i=1}^{k} \dot{m}_{p,i} v_{*,i}}{\sum_{i=1}^{k} \dot{m}_{p,i}} \quad \textbf{effective exhaust velocity} \tag{3.4.4}$$

Mathematically speaking this is an arithmetic mean exhaust velocity. In words, we have the following:

> For parallel staging the effective exhaust velocity is calculated from the mean of exhaust velocities weighted by the respective mass flow rates of the stage thrusters.

If the exhaust velocities are all the same, $v_{*i} = v_{*j}$, Eq. (3.4.4) results in $v_* = v_{*i}$, as expected. With this definition the rocket equation of a parallel-staged rocket is identical to that of a single-stage rocket Eq. (2.2.2)

$$\boxed{\Delta v = v_* \cdot \ln \frac{m_0}{m}} \quad \textbf{parallel-staging rocket equation} \tag{3.4.5}$$

with v_* given by Eq. (3.4.4). Note that owing to this identity any combination of serial staging and parallel staging can be easily calculated.

Advantages and Disadvantages of Parallel Staging Compared with Serial Staging

Advantages

+ The total engine weight is fully used during propulsion time. It does not have to be carried as "dead" payload of the following stage as in serial staging.
+ Thereby in near-Earth space one achieves a higher acceleration and hence a lower gravitational loss (see Sect. 2.3.3).
+ Attaching additional boosters easily adapts a launcher to larger payload masses.
+ Empty tank mass can be minimized by propellant transfer with collecting pipes.
+ Development costs are minimized by standardizing structure and engines.
+ A smaller overall length of a rocket reduces bending moments and longitudinal and lateral oscillations.

Disadvantages

– Structural load is higher after launch.
– This implies higher dynamic drag losses.
– Due to the relatively long combustion time the boosters have to be dimensioned to operate over a higher altitude range. Therefore, the nozzle cross sections cannot be optimally dimensioned for all altitudes implying less thrust. With serial staging the engines can be far better adapted to the respective operative ranges to achieve higher effective exhaust velocities.

After Saturn V, which was a true serial three-stage rocket, NASA quickly passed over to two-stage rockets with parallel staging of the first stage by reflecting on the archetypal Russian R-7 rocket. This mixed type of staging is a good trade-off between the pros and cons of serial and parallel staging. Engines that are just flanged on to the first stage are called *strap-on boosters* or just *boosters*. Today virtually any launcher is build like that: Delta, Soyuz, Ariane, and Shuttle. Parallel staging of only the first stage has the advantage that by using two, four or six boosters the thrust can be easily adapted to varying payloads without changing the rocket design.

3.5 Other Types of Staging

Instead of integrating a tank plus engine in one stage, it is also possible (see Fig. 3.1) to stage only the tank (**tank staging**) or the engines (**engine staging**). These staging types can be combined with serial staging and parallel staging. Tank staging is an interesting alternative, for instance, when a propellant component has a very low density (such as hydrogen), leading to a large and hence heavy empty tank at the end of the launching phase.

A quite interesting option of tank staging has been discussed since the 1970s: Only one type of engine for two different propellants (dual propellant propulsion). Initially, a propellant with a higher density (e.g., kerosene and liquid oxygen) is burnt because of the higher thrust. At burnout, one merely switches to a propellant with lower density, but higher specific impulse, (e.g., liquid hydrogen and oxygen). This staging type would be very effective (just one engine mass) and reliable (no full-stage separation) if the empty propellant tanks, would be attached external to the rocket to be jettisoned.

3.6 Problems

Problem 3.1 *Ignition Sequencing*
Assume a serial two-stage sounding rocket, which ascends vertically and provides a Δv_1 by its first stage and a Δv_2 by its second stage. Now consider the two cases: The sounding rocket ignites its second stage

1. immediately after burn-out of the first stage.
2. after the second partial rocket has reached $v = 0$.

Assuming $g = const$ and the ideal case where all thrust phases have duration $t \to 0$, show that the payload in the first case climbs by $\Delta h = \Delta v_1 \cdot \Delta v_2 / g$ higher than in the second case.

Problem 3.2 *Infinite-Stages Rocket*
Starting from Eq. (3.3.2) show that for infinite many stages equation $\Delta v = -v_* (1 - \varepsilon) \ln \lambda_*$ holds (see Eq. (3.3.6)).

Hint: Define the auxiliary variable $x := 1 - \varepsilon$ and make use of $\lim_{n \to \infty} \left[n \left(\lambda_*^{1/n} - 1 \right) \right] = \ln \lambda_*$.

Problem 3.3 *Space Shuttle Propulsion*
Consider a Space Shuttle with a total launch mass of 2017 t (1 t = 1000 kg); orbiter mass of 111 t; external tank (ET) of total mass 738 t, 705 t of which is H_2/O_2 fuel. There are two external Solid Rocket Boosters (SRB) of total mass 584 t each, of which is 500 t solid fuel, firing at $I_{sp} = 300$ s. The three Space Shuttle Main Engines (SSME) had a $v_* = 4.3$ km s^{-1} and a fuel flow rate of 500 kg s^{-1} each. Finally the orbiter incorporated two thrusters called Orbital Maneuvering Systems (OMS) with $v_* = 3.0$ km s^{-1} and 11 t of UDMH/N_2O_4 fuel.

There are three propulsion phases: during the first 120 s of ascent the SRBs burned in parallel with the three SSMEs. Then the SRBs were jettisoned and the orbiter with the ET continued ascent during this second phase until the ET was empty and the SSMEs were cut off. In the third phase, which included the apogee boost into the target orbit, orbit maintenance maneuvers and deorbit burn, the orbiter fired its OMSs.

Assuming that the mass flow rate is constant during all these phases show that the maximum Δv achievable with this Shuttle system adds up to $\Delta v_{\max} = 9.63$ km s^{-1}.