Chapter 7
Boundary Layer Theory

We have seen in Chap. 6 that singularly perturbed problems can have co-existing regular and singular solutions that scale differently as $\varepsilon \to 0$. In the context of physical systems described by differential equations, such structures yield *multi-scale phenomena*. Everyday life yields countless examples of multi-scale phenomena: violent winds in tornadoes surrounded by relatively calm air over large areas, bands of wake behind ships moving in otherwise still waters, cracks forming in uniform solid materials, spots, stripes and other intricate patterns developing in biological systems. In these, and many other contexts, we can separate the behaviour of the system into regions of rapid variation of quantities of interest compared to other larger scale regions of slow variation. Models that can capture such diverse behaviours will allow for multiple distinguished limits, describing balances between different sets of dominant effects in different regions. The relatively narrow regions of rapid variations are generally called *boundary layers*.

Although boundary layers were originally formulated to describe problems in fluid mechanics and aerodynamics [76] (where they generally occur on the boundary of a solid object passing through a surrounding uniform fluid flow), they also describe solutions in broader sets of contexts.

While attempting to directly find a solution of the full problem on an entire domain directly may be very difficult, constructing "partial solutions" on different regions using perturbation expansions can be straightforward. Following the approach introduced in Sect. 6.5, different dominant balances will re-scale the full model into different forms, leading to different regular or singular solutions. Further analysis is then needed to assemble the partial solutions into a complete solution of the full problem. This is accomplished through *asymptotic matching* and leads to this solution methodology being called *the method of matched asymptotic expansions*.

T. Witelski and M. Bowen, *Methods of Mathematical Modelling*,
Springer Undergraduate Mathematics Series, DOI 10.1007/978-3-319-23042-9_7

7.1 Observing Boundary Layer Structure in Solutions

In order to illustrate the main principles behind boundary layers and matched asymptotics, we first consider a problem for which we can express the solution exactly and examine how it can be separated into pieces stemming from behaviours at different scales.

Consider the linear, constant coefficient ordinary differential equation

$$\varepsilon \frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 0 \qquad \text{for } \varepsilon \to 0, \tag{7.1a}$$

on the domain $0 \le x \le 1$, subject to the boundary conditions

$$y(0) = 0, \qquad y(1) = 1. \tag{7.1b}$$

Singular behaviour in the solution should be expected since setting $\varepsilon = 0$ in (7.1a) reduces the equation to a first order ODE, whose solution can satisfy only one of the boundary conditions.

The exact solution of (7.1) for any $\varepsilon > 0$ is given by

$$y(x) = \frac{\exp\left(\frac{-1+\sqrt{1-\varepsilon}}{\varepsilon}\, x\right) - \exp\left(\frac{-1-\sqrt{1-\varepsilon}}{\varepsilon}\, x\right)}{\exp\left(\frac{-1+\sqrt{1-\varepsilon}}{\varepsilon}\right) - \exp\left(\frac{-1-\sqrt{1-\varepsilon}}{\varepsilon}\right)} \tag{7.2}$$

(see Fig. 7.1). If ε is small ($0 < \varepsilon \ll 1$), we can expand the arguments of the exponentials to yield

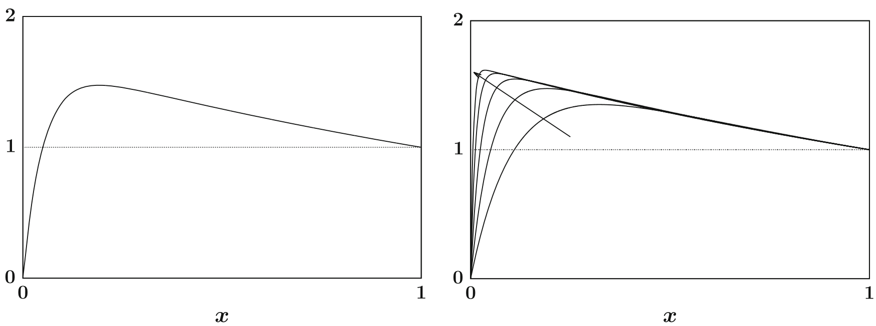


Fig. 7.1 (*Left*) solution (7.2) to problem (7.1) for $\varepsilon = 0.1$. (*Right*) the behaviour of the solution in the limit $\varepsilon \to 0$

$$y(x) = \frac{\exp\Big(-\big[\frac{1}{2}+\frac{\varepsilon}{8}+\cdots\big]x\Big) - \exp\Big(-\big[\frac{2}{\varepsilon}-\frac{1}{2}+\cdots\big]x\Big)}{\exp\Big(-\big[\frac{1}{2}+\frac{\varepsilon}{8}+\cdots\big]\Big) - \exp\Big(-\big[\frac{2}{\varepsilon}-\frac{1}{2}+\cdots\big]\Big)}$$

$$\sim \frac{\exp\left(-\frac{1}{2}x\right) - \exp\left(-\frac{2}{\varepsilon}x\right)}{\exp\left(-\frac{1}{2}\right) - \exp\left(-\frac{2}{\varepsilon}\right)}, \tag{7.3}$$

The size of the domain is independent of ε, and hence one of the spatial scales should be $x = O(1)$ as $\varepsilon \to 0$, as represented by the $e^{-x/2}$ term in (7.3). The other term there depends on a more rapidly varying spatial scale, x/ε, in which a small ($O(\varepsilon)$) change in x yields a $O(1)$ change in the solution. The distinctness of these two scales makes it possible to construct the solution to (7.1) by seeking its dependence on each scale separately.

Each spatial scale in the problem is associated with a limiting process for the solution of (7.1) as $\varepsilon \to 0$. Fixing $x = O(1)$ in the range $0 < x \le 1$ gives

$$\lim_{\varepsilon\to 0} y(x) \sim \frac{e^{-x/2} - \text{e.s.t.}}{e^{-1/2} - \text{e.s.t.}} \sim e^{(1-x)/2}, \tag{7.4}$$

where "e.s.t." refers to *exponentially small terms*, of the form $e^{-\alpha/\varepsilon}$ for $\alpha > 0$, that are smaller than all algebraic powers of ε ($\varepsilon^n \gg e^{-\alpha/\varepsilon}$ for $\varepsilon \to 0$) and are treated as negligible in this context. This limiting form of the solution satisfies the boundary condition at $x = 1$, but not the one at $x = 0$ (7.1b).

Note that we have excluded the case $x = 0$ from consideration in (7.4) so that $e^{-2x/\varepsilon}$ is indeed exponentially small. If instead, we consider a small neighbourhood of the origin, $0 \le x = O(\varepsilon)$, then we must take the dual limit $\varepsilon \to 0$ and $x \to 0$ with the ratio $X = x/\varepsilon$ held fixed. In terms of the new spatial variable X, the limit of (7.3) now becomes

$$\lim_{\varepsilon\to 0} y \sim \frac{e^{-\varepsilon X/2} - e^{-2X}}{e^{-1/2} - e^{-2/\varepsilon}} \sim e^{1/2}(1 - e^{-2X}). \tag{7.5}$$

This limiting form satisfies the boundary condition at $x = 0$ ($X = 0$). The boundary condition at $x = 1$ is not satisfied, but since the right hand boundary position, corresponding to $X = 1/\varepsilon$, violates the assumption $X = O(1)$ made in taking the limit, agreement should not have been expected.

Similar examples are often used in analysis [21] to illustrate functions that have *non-uniform convergence*. In the present context, we see that different limiting properties of the solution are captured by different limiting processes. For the majority of the domain (here, where $x = O(1)$), the solution is given by (7.4) and is called the *outer solution*. In contrast, in the *boundary layer*, or *inner domain*, the solution exhibits singular behaviour that cannot be captured by the outer solution. In this example, the *inner solution* in the boundary layer close to $x = 0$ has a singular derivative, $dy/dx = O(\varepsilon^{-1}) \to \infty$ as $\varepsilon \to 0$.

It is also worth mentioning that problems featuring a separation of scales can be extremely difficult to compute numerically (typically referred to as *stiff problems*). In contrast, solutions of these problems can often be very accurately obtained in terms of inner and outer solutions using perturbation methods.

7.2 Asymptotics of the Outer and Inner Solutions

We now discuss the construction of solutions to singular perturbation problems by calculating perturbation expansions for the outer and inner solutions.

We shall continue to use problem (7.1a, 7.1b) to illustrate the methodology. We begin by attempting to find the outer solution on the *outer domain*, $0 < x \le 1$, in which we assume that $y(x)$ and all of its derivatives are bounded, smooth and $O(1)$. We assume $y(x)$ can be expanded as a regular perturbation expansion of the form

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots \quad \text{as} \quad \varepsilon \to 0. \tag{7.6}$$

Similar to the approach used for (6.24), we substitute this expansion into (7.1a, 7.1b) and separate terms in powers of $\varepsilon \to 0$ yielding the system of sub-problems

$$\begin{array}{lll} O(\varepsilon^0): & 2y_0' + y_0 = 0 & y_0(1) = 1, \\ O(\varepsilon^1): & 2y_1' + y_1 = -y_0'' & y_1(1) = 0, \\ O(\varepsilon^2): & 2y_2' + y_2 = -y_1'' & y_2(1) = 0, \end{array} \tag{7.7}$$

and so on for higher powers of ε. Note that the boundary condition at $x = 0$ from (7.1b) is not included above since $x = 0$ is not within the outer domain. Solving the $O(1)$ problem gives the leading order outer solution

$$y_0(x) = e^{(1-x)/2}, \tag{7.8}$$

which reproduces (7.4). Additional terms in the expansion can be constructed by working through the higher order sub-problems in (7.7) in sequence.

We now turn our attention to constructing the solution in the *inner region*, corresponding to having $x = O(\varepsilon)$. Introducing the rescaling $x = \varepsilon X$ and writing $y(x) = Y(X)$ transforms (7.1a) to

$$\frac{d^2Y}{dX^2} + 2\frac{dY}{dX} + \varepsilon Y = 0, \tag{7.9}$$

with the boundary condition (7.1b) becoming $Y(0) = 0$. Seeking the solution in the form of a regular expansion, $Y(X) = Y_0(X) + \varepsilon Y_1 + \varepsilon^2 Y_2(X) + \cdots$, yields the leading order $O(1)$ problem

$$Y_0'' + 2Y_0' = 0, \qquad Y_0(0) = 0, \tag{7.10}$$

with solution

$$Y_0(X) = A(1 - e^{-2X}). \tag{7.11}$$

Note that one constant, A, remains undetermined in the solution since the ODE in (7.10) is second order, but we are only imposing one side condition. The solutions of the higher order terms, $Y_n(X)$ will similarly add one new undetermined constant at each order in the expansion. Since the inner problem did not uniquely define the inner solution, we examine its relationship with the outer solution in an attempt to fix the unknown constant A.

It is important to note that the inner and outer domains are not mutually exclusive and are valid on regions broader than the strict prescriptions of their spatial scales (here $x = O(\varepsilon)$ and $x = O(1)$ respectively). In the outer solution x is bounded away from zero, but can become small, say $x = O(\sqrt{\varepsilon}) \to 0$. Likewise, in the inner solution x should be small, but $X = x/\varepsilon$ can become large, say $X = O(1/\sqrt{\varepsilon}) \to \infty$.

In fact, it can be shown that there is an *overlap domain*,

$$\text{overlap domain:} \quad \varepsilon \ll x \ll 1, \tag{7.12}$$

between the scales set by the inner and outer domains, where both inner and outer solution are valid (see Fig. 7.2), Loosely speaking, there is a range where x is small (small enough for the inner solution to apply), but not too small (where the outer solution would not apply). Since the original full problem (7.1a, 7.1b) has a unique solution; if the inner and outer solutions are both valid in the overlap domain, they cannot be two distinct solutions and must in fact be two different asymptotic representations of the same solution. This relation is expressed in terms of limits derived from (7.12): (i) $x \ll 1$, the outer variable must approach the inner domain, $x \to 0$ and (ii) $\varepsilon \ll x$ (or after dividing across by ε: $1 \ll X$), the inner variable must approach the outer domain, $X \to \infty$. The resulting limit requirement

$$\lim_{X\to\infty} Y_0(X) = \lim_{x\to 0} y_0(x), \tag{7.13}$$

is called the leading order *asymptotic matching condition* [60, 101]. This principle can be paraphrased as

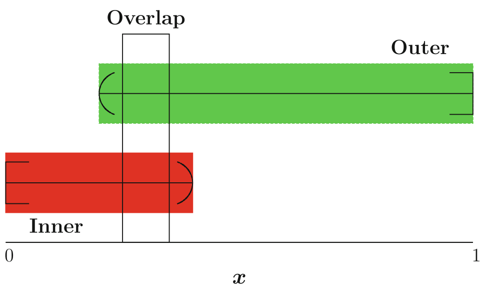


Fig. 7.2 A schematic representation of the inner, outer and overlap domains for problem (7.1a, 7.1b)

$$\text{“The outer limit of the inner solution equals the inner limit of the outer solution.”} \tag{7.14}$$

Applying (7.13) to Y_0 given by (7.11) and y_0 by (7.8) yields

$$\lim_{X\to\infty} A(1-e^{-2X}) = A = \lim_{x\to 0} e^{(1-x)/2} = e^{1/2}, \tag{7.15}$$

thereby determining the constant $A = e^{1/2}$. With this value for A the inner solution (7.11) reproduces the limit (7.5) found from the exact solution.

While we have now determined the leading order inner and outer solutions completely, only a little more work is needed combine the outer and inner solutions to form a *composite* representation of the leading order solution, denoted here by $y_{\text{comp}}(x)$, valid over the entire domain $0 \le x \le 1$. An appropriate form for $y_{\text{comp}}(x)$ is given by the expression

$$y_{\text{comp}}(x) = y_0 + Y_0 - (\text{overlap from matching}), \tag{7.16}$$

where the overlap is simply the contribution found through matching in (7.15). At a formal level, on most of the domain, we have $y \sim y_0$, with the inner solution only becoming significant in the inner domain. Where Y_0 becomes important, we gain equal contributions from both the inner and outer solutions, and so to effectively prevent “double-counting”, we must subtract off the overlap.

Another way of expressing (7.16) is to write

$$y_{\text{comp}} = y_0 + Y_{\text{BLC}}, \tag{7.17}$$

where the *boundary layer correction*, Y_{BLC}, is the adjustment to the outer solution made by the boundary layer to satisfy the boundary condition, with

$$Y_{\text{BLC}} \equiv Y_0 - (\text{overlap from matching}), \tag{7.18}$$

where we expect $Y_{\text{BLC}} \to 0$ as $X \to \infty$.

Writing the inner solution (7.11) as $Y_0 = e^{1/2} - e^{(1-4x/\varepsilon)/2}$ and using the overlap from (7.15), the leading order boundary layer correction is

$$Y_{\text{BLC}} = e^{1/2} - e^{(1-4x/\varepsilon)/2} - e^{1/2} = -e^{(1-4x/\varepsilon)/2}, \tag{7.19}$$

which does indeed vanish as $X \to \infty$. Hence, (7.17) gives the leading order solution

$$y_{\text{comp}} = e^{(1-x)/2} - e^{(1-4x/\varepsilon)/2} = \frac{e^{-x/2} - e^{-2x/\varepsilon}}{e^{-1/2}}, \tag{7.20}$$

on the entire domain $0 \le x \le 1$. This is comparable to the exact solution (7.2), except for the absence of the exponentially small $e^{-2/\varepsilon}$ term (which cannot be captured in regular expansions such as (7.6)).

7.3 Constructing Boundary Layer Solutions

The presentation in the previous section made use of some prior knowledge of the form of the solution to reduce the overall problem to that of determining the expansions of the inner and outer solutions. In this section, we describe the further steps required to investigate boundary layer problems without any additional given information about the form of the solution.

The full process of constructing a solution involving matched asymptotic expansions requires examining the two additional questions:

- What are the scalings for the inner (and outer) solution(s)?
- Where are the boundary layer(s) located?

The answers to these questions ultimately determine the form of the overall solution, and control which boundary conditions apply to the inner and outer solutions.

For many singularly perturbed differential equations, solutions can be constructed by a step-by-step process:

(1) *The outer solution*: If the problem is in standard form, try a regular expansion, $y(x) = y_0(x) + \varepsilon y_1(x) + \cdots$, for the outer solution. If all of the boundary conditions can be satisfied by this solution, then the problem is complete; otherwise, inner regions will be necessary.

(2) *Find the dominant balances*: The appropriate forms for all of the regular (outer) and singular (inner) solutions of the ODE will be determined by the distinguished limits of the problem. In general, both the independent and dependent variables may need to be scaled to obtain all of the dominant balances,

$$y = \varepsilon^{\beta} Y(X), \qquad X = \frac{x - x_*}{\varepsilon^{\alpha}} \quad \Leftrightarrow \quad x = x_* + \varepsilon^{\alpha} X, \tag{7.21}$$

where the powers α, β and the assumed position of the boundary layer, x_* must all be determined (there may be multiple valid locations for x_*). The outer solution in step (1) assumes $\alpha = 0$, $\beta = 0$.

(3) *The inner solution*: For the singular distinguished limit, write the problem as a rescaled regular problem and seek the solution in the form of an appropriate regular expansion, $Y(X) = Y_0(X) + \varepsilon Y_1(x) + \cdots$.

(4) *Asymptotic matching*: Apply asymptotic matching between the inner/outer solutions (typically via (7.13)) to confirm the consistency of the asymptotic expansion and determine any remaining unknown parameters in the solution.

(5) *The composite solution*: Writing the outer and inner solutions in terms of the original variables and subtracting the overlaps from the matching process to

prevent “double-counting” will produce the leading order solution on the entire domain (7.16). So, in an example with boundary layers at both the left and right boundaries, we write

$$y_{\text{comp}}(x) \sim y_0(x) + Y^L_{\text{BLC}} + Y^R_{\text{BLC}}. \tag{7.22}$$

This description covers broad classes of problems, but as will be seen, in some cases, steps (2, 3, 4) may become a bit intertwined. We also note:

- When boundary layers are necessary, which boundary conditions apply to the outer solution may not be immediately apparent. Hence a general form for the outer solution will be needed initially.
- The boundary conditions as well as the ODE play a role in determining the dominant balances.
- The location of the boundary layer and which boundary conditions apply to the inner solution might not be determined until matching is applied.
- If the inner/outer solutions are not matchable (either limit does not exist, or equation (7.13) cannot be satisfied) then the assumed choice of boundary layer position x_* or dominant balance may be not be right.
- While the term “boundary layer” stems from the fact that the inner domain often occurs at a boundary, in some cases, they can also occur within the domain of a problem, in which case they are sometimes called *interior layers*.

Consequently, the reader should consider steps (1)–(5) as “guidelines” that may need to be adjusted depending on the given problem; this is one of the challenging (and interesting) points of matched asymptotic expansions.

We use an example to illustrate the aspects of the above procedure. Consider the boundary value problem

$$\varepsilon\frac{d^2y}{dx^2} + \frac{dy}{dx} = \cos x \tag{7.23a}$$

in the limit $\varepsilon \to 0$ on the domain $0 \le x \le \pi$, subject to the boundary conditions

$$y(0) = 2, \qquad y(\pi) = -1. \tag{7.23b}$$

7.3.1 The Outer Solution

Assuming that $y(x)$ and its derivatives are bounded as $\varepsilon \to 0$, we write the outer solution as $y \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) \cdots$. Substituting into (7.23a) gives the sequence of equations

$$\begin{aligned} O(\varepsilon^0):& \quad y_0' = \cos x,\\ O(\varepsilon^1):& \quad y_1' + y_0'' = 0,\\ O(\varepsilon^2):& \quad y_2' + y_1'' = 0, \end{aligned}$$

and so on for higher order equations. The $O(1)$ problem yields $y_0 = \sin x + A$ and substituting this into the $O(\varepsilon)$ equation gives $y_1(x) = -\cos x + B$. We can proceed in this way to determine as many terms as desired in the expansion of the general outer solution

$$y_{\text{out}} = (\sin x + A) + \varepsilon(-\cos x + B) + O(\varepsilon^2). \tag{7.24}$$

At each order, there is only a single constant of integration, $A, B, \ldots$. Imposing the condition at $x = 0$ from (7.23b) selects $A = 2$, while the condition at $x = \pi$ picks $A = -1$; the outer solution cannot satisfy both at once, and hence a boundary layer will be required. In summary, at this point, we do not know which boundary conditions will apply to the outer and which to the inner solutions.

7.3.2 *The Distinguished Limits*

To determine the relevant scaling of the singular solution, we write $y(x) = \varepsilon^\beta Y(X)$ and $X = (x - x_*)/\varepsilon^\alpha$ and assume that $Y(X) = O(1)$ for $X = O(1)$. Substituting into (7.23a) yields

$$\underbrace{\varepsilon^{1-2\alpha+\beta} Y''}_{(1)} + \underbrace{\varepsilon^{-\alpha+\beta} Y'}_{(2)} = \underbrace{\cos(x_* + \varepsilon^\alpha X)}_{(3)}, \tag{7.25}$$

where $0 \le x_* \le \pi$. We also note that both boundary conditions (7.23b) take the form $\varepsilon^\beta Y = O(1)$, and hence any solution local to a boundary must have $\beta = 0$. It remains to determine α from the possible dominant balances:

$$\begin{array}{llll} \text{(a) Terms (2, 3):} & \varepsilon^{-\alpha} = \varepsilon^0 & \Rightarrow & \alpha = 0, \\ \text{(b) Terms (1, 3):} & \varepsilon^{1-2\alpha} = \varepsilon^0 & \Rightarrow & \alpha = 1/2, \\ \text{(c) Terms (1, 2):} & \varepsilon^{1-2\alpha} = \varepsilon^{-\alpha} & \Rightarrow & \alpha = 1. \end{array}$$

Option (a) is the regular distinguished limit that corresponds to the outer solution. Option (b) is not a valid balance since the neglected term (2) is not sub-dominant, $O(\varepsilon^{-1/2}) \gg O(1)$. Consequently, the boundary layer must take the form given by (c) where the neglected term (3) is sub-dominant to the leading balance with $O(1) \ll O(\varepsilon^{-1})$. We note that in some problems, the dominant balances can change for different assumed positions of the boundary layer, x_* (most notably for non-autonomous equations), but here term (3) uniformly satisfies $|\cos(x)| \le 1 = O(\varepsilon^0)$. Hence our scaled equation for the inner solution is given by

$$\frac{d^2Y}{dX^2} + \frac{dY}{dX} = \varepsilon \cos(x_* + \varepsilon X), \tag{7.26}$$

where x_* has not yet been determined.

7.3.3 The Inner Solution

Having the inner problem in regular perturbation form, we expand $Y(X)$ as $Y \sim Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \cdots$ and substitute into (7.26) to give the system of equations

$$\begin{aligned} O(\varepsilon^0): &\quad Y_0'' + Y_0' = 0, \\ O(\varepsilon^1): &\quad Y_1'' + Y_1' = \cos(x_*), \\ O(\varepsilon^2): &\quad Y_2'' + Y_2' = -\sin(x_*)X, \ldots . \end{aligned} \tag{7.27}$$

The $O(1)$ equation yields the leading order inner solution,

$$Y_0(X) = D + Ce^{-X}, \tag{7.28}$$

with the higher order problems producing smaller corrections to this result. The constants of integration C, D must be determined by boundary conditions or by matching with the outer solution, but this, in turn, depends on the location of x_*.

Consider the forms of the inner domain in terms of $X = (x - x_*)/\varepsilon$ for different possible values of x_*

(i) Left boundary $(x_* = 0): x \geq 0 \quad \Rightarrow \quad 0 \leq X < o(1/\varepsilon)$
(ii) Interior $: 0 < x_* < \pi \quad \Rightarrow \quad -o(1/\varepsilon) < X < o(1/\varepsilon)$
(iii) Right boundary $(x_* = \pi): x \leq \pi \quad \Rightarrow \quad -o(1/\varepsilon) < X \leq 0.$

These options correspond to three possible forms of the composite solution (see Fig. 7.3): (a) a left boundary layer satisfying $y(0) = 2$ matching with an outer solution which has $A = 1$ in order to satisfy the right boundary condition, (b) a narrow interior transition region connecting two outer solutions, with $A^L = 2$ and $A^R = -1$, and (c) a right boundary layer satisfying $y(\pi) = -1$, with an outer solution satisfying $y(0) = 2$.

The location of the boundary layer will be determined by the structure of (7.28) and its limiting behaviour. The exponential term e^{-X} in (7.28) diverges if X is allowed to become large and negative. Such *exponentially diverging* terms *cannot* satisfy the

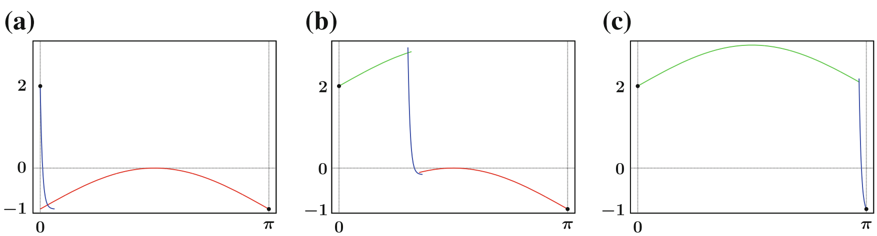


Fig. 7.3 Three hypothetical sketches of the conjectured inner/outer solutions for (7.23a, 7.23b) with a boundary layer **a** at the (*Left*), **b** in the (*Interior*), **c** at the (*Right*) *edge* of the domain

asymptotic matching condition (7.13) (with $X \to -\infty$ being the appropriate form of the 'outer limit' process) and are *un-matchable*.

Consequently options (ii) and (iii) are not feasible, and we conclude that the boundary layer must be at $x_* = 0$ with the left boundary condition from (7.23b) being relevant, namely $Y(0) = 2$. Applying this condition reduces (7.28) to

$$Y_0(X) = 2 + C(e^{-X} - 1), \tag{7.29}$$

where C remains to be determined.

7.3.4 Asymptotic Matching

Having identified the position of the boundary layer as $x_* = 0$, we have the leading order inner solution (7.29), valid on $0 \le x < O(\varepsilon)$, and the outer solution (7.24), valid on $0 < x \le \pi$. Since the right boundary lies in the outer domain, that boundary condition determines $A = -1$ in (7.24), leaving C from (7.29) as the last remaining unknown.

Applying the matching condition (7.13) for $Y_0(X \to \infty)$ and $y_0(x \to 0)$ yields

$$\lim_{X\to\infty} 2 + C(e^{-X} - 1) = 2 - C = \lim_{x\to 0} \sin(x) - 1 = -1 \tag{7.30}$$

and hence $C = 3$.

7.3.5 The Composite Solution

The overlap shared in common by the leading order inner and outer solutions above is -1. Therefore we can form the boundary layer correction as

$$Y_{\text{BLC}} = Y_0 - (-1) = 3e^{-X}.$$

Finally, adding this correction to the outer solution yields the leading order composite solution on $0 \le x \le \pi$ (see Fig. 7.4),

$$y_{\text{comp}} = -1 + \sin(x) + 3e^{-x/\varepsilon}. \tag{7.31}$$

This is in agreement with the exact solution (valid for all $\varepsilon > 0$),

$$y = -1 + \frac{\sin(x) - \varepsilon[1 + \cos(x)]}{1 + \varepsilon^2} + \left(3 + \frac{2\varepsilon}{1 + \varepsilon^2}\right) e^{-x/\varepsilon} + \text{e.s.t.}$$

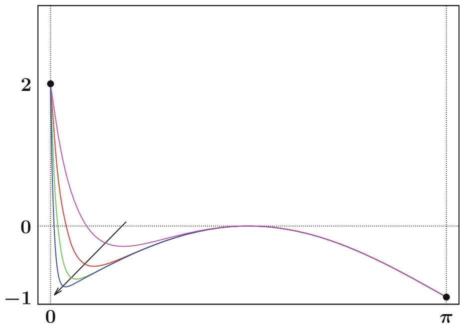


Fig. 7.4 A plot of $y_{\text{comp}}(x)$ (7.31) for a sequence of $\varepsilon \to 0$. The boundary layer becomes narrower as ε decreases

7.4 Further Examples

We present further examples to illustrate other aspects of the method of matched asymptotic expansions.

Consider the problem of obtaining the leading order solution to the ODE problem on $0 \le x \le 1$,

$$\varepsilon \frac{d^2 y}{dx^2} - (2 - x^2) y = -1 \qquad \text{for } \varepsilon \to 0, \tag{7.32a}$$

with boundary conditions

$$y'(0) = 0, \qquad y(1) = 0. \tag{7.32b}$$

We begin by seeking the outer solution as a regular perturbation expansion, $y(x) \sim y_0 + \varepsilon y_1(x) + \varepsilon^2 y_2 + \cdots$. The equation for the leading order term is

$$-(2 - x^2) y_0 = -1 \quad \Longrightarrow \quad y_0(x) = \frac{1}{2 - x^2}. \tag{7.33}$$

The leading order outer solution satisfies the $y'(0) = 0$ boundary condition, i.e. $y_0'(0) = 0$, hence no boundary layer is needed there.

The outer solution has no free parameters, and it does not satisfy the boundary condition at $x = 1$. Therefore there must be a boundary layer at $x_* = 1$.

So we seek a singular solution in the form $y(x) = \varepsilon^\beta Y(X)$ with $X = (x-1)/\varepsilon^\alpha$ for $X \le 0$. Unlike the inhomogeneous conditions in (7.23b), the homogeneous boundary condition $y(1) = 0$ does not provide us information on β since $\varepsilon^\beta Y(0) = \varepsilon^\beta 0 = 0$ for any β. However, if we can determine β using some alternative means, it will simplify the process of finding the distinguished limit for the inner solution. Turning to the asymptotic matching condition provides help; here this condition will take the form

$$\lim_{x \to x_*} y_0(x) = \lim_{X \to -\infty} \varepsilon^\beta Y_0(X). \tag{7.34}$$

While we have not determined $Y_0(X)$, it is assumed to be $O(1)$, and for $x \to 1$, we have $y_0(1) = 1$. Since the limit of the outer solution is $O(1)$, so must be the limit of the inner solution, hence $\beta = 0$.

Substituting $y(x) = Y(X)$ and $x = 1 + \varepsilon^\alpha X$ into (7.32a) yields

$$\varepsilon^{1-2\alpha} Y'' - (2 - (1 + \varepsilon^\alpha X)^2) Y = -1$$

and in final form:

$$\underbrace{\varepsilon^{1-2\alpha} Y''}_{(1)} - \underbrace{(1 - 2\varepsilon^\alpha X - \varepsilon^{2\alpha} X^2) Y}_{(2)} = \underbrace{-1}_{(3)}. \tag{7.35}$$

On a finite domain, $\alpha \geq 0$ with $\alpha > 0$ describing boundary layers that narrow like $O(\varepsilon^\alpha)$ as $\varepsilon \to 0$. We note that for $\alpha > 0$ the sum of terms in the parentheses in term (2) have leading order term $(2 - x^2) \sim 1$, so the higher order terms there cannot contribute to a consistent dominant balance.

Balancing (2, 3) yields the distinguished limit for the outer solution, $\alpha = 0$, with term (1) being sub-dominant, $O(\varepsilon) \ll O(1)$. The other distinguished limit is $\alpha = 1/2$, which balances all three terms at $O(1)$.

We now have the form of the inner problem as

$$Y'' - (1 - 2\varepsilon^{1/2} X - \varepsilon X^2) Y = -1, \qquad Y(0) = 0, \tag{7.36}$$

for $X \leq 0$. The presence of the $\varepsilon^{1/2}$ suggests the expansion of the solution should take the form $Y(X) \sim Y_0 + \varepsilon^{1/2} Y_1 + \varepsilon Y_2 + \cdots$. The leading order ODE is

$$Y_0'' - Y_0 = -1, \tag{7.37}$$

with solution

$$Y_0(X) = Ae^{-X} + Be^X + 1. \tag{7.38}$$

For this solution to be matchable to the outer solution as $X \to -\infty$, we must take $A = 0$. Applying the boundary condition, $Y_0(0) = B + 1 = 0$ then gives us the solution,

$$Y_0(X) = 1 - e^X. \tag{7.39}$$

Since neither the leading order inner or outer solutions have any undetermined constants they should match automatically. This is indeed the case, and (7.34) applied to (7.33) and (7.39) shows they match with an overlap limit of 1. Hence we can construct the leading order composite solution (see Fig. 7.5)

$$y_{\text{comp}} \sim \frac{1}{2 - x^2} - e^{(x-1)/\sqrt{\varepsilon}}. \tag{7.40}$$

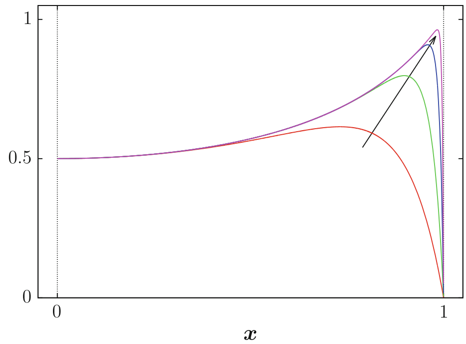


Fig. 7.5 Plot of (7.40) for $\varepsilon = 10^{-n}$ with $n = 2, 3, 4, 5$

Getting a solution accurate to higher orders would involve obtaining further terms in the expansions of the inner and outer solutions. Despite the fact that the expansions have different gauge function (ε^n vs. $\varepsilon^{m/2}$) the solutions must match together. One approach for performing matching to higher order is given in Exercise 7.6.

We now make one change to (7.32a) and illustrate how dramatically the structure of the solution is affected; consider the ODE problem on $0 \le x \le 1$,

$$\varepsilon \frac{d^2y}{dx^2} - (1 - x^2)y = -1 \qquad \text{for } \varepsilon \to 0, \tag{7.41a}$$

with boundary conditions

$$y'(0) = 0, \qquad y(1) = 0. \tag{7.41b}$$

The only change from the previous example is that the coefficient $(2 - x^2)$ in (7.32a) has been replaced by $(1 - x^2)$.

As before, the outer solution can be expressed as a regular expansion and the leading order solution is given by an algebraic equation,

$$y_0 = \frac{1}{1 - x^2}, \tag{7.42}$$

which blows up as $x \to 1$ and does not satisfy the boundary condition $y(1) = 0$. We again conclude that there must be a boundary layer at $x_* = 1$, but now face the problem of determining a boundary layer solution that can match to a diverging outer solution.

We seek an inner solution in the scaled form $y(x) = \varepsilon^{\beta} Y(X)$ with $X = (x-1)/\varepsilon^{\alpha}$, where we expect $\beta < 0$ to capture the singular nature of the magnitude of the solution and $\alpha > 0$ for a narrow boundary layer. Substituting into (7.41a) yields

$$\underbrace{\varepsilon^{1-2\alpha+\beta} Y''}_{(1)} + \underbrace{\varepsilon^{\alpha+\beta} X(2 + \varepsilon^{\alpha} X)Y}_{(2)} = \underbrace{-1}_{(3)}. \tag{7.43}$$

We now consider the options for two-term dominant balances in this equation:

(a) Terms (1, 2) balance if $1 - 2\alpha + \beta = \alpha + \beta$, namely $\alpha = 1/3$. To ensure that the balance is consistent, and these terms are larger than term (3), we need $\alpha + \beta < 0$, i.e. $\beta < -1/3$.
(b) Terms (1, 3) balance if $1 - 2\alpha + \beta = 0$, yielding $\beta = 2\alpha - 1$. Term (2) is sub-dominant if $\alpha + \beta > 0$. Consequently, this and the condition $\beta < 0$ determine the range $1/3 < \alpha < 1/2$.
(c) Terms (2, 3) balance if $\alpha + \beta = 0$, hence $\beta = -\alpha$. Term (1) is sub-dominant when $1 - 2\alpha + \beta > 0$, yielding $0 < \alpha < 1/3$.

It can be useful to visualise these relations in the (α, β) parameter plane in what is called a Newton–Kruskal diagram [105], see Fig. 7.6.

The leading order equations proposed by each of the above respective cases are:

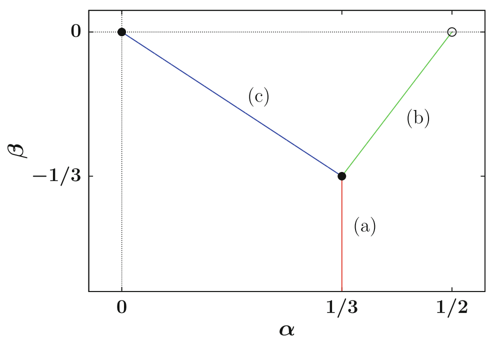
$$Y_0'' + 2XY_0 = 0, \tag{7.44a}$$
$$Y_0'' = -1, \tag{7.44b}$$
$$2XY_0 = -1. \tag{7.44c}$$

Each of these equations can be shown to have some deficiency in trying to describe the inner solution. The solution of (7.44c), $Y_0 = -1/(2X)$, cannot satisfy the boundary condition at $X = 0$. The solution of (7.44b) is a parabola that cannot satisfy the asymptotic matching condition for $X \to -\infty$. Equation (7.44a) is less straightforward; it is a version of Airy's differential equation [11, 105] but it can likewise be shown that its solutions also cannot satisfy the matching condition (7.34).

The above balances are self-consistent, but because they are not the most general dominant balance, they actually have a limited range of validity with the inner domain (it can be shown that (a) holds for $X = O(1)$, (b) holds for $X \to 0$, (c) holds for $X \to -\infty$). The distinguished limit for inner problem is given by the intersection of the three cases, $\alpha = 1/3$, $\beta = -1/3$, with all three terms in (7.43) balancing,

$$Y_0'' + 2XY_0 = -1, \qquad Y_0(0) = 0. \tag{7.45}$$

Fig. 7.6 Newton–Kruskal diagram for (7.43) showing possible two-term dominant balances for cases (a, b, c) as line segments



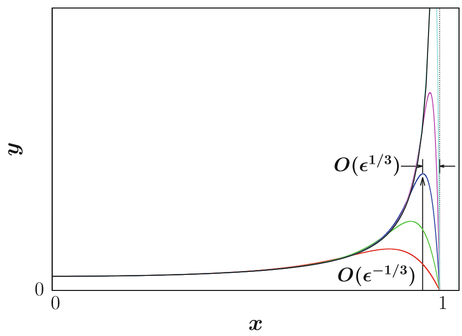


Fig. 7.7 Numerical solutions of (7.41a, 7.41b) for a sequence of $\varepsilon_n \to 0$ (*colour curves*) and the outer solution (7.42) (*black curve*)

Another way to come to this choice of scalings is to use the matching condition (7.34) with the outer solution (7.42) written in terms of X as,

$$y_0 = \frac{1}{1-(1+\varepsilon^{\alpha}X)^2} \sim -\frac{1}{2\varepsilon^{\alpha}X} \sim \varepsilon^{\beta}Y_0, \tag{7.46}$$

which we can recognise as case (c) above (for $X \to -\infty$) with $\beta = -\alpha$. This would reduce (7.43) to an equation for α, having two distinguished limits: $\alpha = 0$ (the outer solution), and the boundary layer given by $\alpha = 1/3$.

Constructing the composite solution would require solving (7.45) (it is an inhomogeneous version of Airy's equation) and carrying out the matching from (7.46) using the approach of Exercise 7.6. Instead, in Fig. 7.7 we show that the numerical solution of the full problem is well characterised by the outer solution on most of the domain with the maximum value of the solution and the width of the boundary layer being well-predicted by the scaling of the inner solution.

7.5 Further Directions

The problems we have considered above provide some insight into how boundary layers and matched asymptotic expansions can separate out some of the delicate behaviours of solutions of singularly perturbed problems. The examples have shown that while the steps outlined in Sect. 7.4 are a good guide, they may be coupled to each other in different ways in each problem—for example: are the scalings of the inner solution determined by the ODE, the boundary conditions, or by matching? is matching needed to set undetermined coefficients in the inner or outer solution (or neither or both)? Many problems require creative application of these steps. Analysis of some more challenging problems remain open research problems.

The form of the solutions obtained to such problem generally provides greater insight into the nature of the system. The dominant balance that determines the leading outer solution provides the simplest essential approximation of the behaviour in the problem. This approximation will be valid everywhere apart from the boundary layers. Matched asymptotics provides understanding of whether the boundary layers are necessary to pin-down properties of the outer solution or merely correct the solution in narrow regions. Often, knowledge of the physical system being modelled can guide expectations on boundary layer positions and dominant balances; this can sometimes simplify the mathematical steps. Sometimes, the matched asymptotics will uncover unexpected dominant balances that can highlight novel and important behaviours in the problem.

This chapter only hints at the broad array of models that can be studied using matched asymptotics and the types of behaviours that can result. Some of these include boundary layers within boundary layers (nested layers or "triple decks"), boundary layers that begin at higher orders ("corner layers"), and problems with unusual gauge functions. There is a wide array of books that give further studies of boundary layer problems [47, 48, 58, 78, 92] and some primary sources on the theory of asymptotic matching are [60, 101].

7.6 Exercises

7.1 Evaluate $\lim_{\varepsilon\to 0} e^{-1/\varepsilon}/\varepsilon^n$ to demonstrate that exponentially small terms are smaller than all algebraic terms.

7.2 Determine the three possible dominant balances for (7.25) that could occur for $x_* = \pi/2$, noting that $\cos(x) \to 0$ as $x \to x_*$.

7.3 Consider the problem for $y(x)$ on $0 \le x \le 1$ with $\varepsilon \to 0$,

$$\varepsilon\frac{d^2y}{dx^2} - (4 - x^2)y = \cos\left(\tfrac{\pi}{2}x\right) \qquad y(0) = -1 \qquad y(1) = 2$$

(a) Determine the two distinguished limits for this problem.
(b) Write the leading order outer solution $y_0(x)$.
(c) Find the leading order inner solutions $Y_0(X)$.
(d) Write the leading order uniformly-valid solution.

7.4 Consider the initial value problem for $y(x)$ on $0 \le x$ for $\varepsilon \to 0$,

$$\varepsilon\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 6y = 5x, \qquad y(0) = 0, \qquad y'(0) = \frac{4}{\varepsilon^2}.$$

You are given that the solution has a boundary layer at $x_* = 0$.

(a) Determine the leading order inner solution.
(b) Write the outer limit of the inner solution to determine a necessary matching condition on the outer solution.
(c) Determine the leading order outer solution for $x > 0$.

7.5 Consider the problem for $y(x)$ on $0 \le x \le 1$ with $\varepsilon \to 0$,

$$\varepsilon \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + e^y = 0, \qquad y(0) = 0, \qquad y(1) = 0.$$

(a) Find the general leading order outer solution $y_0(x)$.
(b) Find the leading order inner solution $Y_0(X)$ and determine where the boundary layer occurs.
(c) Write the leading order composite solution.
(d) Determine the next term in the outer solution, $y_1(x)$.

7.6 Consider the problem for $y(x)$ on $0 \le x \le 1$ with $\varepsilon \to 0$,

$$\varepsilon \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 2x, \qquad y(0) = -2, \qquad y(1) = 1.$$

(a) Determine the distinguished limit for the inner problem for this equation. By solving the leading order inner problem, determine where the boundary layer occurs.
(b) Obtain the *first two terms* in the expansion of the inner solution (with the appropriate boundary conditions imposed),

$$Y(X) \sim Y_0(X) + \varepsilon Y_1(X).$$

(c) Determine the *first two terms* in the expansion of the outer solution (with the appropriate boundary conditions imposed),

$$y(x) \sim y_0(x) + \varepsilon y_1(x).$$

(d) The inner solution will have undetermined constants. These constants can be determined using *higher-order matching using intermediate variables* [60, 101] via the following steps:

(i) Analogous to (7.12), define small parameter η with $\varepsilon^\alpha \ll \eta \ll 1$.
(ii) Define the intermediate variable $\hat{x}$ as $\hat{x} = (x - x_*)/\eta$.
(iii) Use the relations

$$x = x_* + \eta\hat{x}, \qquad X = \frac{\eta}{\varepsilon^\alpha}\hat{x}$$

to write the outer and inner solutions both in terms of the intermediate variable $\hat{x}$ and ε, η.
(iv) Use the relations $\varepsilon^\alpha \ll \eta \ll 1$ to expand out exponential functions or eliminate small terms in the solutions on the overlap region, with $\hat{x} = O(1)$.
(v) Arrange the remaining terms as an ordered asymptotic expansion involving η, ε and determine the remaining constants through matching of terms.

(e) Write the uniform solution valid up through $O(\varepsilon)$ terms.

7.7 Consider the problem for $y(x)$ on $0 \le x \le 2$ with $\varepsilon \to 0$,

$$x^2 + y^2 = 4 - \varepsilon \frac{dy}{dx}, \qquad y(0) = \frac{3}{\varepsilon}.$$

(a) Determine the first two terms in the outer solution, $y \sim y_0 + \varepsilon y_1$.
(b) Boundary layers can occur at $x_* = 0$ and $x_* = 2$. For each case, determine the α, β for each distinguished limit and write its corresponding leading order equation for $Y_0(X)$. Note that there are two singular distinguished limits at $x_* = 0$.

7.8 Consider the problem for $y(x)$ on $0 \le x \le 1$ for $\varepsilon \to 0$,

$$\varepsilon \frac{d^2y}{dx^2} - y = -4 + \frac{\varepsilon^2 y}{(x-1)^3}, \qquad y(0) = 0, \qquad y(1) = 0.$$

The leading order outer solution is $y_0(x) \equiv 4$.

(a) Determine the leading order inner solution for the boundary layer at $x_* = 0$.
(b) At $x_* = 1$, there are <u>*two*</u> different distinguished limits. Determine α for each and obtain the respective leading order equations for each $Y_0(X)$.
Since these solutions have different α's, one layer is nested inside the other. The "inner-inner" layer solution should satisfy boundary condition at $x = 1$. The (wider) "intermediate inner" layer should asymptotically match to the outer and inner-inner solutions for the limits $X \to -\infty$ and $X \to 0$ respectively in this *triple deck* problem.