Chapter 8
Long-Wave Asymptotics for PDE Problems

In this chapter we will study matched asymptotics and boundary layer theory applied to some classes of multi-dimensional problems for partial differential equations (PDE). Perturbation methods offer an interesting alternative way to construct solutions that will provide different insight into the structure of some PDE problems. We will also see that matched asymptotics will allow us to solve problems that cannot be tackled directly by classical methods.

We will illustrate the analysis in the context of problems for Laplace's equation,

$$\nabla^2 \mathsf{U} \equiv \frac{\partial^2 \mathsf{U}}{\partial \mathsf{X}^2} + \frac{\partial^2 \mathsf{U}}{\partial \mathsf{Y}^2} = 0. \tag{8.1}$$

The Laplacian operator ∇^2 is a fundamental element of equations describing various phenomena such as diffusion, wave propagation, and equilibrium potentials

$$\frac{\partial \mathsf{U}}{\partial \mathsf{T}} = \nabla^2 \mathsf{U}, \qquad \frac{\partial^2 \mathsf{U}}{\partial \mathsf{T}^2} = \nabla^2 \mathsf{U}, \qquad \mathsf{F}(\mathsf{X}, \mathsf{Y}) = \nabla^2 \mathsf{U}.$$

These equations arise in electromagnetism, heat conduction, mass transfer, fluid flow, solid mechanics, and many other areas.

We will focus on electrostatics as an example application—in this case $\mathsf{U}(\mathsf{X}, \mathsf{Y})$ represents the electric potential (voltage) that drives the flow of charges and currents in a conductor. In particular, we will describe the current flow and electric charge density in a piece of wire by solving Eq. (8.1) on a long slender domain, subject to various boundary conditions.

8.1 The Classic Separation of Variables Solution

We begin by briefly reviewing the traditional separation of variables approach to solving boundary value problems for Laplace's equation [23, 44].

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For elliptic PDE such as Laplace's equation, to uniquely determine a solution, a problem must provide boundary conditions around the entire boundary of the domain. In the context of electrostatics, Dirichlet boundary conditions, where the value of the solution is specified, correspond to imposing a known voltage on the edge of the domain, e.g. $\mathsf{U}(\mathsf{X}=0,\mathsf{Y})=\bar{\mathsf{U}}$. In contrast, Neumann boundary conditions provide the value of the directional derivative of the solution normal to the boundary

$$\frac{\partial \mathsf{U}}{\partial n} \equiv \hat{\mathbf{n}}\cdot\nabla\mathsf{U}, \tag{8.2}$$

where $\hat{\mathbf{n}}$ is the unit normal vector, perpendicular to the boundary, in the outward direction, e.g. $\hat{\mathbf{n}}=-\hat{\mathbf{i}}$ is the normal to the left boundary of a domain given by $\mathsf{X}\geq 0$,

$$\left.\frac{\partial \mathsf{U}}{\partial n}\right|_{\mathsf{X}=0} = -\hat{\mathbf{i}}\cdot(\partial_{\mathsf{X}}\mathsf{U},\partial_{\mathsf{Y}}\mathsf{U}) = -\left.\frac{\partial \mathsf{U}}{\partial \mathsf{X}}\right|_{\mathsf{X}=0} = \mathsf{J}(\mathsf{Y}).$$

Physically, Neumann conditions specify the current, or *flux* of the solution, out of the domain.

Consider Laplace's equation on a rectangular domain,

$$\frac{\partial^2\mathsf{U}}{\partial\mathsf{X}^2}+\frac{\partial^2\mathsf{U}}{\partial\mathsf{Y}^2}=0 \qquad 0\leq\mathsf{X}\leq\mathsf{L} \qquad 0\leq\mathsf{Y}\leq\mathsf{H} \tag{8.3a}$$

subject to Dirichlet boundary conditions,

$$\mathsf{U}(\mathsf{X},0)=0,\qquad \mathsf{U}(\mathsf{L},\mathsf{Y})=0,\qquad \mathsf{U}(0,\mathsf{Y})=0,\qquad \mathsf{U}(\mathsf{X},\mathsf{H})=\mathsf{F}(\mathsf{X}). \tag{8.3b}$$

This is an elementary 'building-block' problem where the form of the solution will be entirely due to the one inhomogeneous boundary condition. Solutions to problems on the same domain, but with inhomogeneous boundary conditions on other edges, can be built-up from linear superposition of combinations of such building block solutions.

Since this is a linear problem, the overall solution can be obtained as a superposition of linearly independent trial solutions. The trial solutions can be sought in *separation of variables* form, as a product of functions of the independent variables:

$$\mathsf{U}(\mathsf{X},\mathsf{Y})=\sum_{n=1}^{\infty}\mathsf{c}_n\mathsf{U}_n(\mathsf{X},\mathsf{Y}) \quad\text{with}\quad \mathsf{U}_n(\mathsf{X},\mathsf{Y})=\mathsf{A}_n(\mathsf{X})\mathsf{B}_n(\mathsf{Y}). \tag{8.4}$$

Substituting (8.4) into the homogeneous boundary condition $\mathsf{U}(0,\mathsf{Y})=0$ for $0\leq\mathsf{Y}\leq\mathsf{H}$ with the assumptions that the solution U is nontrivial (not all coefficients $\mathsf{c}_n=0$) and that the U_n trial solutions are linearly independent yields boundary conditions on the $\mathsf{A}_n(\mathsf{X})$ functions for $n=1,2,3,\ldots,$

$$\sum\mathsf{c}_n\mathsf{A}_n(0)\mathsf{B}_n(\mathsf{Y})=0 \qquad\text{for}\quad 0<\mathsf{Y}<\mathsf{H} \qquad\Longrightarrow\qquad \mathsf{A}_n(0)=0.$$

Similarly for the other homogeneous boundary conditions, $\mathsf{U}(\mathsf{L},\mathsf{Y}) = 0$ yields $\mathsf{A}_n(\mathsf{L}) = 0$ while $\mathsf{U}(\mathsf{X},0) = 0$ gives $\mathsf{B}_n(0) = 0$. The inhomogeneous boundary condition in (8.3b) will be approached differently.

Substituting $\mathsf{U}_n(\mathsf{X},\mathsf{Y})$ from (8.4) into (8.3a) and requiring each trial solution to satisfy Laplace's equation yields

$$\frac{d^2\mathsf{A}_n}{d\mathsf{X}^2}\mathsf{B}_n(\mathsf{Y}) + \mathsf{A}_n(\mathsf{X})\frac{d^2\mathsf{B}_n}{d\mathsf{Y}^2} = 0$$

which can be re-arranged to give

$$\frac{\mathsf{A}_n''(\mathsf{X})}{\mathsf{A}_n(\mathsf{X})} = -\frac{\mathsf{B}_n''(\mathsf{Y})}{\mathsf{B}_n(\mathsf{Y})} = \mathsf{s}_n = \pm\lambda_n.$$

Since the equality must hold for all independent values of X and Y, both sides must take the same constant value s_n, called a *separation constant*, which here is written as a sign times the undetermined positive constant $\lambda_n \geq 0$, where the subscript indicates that the separation constants are generally distinct between different trial solutions. The main result of this separation of variables approach is to split the PDE into two separate ODEs for $\mathsf{A}_n(\mathsf{X})$ and $\mathsf{B}_n(\mathsf{Y})$ that are linked only through λ_n.

In order to make further progress it is convenient to begin by analysing the ODE problem satisfying homogeneous boundary conditions. In this case this selects the problem in X-direction for $\mathsf{A}_n(\mathsf{X})$. It can be shown that subject to the boundary conditions, obtaining nontrivial solutions of $\mathsf{A}_n'' - \mathsf{s}_n\mathsf{A}_n = 0$ forces s_n to be negative, $\mathsf{s}_n = -\lambda_n$ [23, 44]. The problem for $\{\mathsf{A}_n, \lambda_n\}$ is an *eigenvalue problem*,

$$\mathsf{A}_n'' + \lambda_n\mathsf{A}_n = 0, \qquad \mathsf{A}_n(0) = 0, \qquad \mathsf{A}_n(\mathsf{L}) = 0, \tag{8.5}$$

yielding an infinite sequence of oscillatory eigenfunction solutions

$$\mathsf{A}_n(\mathsf{X}) = \sin\left(\frac{n\pi}{\mathsf{L}}\mathsf{X}\right), \qquad \lambda_n = \frac{n^2\pi^2}{\mathsf{L}^2}, \qquad n = 1, 2, 3, \ldots. \tag{8.6}$$

Having obtained the separation constants, the ODE for $\mathsf{B}_n(\mathsf{Y})$ is now completely specified,

$$\mathsf{B}_n'' - \tfrac{n^2\pi^2}{\mathsf{L}^2}\mathsf{B}_n = 0 \qquad \Longrightarrow \qquad \mathsf{B}_n(\mathsf{Y}) = \mathsf{C}_1\sinh(\tfrac{n\pi}{\mathsf{L}}\mathsf{Y}) + \mathsf{C}_2\cosh(\tfrac{n\pi}{\mathsf{L}}\mathsf{Y}). \tag{8.7}$$

Imposing the $B_n(0) = 0$ boundary condition reduces the general solution to $\mathsf{B}_n(\mathsf{Y}) = \sinh(\tfrac{n\pi}{\mathsf{L}}\mathsf{Y})$. The full solution (8.4) then takes the form

$$\mathsf{U}(\mathsf{X},\mathsf{Y}) = \sum_{n=1}^{\infty}\mathsf{c}_n\sinh(\tfrac{n\pi}{\mathsf{L}}\mathsf{Y})\sin(\tfrac{n\pi}{\mathsf{L}}\mathsf{X}); \tag{8.8}$$

all that remains is to determine the coefficients, c_n. Applying the inhomogeneous boundary condition to the series at $\mathsf{Y} = \mathsf{H}$ yields,

$$\mathsf{U}(\mathsf{X},\mathsf{H}) = \sum_{n=1}^{\infty} \underbrace{\mathsf{c}_n \sinh(\tfrac{n\pi}{\mathsf{L}}\mathsf{H})}_{f_n} \sin(\tfrac{n\pi}{\mathsf{L}}\mathsf{X}) = \mathsf{F}(\mathsf{X}).$$

This equation can be interpreted as the Fourier sine series for the function $\mathsf{F}(\mathsf{X})$ (see Appendix A) and determines the coefficients in the expansion,

$$\sum_{n=1}^{\infty} f_n \sin(\tfrac{n\pi}{\mathsf{L}}\mathsf{X}) = \mathsf{F}(\mathsf{X}) \qquad \Longrightarrow \qquad f_n = \frac{2}{\mathsf{L}}\int_0^{\mathsf{L}} \mathsf{F}(\mathsf{X}) \sin(\tfrac{n\pi}{\mathsf{L}}\mathsf{X})\, d\mathsf{X}.$$

Finally, expressing c_n in terms of f_n, we obtain the solution in the form of an infinite series

$$\mathsf{U}(\mathsf{X},\mathsf{Y}) = \sum_{n=1}^{\infty} \left(\frac{2}{\mathsf{L}\sinh(\tfrac{n\pi}{\mathsf{L}}\mathsf{H})} \int_0^{\mathsf{L}} \mathsf{F}(\tilde{\mathsf{X}}) \sin(\tfrac{n\pi}{\mathsf{L}}\tilde{\mathsf{X}})\, d\tilde{\mathsf{X}} \right) \sinh(\tfrac{n\pi}{\mathsf{L}}\mathsf{Y}) \sin(\tfrac{n\pi}{\mathsf{L}}\mathsf{X}). \tag{8.9}$$

8.2 The Dirichlet Problem on a Slender Rectangle

Solution (8.9) holds for all choices of H, L but working with an infinite series can be somewhat cumbersome. We will now see how a much more compact, but equivalent form of the solution can be obtained for slender rectangles, where the *aspect ratio* $\mathsf{H}/\mathsf{L} \ll 1$, with, for the most-part, a dramatic simplification of the PDE problem to a sequence of simple ODE problems.

Consider a problem for Laplace's equation on the rectangular domain, $0 \le \mathsf{X} \le \mathsf{L}$, $0 \le \mathsf{Y} \le \mathsf{H}$ (see Fig. 8.1):

$$\frac{\partial^2 \mathsf{U}}{\partial \mathsf{X}^2} + \frac{\partial^2 \mathsf{U}}{\partial \mathsf{Y}^2} = 0, \tag{8.10a}$$

subject to the boundary conditions

$$\mathsf{U}(\mathsf{X},0) = 0, \qquad \mathsf{U}(\mathsf{X},\mathsf{H}) = \bar{\mathsf{U}} f(\mathsf{X}/\mathsf{L}), \tag{8.10b}$$

$$\mathsf{U}(0,\mathsf{Y}) = \bar{\mathsf{U}} g_0(\mathsf{Y}/\mathsf{H}), \quad \mathsf{U}(\mathsf{L},\mathsf{Y}) = \bar{\mathsf{U}} g_1(\mathsf{Y}/\mathsf{H}), \tag{8.10c}$$

where $f(x)$, $g_0(y)$, $g_1(y)$ are given functions.

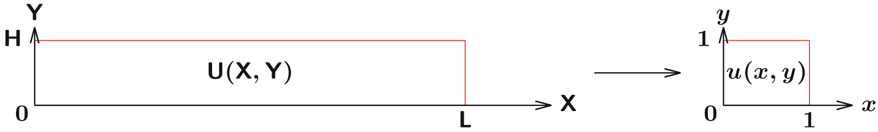

Fig. 8.1 The domain for problem (8.10), a slender rectangle in dimensional coordinates (*Left*) and the rescaled dimensionless domain (*Right*)

We nondimensionalize using the scalings

$$\mathsf{X} = \mathsf{L}x, \qquad \mathsf{Y} = \mathsf{H}y, \qquad \mathsf{U} = \bar{\mathsf{U}}u(x, y), \tag{8.11}$$

yielding the scaled problem on $0 \le x \le 1$ and $0 \le y \le 1$,

$$\varepsilon^2 u_{xx} + u_{yy} = 0 \tag{8.12a}$$

$$u(x, 0) = 0, \quad u(x, 1) = f(x) \qquad \text{for } 0 < x < 1 \tag{8.12b}$$

$$u(0, y) = g_0(y), \qquad u(1, y) = g_1(y) \qquad \text{for } 0 < y < 1 \tag{8.12c}$$

where the aspect ratio, or *slenderness parameter* $\varepsilon = \mathsf{H}/\mathsf{L}$ is small, $\varepsilon \to 0$, corresponding to a long, thin domain.

Using ε as a perturbation parameter, we begin by seeking an outer solution of the form

$$u(x, y) \sim u_0(x, y) + \varepsilon^2 u_1(x, y) + \varepsilon^4 u_2(x, y) + \cdots . \tag{8.13}$$

Note that for this problem it is sufficient to use an expansion having only even powers of ε because the perturbation parameter appears in the problem only as ε^2 in (8.12a). Substituting into (8.12a, 8.12b), we obtain a sequence of problems which are each essentially ODE boundary value problems in the y-direction with x entering only as a secondary parameter. At leading order we get

$$u_{0yy} = 0, \qquad u_0(y = 0) = 0, \qquad u_0(y = 1) = f(x), \tag{8.14a}$$

where the differential equation determines u_0 to be linear with respect to y, $u_0 = C_1 y + C_2$. The C coefficients need not be constants; they must be independent of y but can depend on any other variable(s) present in the problem, namely $u_0(x, y) = C_1(x)y + C_2(x)$. Applying the boundary conditions from (8.14a), we arrive at the leading order solution

$$u_0(x, y) = f(x)y. \tag{8.14b}$$

Similarly, at higher orders,

$$O(\varepsilon^2): \quad u_{1yy} = -u_{0xx}, \quad u_1(y=0) = 0, \quad u_1(y=1) = 0$$

$$\Longrightarrow \quad u_1(x, y) = \tfrac{1}{6} f''(x)(y - y^3).$$

$$O(\varepsilon^4): \quad u_{2yy} = -u_{1xx}, \quad u_2(y=0) = 0, \quad u_2(y=1) = 0$$

$$\Longrightarrow \quad u_2(x, y) = \tfrac{1}{360} f''''(x)(7y - 10y^3 + 3y^5).$$

If the function $f(x)$ in the boundary condition at the top of the domain is smooth, this expansion can be continued to all orders so as to obtain the outer solution in terms of polynomials in y times derivatives of $f(x)$. However, this solution will not in general satisfy the boundary conditions (8.12c) at $x = 0$ and $x = 1$, and hence boundary layer corrections will be needed.

Noting that (8.12a) has a small parameter multiplying the highest order derivative in the x-direction, we recognise it as a singular perturbation problem of the type treated in Chap. 7 and we seek boundary layers at $x = 0$ and $x = 1$. In order to analyse the structure of the inner solution with respect to x, we assume a scaled solution of the form (7.21),

$$X = \frac{x - x_*}{\varepsilon^\alpha}, \qquad u = U(X, y), \tag{8.15}$$

where we have already made use of the Dirichlet boundary conditions, $u = g = O(1)$ for the scaling of U. Substituting this form into (8.12a) yields

$$\varepsilon^{2-2\alpha} U_{XX} + U_{yy} = 0,$$

which selects the distinguished limit $\alpha = 1$ for the inner solution at both $x_*^L = 0$ for the left boundary and $x_*^R = 1$ for the right boundary layer. We will describe the solution for $x_*^L = 0$ with the construction for the right boundary layer following analogously.

While the inner problem with $\alpha = 1$ has brought us back to solving the full Laplace's equation,

$$U_{XX} + U_{yy} = 0, \tag{8.16a}$$

there are subtle changes in the domain and boundary conditions that in this context make the inner problem more tractable than the original full problem. The domain for this inner problem is a semi-infinite strip, $0 \le y \le 1$ with $X \ge 0$. Since $x = \varepsilon X$ with $X = O(1)$ in the left boundary layer, we can re-write the $y = 1$ boundary condition as

$$U(X, 1) = f(\varepsilon X) = f(0) + \varepsilon f'(0) X + \tfrac{1}{2}\varepsilon^2 f''(0) X^2 + \cdots$$

yielding boundary conditions on the leading order inner solution as constant values

$$U_0(X, 0) = 0, \qquad U_0(X, 1) = f(0). \tag{8.16b}$$

The boundary condition at $X = 0$ is unchanged from (8.12c), but now the remaining boundary condition is supplied by the analogue of the asymptotic matching condition with the outer solution (see (7.13))

$$\lim_{X\to\infty} U_0(X, y) = \lim_{x\to 0} u(x, y). \tag{8.16c}$$

Consequently (8.12c), (8.16c) and (8.14b) specify the left and right boundary conditions as

$$U_0(0, y) = g_0(y), \qquad U_0(X \to \infty, y) = f(0)y. \tag{8.16d}$$

Noting the term $f(0)y$ from the outer solution is present in both (8.16b, 8.16d), we observe that the problem can be expressed in terms of the boundary layer correction function $V(X, y)$ defined by

$$U_0(X, y) = V(X, y) + f(0)y, \tag{8.17}$$

where $V(X, y)$ satisfies the problem

$$V_{XX} + V_{yy} = 0, \tag{8.18a}$$

$$V(X, 0) = 0, \qquad V(X, 1) = 0, \tag{8.18b}$$

$$V(0, y) = g_0(y) - f(0)y, \qquad V(X \to \infty, y) = 0. \tag{8.18c}$$

This matches the form of the Dirichlet "building-block" problem with a single inhomogeneous boundary condition described in Sect. 8.1. Here the homogeneous boundary conditions determine the oscillatory eigenfunctions to be in the y-direction with $\lambda_n = n\pi$ yielding

$$V(X, y) = \sum_{n=1}^{\infty} c_n e^{-n\pi X} \sin(n\pi y), \tag{8.19a}$$

where

$$c_n = 2\int_0^1 [g_0(y) - f(0)y] \sin(n\pi y)\, dy. \tag{8.19b}$$

We note that the $e^{-n\pi X}$ factors resulted from solving the ODE $A_n'' - n^2\pi^2 A_n = 0$ yielding general solutions $A_n(X) = d_0 e^{n\pi X} + d_1 e^{-n\pi X}$ and enforcing the $X \to \infty$ boundary condition to then eliminate the un-matchable exponentially growing modes.

In conclusion, the leading order uniform solution can then be composed from the outer solution and the boundary layer corrections,

$$u_{0\text{unif}}(x, y) = u_0(x, y) + V^L(X^L, y) + V^R(X^R, y), \tag{8.20}$$

where $V^L(X^L, y)$ is given by (8.19a, 8.19b) with $X^L = x/\varepsilon$ and $V^R(X^R, y)$ analogously by the right boundary layer correction with $X^R = (x-1)/\varepsilon$.

8.3 The Insulated Wire

Current flow through an insulated wire can be represented by replacing the Dirichlet conditions on the upper and lower boundaries with no-flux Neumann boundary conditions (8.2). Using the same scalings as in the previous example (8.11), the nondimensionalized problem on $0 \le x \le 1$ and $0 \le y \le 1$ takes the form,

$$\varepsilon^2 u_{xx} + u_{yy} = 0, \tag{8.21a}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \qquad \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0 \qquad \text{for} \quad 0 < x < 1, \tag{8.21b}$$

$$u(x=0) = g_0(y), \qquad u(x=1) = g_1(y) \qquad \text{for} \quad 0 < y < 1. \tag{8.21c}$$

This problem describes current flow in a straight insulated wire with prescribed end-voltages.

We begin by constructing the outer solution in the form $u \sim u_0 + \varepsilon^2 u_1 + \varepsilon^4 u_2$. The leading order problem for $u_0(x, y)$ is then given by

$$u_{0yy} = 0, \qquad u_{0y}(y=0) = 0, \qquad u_{0y}(y=1) = 0, \tag{8.22}$$

where the general solution of the differential equation is again a linear function of y with x-dependent coefficients, $u_0(x, y) = C_1(x)y + C_2(x)$. Applying the boundary conditions yields

$$u_0(x, y) = C_2(x), \tag{8.23}$$

and so the solution has been shown to be independent of y, but is given by some as-yet undetermined function of x. Often in perturbation methods, undetermined parts of solutions will get pinned down by conditions needed for consistency appearing at higher orders, or from matching to solutions on other parts of the domain. At order $O(\varepsilon^2)$ we have the problem for $u_1(x, y)$,

$$u_{1yy} = -u_{0xx}, \qquad u_{1y}(x, 0) = 0, \qquad u_{1y}(x, 1) = 0. \tag{8.24}$$

Substituting (8.23) for u_0 into this problem, we obtain the general solution

$$u_1 = -\tfrac{1}{2}C_2''(x)y^2 + C_3(x)y + C_4(x).$$

Applying the boundary condition at $y = 0$ determines $C_3(x) \equiv 0$; applying the boundary condition at $y = 1$ yields

$$\frac{d^2C_2}{dx^2} = 0 \qquad \text{for } 0 < x < 1; \tag{8.25}$$

this is an ordinary differential equation for the leading order solution. Such equations determining consistency conditions on lower-order solutions, coming out of parts of higher order problems, are often called *solvability conditions*.

Equation (8.25) has a simple linear general solution,

$$C_2(x) = D_1x + D_2,$$

but in order to determine the constants $D_{1,2}$, we need to impose boundary conditions. Our original problem indeed had boundary conditions at $x = 0$ and $x = 1$, (8.21c), but for general imposed functions $g_0(y)$ and $g_1(y)$, $C_2(x)$ cannot possibly satisfy those y-dependent conditions. Hence we must construct boundary layers to satisfy (8.21c) while determining the effective boundary conditions on $C_2(x)$.

Following the same scaling (8.15) as in the previous problem, we determine the inner distinguished limit $\alpha = 1$ for the boundary layers both at $x_*^L = 0$ and $x_*^R = 1$. We again recover Laplace's equation as the inner problem,

$$U_{XX} + U_{yy} = 0. \tag{8.26a}$$

Analysing the boundary layer at $x_*^L = 0$ (the analysis at $x_*^R = 1$ is analogous), the three boundary conditions we can draw from (8.21b, 8.21c) are straightforward,

$$U(0, y) = g_0(y), \qquad U_y(X, 0) = 0, \qquad U_y(X, 1) = 0, \tag{8.26b}$$

and the final boundary condition is obtained from asymptotic matching to the outer solution, using (8.16c),

$$U(X \to \infty, y) = D_2. \tag{8.26c}$$

Again, it is convenient to recast this problem in terms of the boundary layer correction in order to separate out the overlap from matching, so we write

$$U(X, y) = V(X, y) + D_2, \tag{8.27}$$

with $V(X, y)$ satisfying

$$V_{XX} + V_{yy} = 0, \tag{8.28a}$$

$$V_y(X,0)=0, \qquad V_y(X,1)=0, \tag{8.28b}$$
$$V(0,y)=g_0(y)-D_2, \qquad V(X\to\infty,y)=0. \tag{8.28c}$$

Applying separation of variables to this Neumann boundary value problem yields the solution as a cosine series,

$$V(X,y)=\sum_{n=0}^{\infty} c_n e^{-n\pi X}\cos(n\pi y), \tag{8.29}$$

where the matching condition eliminated the exponentially growing modes, $e^{n\pi X}$. The c_n's are then given by the Fourier coefficients of the $X=0$ boundary condition,

$$c_0=\int_0^1 [g_0(y)-D_2]\,dy, \quad \text{and} \quad c_n=2\int_0^1 [g_0(y)-D_2]\cos(n\pi y)\,dy \tag{8.30}$$

for $n=1,2,\ldots$. For $n>0$ all of the $c_n e^{-n\pi X}$ factors in (8.29) decay to zero as $X\to\infty$; any finite values for c_n would be compatible with the remaining boundary condition, that $V(X\to\infty)=0$. The only term that contributes to the value of V for $X\to\infty$ is the $n=0$ term, the uniform constant, $V(X\to\infty)\sim c_0$. In order to satisfy the matching condition (8.28c), we must have $c_0=0$ in (8.30). This determines the boundary condition on the outer solution (8.25) in terms of the average of the g_0 boundary function,

$$C_2(0)=D_2=\int_0^1 g_0(y)\,dy.$$

Analogous analysis of the boundary layer at $x_*=1$ yields the boundary condition for $C_2(1)$ in terms of the average value of g_1 and determines the outer solution as

$$u_0(x,y)=x\left(\int_0^1 g_1(y)-g_0(y)\,dy\right)+\left(\int_0^1 g_0(y)\,dy\right), \tag{8.31}$$

with the leading order uniform solution taking the same form as (8.20).

It is interesting to compare the 'information flow' or 'structural dependence' of the perturbation solution for the past two problems. For the Dirichlet problem, the outer solution can be determined independently of the boundary layers, and sets matching conditions for the boundary layers,

Problem (8.12): | BL ← Outer → BL |

while in the Neumann problem, the outer solution cannot be completely specified until the boundary layer solutions have been calculated,

$$\text{Problem (8.21):} \quad \boxed{\text{BL} \quad \rightarrow \quad \text{Outer} \quad \leftarrow \quad \text{BL}}$$

8.4 The Nonuniform Insulated Wire

We now extend our previous analysis to consider the problem of an insulated wire whose cross-section is not of constant width,[1] see Fig. 8.2. The main consequence of this change is that the domain is no longer separable[2] so the method of separation of variables from Sect. 8.1 can not be used to construct an exact solution of the whole problem. However, boundary layer theory and the method of matched asymptotics again works, with just minor extensions.

We begin with the dimensional problem for Laplace's equation on a domain with $0 \le \mathsf{X} \le \mathsf{L}$ and $0 \le \mathsf{Y} \le \mathsf{H} f(\mathsf{X}/\mathsf{L})$ where $f(x)$ is a given function describing the width of the wire,

$$\mathsf{U}_{\mathsf{XX}} + \mathsf{U}_{\mathsf{YY}} = 0 \tag{8.32a}$$

along with the same imposed-voltage Dirichlet boundary conditions at $\mathsf{X} = 0$ and $\mathsf{X} = \mathsf{L}$, but it is now notable that the lengths of the ends may differ

$$\mathsf{U}(0, \mathsf{Y}) = \bar{U} g_0(\mathsf{Y}/\mathsf{H}) \qquad \text{for } 0 \le \mathsf{Y} \le Hf(0), \tag{8.32b}$$

$$\mathsf{U}(\mathsf{L}, \mathsf{Y}) = \bar{U} g_1(\mathsf{Y}/\mathsf{H}) \qquad \text{for } 0 \le \mathsf{Y} \le Hf(1). \tag{8.32c}$$

The lower edge of the domain remains straight and the associated no-flux boundary condition corresponds to the first condition in (8.21b)

$$\mathsf{U}_{\mathsf{Y}}(\mathsf{X}, 0) = 0 \qquad \text{for} \quad 0 \le \mathsf{X} \le \mathsf{L}. \tag{8.32d}$$

We must be a little more careful in order to properly determine the form of the no-flux condition, $\hat{\mathbf{n}} \cdot \nabla \mathsf{U} = 0$ along the upper boundary. Recalling from multivariable calculus that the normal to a constant-value contour[3] of a function $\mathsf{B}(\mathsf{X}, \mathsf{Y})$ is given by the gradient of B, we construct $\mathsf{B}(\mathsf{X}, \mathsf{Y}) = \mathsf{Y} - \mathsf{H} f(\mathsf{X}/\mathsf{L})$. This defines our boundary as the zero-level contour of $\mathsf{B}(\mathsf{X}, \mathsf{Y})$. To obtain a unit normal vector, we calculate the normalised gradient:

[1] This is sometimes called a "domain perturbation" [61], also see Hinch [47].

[2] A separable domain being expressible as Cartesian products of intervals in independent variables, for example $(\mathsf{X} \in [0, \mathsf{L}]) \times (\mathsf{Y} \in [0, \mathsf{H}])$.

[3] Sometimes called a level set.

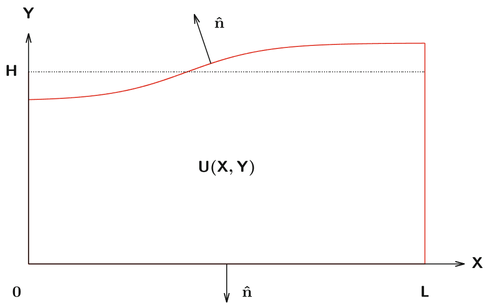

Fig. 8.2 The domain for the non-uniform insulated wire, problem (8.32)

$$\hat{\mathbf{n}} = \pm\frac{\nabla \mathsf{B}}{|\nabla \mathsf{B}|} = \pm\frac{(-(\mathsf{H}/\mathsf{L})f'(\mathsf{X}/\mathsf{L}), 1)}{\sqrt{1+(\mathsf{H}/\mathsf{L})^2 f'(\mathsf{X}/\mathsf{L})^2}},$$

where the $\pm$ sign is chosen so as to correspond to the unit outward normal, pointing out of the domain at the boundary (in this case $+$ for the upper boundary). Finally, taking the dot product with the gradient $\nabla \mathsf{U}$ yields the condition

$$-\frac{\mathsf{H}}{\mathsf{L}}f'(\mathsf{X}/\mathsf{L})\frac{\partial \mathsf{U}}{\partial \mathsf{X}}\bigg|_{(\mathsf{X},\mathsf{H}f(\mathsf{X}/\mathsf{L}))} + \frac{\partial \mathsf{U}}{\partial \mathsf{Y}}\bigg|_{(\mathsf{X},\mathsf{H}f(\mathsf{X}/\mathsf{L}))} = 0. \tag{8.32e}$$

Employing the previously used scalings (8.11) we arrive at the nondimensional problem on $0 \le x \le 1$ and $0 \le y \le f(x)$,

$$\varepsilon^2 u_{xx} + u_{yy} = 0, \tag{8.33a}$$

subject to the boundary conditions

$$u(0, y) = g_0(y) \qquad 0 \le y \le f(0), \tag{8.33b}$$

$$u(1, y) = g_1(y) \qquad 0 \le y \le f(1), \tag{8.33c}$$

and for $0 \le x \le 1$

$$u_y(x, 0) = 0, \qquad u_y(x, f(x)) - \varepsilon^2 f'(x) u_x(x, f(x)) = 0. \tag{8.33d}$$

Following as in Sect. 8.3, we begin by constructing the outer solution, and at leading order obtain the problem,

$$u_{0yy} = 0, \qquad u_{0y}(x, 0) = 0, \qquad u_{0y}(x, f(x)) = 0, \tag{8.34}$$

yielding $u_0(x, y) = C_2(x)$ as in (8.23). To determine $C_2(x)$ we look to the next order problem,

$$u_{0xx} + u_{1yy} = 0, \qquad u_{1y}(x, 0) = 0, \qquad u_{1y}(x, f) = f'(x)u_{0x}(x, f). \tag{8.35}$$

Substituting in for u_0, we find the general solution,

$$u_1(x, y) = -\tfrac{1}{2}C_2''(x)y^2 + C_3(x)y + C_4(x). \tag{8.36}$$

Imposing the $y = 0$ boundary condition determines $C_3 \equiv 0$. The top boundary condition at $y = f(x)$ then takes the form

$$u_{1y}(x, f(x)) = f'(x)C_2'(x);$$

equating this expression with the y-derivative of the general $u_1(x, y)$ solution given by (8.36) evaluated at $y = f$ yields the compatibility condition

$$-C_2''(x)f(x) = f'(x)C_2'(x).$$

Using the product rule, this result can be expressed more compactly as

$$\frac{d}{dx}\left(f(x)\frac{dC_2}{dx}\right) = 0; \tag{8.37}$$

this is a generalisation of (8.25) for non-constant $f(x)$. Boundary conditions for (8.37) are then determined by solving for the boundary layers at $x_*^L = 0$ and $x_*^R = 1$ using separation of variables as was done in Sect. 8.3 to construct the uniform solution on the whole domain (8.20).

8.5 Further Directions

The main asymptotic reduction considered in this chapter is based on having the aspect ratio as the limiting small parameter, $\varepsilon = H/L \to 0$. Stemming from the historical uses of this approach in the context of many problems in fluid dynamics [1, 61] (involving the flow of oil between machine parts, and other thin layers like tears films on the eye), these types of problems are sometimes called *lubrication models* [51, 79, 96]. They offer valuable simplifications to problems by reducing systems to lower-dimensional domains (in our case from a PDE on a two-dimensional region to an ODE on a one-dimensional interval) by separating out the "thin" directions to leave an underlying governing model on the "wide" spatial directions with the appropriate boundary conditions incorporated. These problems are also sometimes called *slender body asymptotics*, and *reduced-dimension models*.

8.6 Exercises

8.1 Consider the problem for Laplace's equation on $0 \le X \le L$ and $0 \le Y \le H$,

$$U_{XX} + U_{YY} = 0,$$

with boundary conditions

$$U(0, Y) = 0, \qquad U(L, Y) = 0 \qquad \text{for} \quad 0 \le Y \le H,$$
$$U(X, 0) = 0, \qquad \left.\frac{\partial U}{\partial Y}\right|_{Y=H} = e^{-3X/L} \quad \text{for} \quad 0 \le X \le L.$$

(a) Use separation of variables to construct the exact solution $U(X, Y)$ (valid for any values of $H, L > 0$).
(b) Nondimensionalize the problem and consider the slender limit of $\varepsilon = H/L \to 0$ to find the leading order outer solution $u_0(x, y)$.
(c) Rescale the solution from (a) by the natural lengthscales and show that it is equivalent to the solution from (b) in the limit $\varepsilon \to 0$.

8.2 Consider the problem for Laplace's equation on $0 \le X \le L$ and $0 \le Y \le H$ in the limit that $H \to 0$ with $L = O(1)$,

$$U_{XX} + U_{YY} = 0,$$

with boundary conditions

$$U(0, Y) = \sin\left(\frac{\pi Y}{2H}\right), \qquad U(L, Y) = -(Y/H)^2 \qquad \text{for} \quad 0 \le Y \le H,$$
$$U(X, 0) = \sin\left(\frac{5\pi X}{L}\right), \qquad U(X, H) = \cos\left(\frac{3\pi X}{L}\right) \qquad \text{for} \quad 0 \le X \le L.$$

(a) Determine the leading-order outer solution.
(b) Determine the boundary layer corrections for the left and right leading-order inner solutions.
(c) Determine the leading order uniformly valid solution.
(d) Use (c) to show that the boundary layer is crucial for calculating the leading order value of the average flux at the right end of the domain:

$$J^R = \frac{1}{H} \int_0^H \left(\left.\frac{\partial U}{\partial X}\right|_{X=L} \right) dY.$$

Hint: Your solution will involve a sum given by the Riemann zeta function, $\sum_{k=0}^{\infty} 1/(2k+1)^3 = \frac{7}{8}\zeta(3) \approx 1.0518$.

8.3 Consider the problem for Laplace's equation on the non-uniform slender domain, $0 \le \mathsf{X} \le \mathsf{L}$ and $0 \le \mathsf{Y} \le F(\mathsf{X})$ where $F(\mathsf{X}) = 15 + 5\cos(3\pi \mathsf{X}/\mathsf{L})$ in the limit $\mathsf{L} \to \infty$ with $\mathsf{H} = 1$,

$$\mathsf{U}_{\mathsf{XX}} + \mathsf{U}_{\mathsf{YY}} = 0,$$

with boundary conditions

$$\mathsf{U}(0,\mathsf{Y}) = -\tfrac{1}{100}\mathsf{Y}^3, \qquad \mathsf{U}(\mathsf{L},\mathsf{Y}) = 3\mathsf{Y}^2 \qquad \text{for} \quad 0 \le \mathsf{Y} \le F(\mathsf{X}),$$

$$\left.\frac{\partial \mathsf{U}}{\partial \mathsf{Y}}\right|_{\mathsf{Y}=0} = 0, \qquad \left.\hat{\mathbf{n}}\cdot\nabla\mathsf{U}\right|_{\mathsf{Y}=F(\mathsf{X})} = 0 \qquad \text{for} \quad 0 \le \mathsf{X} \le \mathsf{L}.$$

(a) Nondimensionalize the problem.
(b) Write the equations for the outer solution up to $O(\varepsilon^2)$. Determine the first two terms in the expansion of the outer solution.
(c) Determine the boundary layer corrections for the left and right leading-order inner solutions. Hint: What is y^{BL} in $0 \le y \le y^{\mathrm{BL}}$ in each boundary layer?
(d) Use matching to determine the x-boundary conditions for the outer solution and then obtain the leading-order outer solution.

8.4 Consider the problem for Laplace's equation for a curved semicircular arc of wire of varying width on $0 \le \theta \le \pi$ with $\bar{\mathsf{R}} - \mathsf{W}f(\theta) \le \mathsf{R} \le \bar{\mathsf{R}} + \mathsf{W}f(\theta)$ in the limit of a large radius of curvature, $\bar{\mathsf{R}} \to \infty$,

$$\frac{\partial^2 \mathsf{U}}{\partial \mathsf{R}^2} + \frac{1}{\mathsf{R}}\frac{\partial \mathsf{U}}{\partial \mathsf{R}} + \frac{1}{\mathsf{R}^2}\frac{\partial^2 \mathsf{U}}{\partial \theta^2} = 0,$$

with boundary conditions

$$\left.\hat{\mathbf{n}}\cdot\nabla\mathsf{U}\right|_{\mathsf{R}=\bar{\mathsf{R}}-\mathsf{W}f(\theta)} = 0, \qquad \left.\hat{\mathbf{n}}\cdot\nabla\mathsf{U}\right|_{\mathsf{R}=\bar{\mathsf{R}}+\mathsf{W}f(\theta)} = 0 \qquad \text{for} \quad 0 \le \theta \le \pi,$$

$$\mathsf{U}(\mathsf{R},0) = \bar{\mathsf{U}}g_0(\mathsf{R}/\bar{\mathsf{R}}), \qquad \mathsf{U}(\mathsf{R},\pi) = \bar{\mathsf{U}}g_1(\mathsf{R}/\bar{\mathsf{R}}) \qquad \text{for} \quad 0 \le \mathsf{X} \le \mathsf{L}.$$

(a) Nondimensionalize the problem using $\mathsf{R} = \bar{\mathsf{R}}r$, but show that this does not yield a scaled problem with a small parameter.
(b) Let $r = 1 + \varepsilon y$. What should be used for the aspect ratio ε? Write the complete problem satisfied by $u(y,\theta)$.
(c) Write the problems for the outer solution, $u(y,\theta) \sim u_0(y,\theta) + \varepsilon u_1(y,\theta) + \varepsilon^2 u_2(y,\theta)$.

8.5 The derivation of the *Korteweg de Vries* (KdV) equation for waves on the surface of shallow layers of water follows from similar asymptotic reductions of Laplace's equation with appropriate boundary conditions [31, 58, 80]. Consider the following nondimensionalized problem for a time-dependent potential $\phi(x,y,t)$ and the shape of waves on the top-surface of a fluid layer, $F(x,t)$, on $-\infty < x < \infty$,

$$\varepsilon^2\phi_{xx} + \phi_{yy} = 0 \quad \text{on } 0 \le y \le 1 + \varepsilon^2 F \tag{8.38a}$$

$$\phi_y = 0 \quad \text{at } y = 0 \tag{8.38b}$$

$$\phi_t + \tfrac{1}{2}(\varepsilon^2\phi_x^2 + \phi_y^2) + F = 0 \quad \text{at } y = 1 + \varepsilon^2 F \tag{8.38c}$$

$$\varepsilon^2 F_t + \varepsilon^4\phi_x F_x = \phi_y \quad \text{at } y = 1 + \varepsilon^2 F \tag{8.38d}$$

The first steps in the derivation are:

(a) Expand $\phi = \phi_0 + \varepsilon^2\phi_1 + \varepsilon^4\phi_2 + O(\varepsilon^6)$ and $F = F_0 + \varepsilon^2 F_1 + O(\varepsilon^4)$ for $\varepsilon \to 0$ and use (8.38a, 8.38b) to determine ϕ_0, ϕ_1, ϕ_2 in terms of polynomials in y and three un-determined functions $C_0(x,t)$, $C_1, (x,t)$, $C_2(x,t)$.
(b) Determine the leading order equations for C_0, f_0 obtained from substituting the expansions for ϕ, F into (8.38c, 8.38d).
(c) Determine the equations for the $O(\varepsilon^2)$ next-order equations obtained from (8.38c, 8.38d). (To be concluded in Exercise 9.12.)

8.6 The derivation of the *porous medium equation*, introduced earlier as Eq. (5.21),

$$\frac{\partial h}{\partial t} = \frac{1}{3}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right),$$

follows from a long-wave analysis for the evolution of the height of a layer of a viscous fluid $y = h(x,t)$ [1, 79]. Consider the $\varepsilon \to 0$ limit of the problem

$$\varepsilon^2 u_{xx} + u_{yy} = -h_x \quad \text{on } 0 \le y \le h \tag{8.39a}$$

$$u_x + v_y = 0 \quad \text{on } 0 \le y \le h \tag{8.39b}$$

$$u = v = 0 \quad \text{at } y = 0 \tag{8.39c}$$

$$u_y = 0 \quad \text{at } y = h \tag{8.39d}$$

$$h_t + u h_x = v \quad \text{at } y = h \tag{8.39e}$$

where $u(x,y,t)$, $v(x,y,t)$ are the horizontal and vertical components of the fluid velocity respectively and the last three equations are boundary conditions on the first two.

(a) Show that by using (8.39b, 8.39c) to express v in terms of u, Eq. (8.39e) can be written as

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^h u\,dy\right) = 0. \tag{8.40}$$

(b) Determine the leading order horizontal velocity, $u_0(x,y,t)$ in terms of h from (8.39a, 8.39c, 8.39d) to then obtain the porous medium equation from (8.40).

8.7 Note that in all of the problems considered here we assumed a perturbation expansion for the outer solution given in powers of ε^2, (8.13). Sometimes this must be modified. Consider problem (8.12) with the Dirichlet condition $u(x, 1) = x$ on the top boundary. Examine the expansion of the boundary layers to show that if we wish to get a solution that is uniformly valid to $O(\varepsilon^2)$ then the expansion $u \sim u_0 + \varepsilon u_1 + \varepsilon^2 u_2$ is needed.