Chapter 6
Perturbation Methods

As we described in Chap. 4, physical problems can always be scaled and restated as dimensionless models. The scaling process identifies the relative importance of different physical effects in terms of the magnitudes of the dimensionless parameters that appear. In the absence of actual parameter values, problem-specific analytical and/or numerical methods are typically necessary to make progress towards a general solution. However, if any dimensionless parameters are known to be relatively large or small, then so-called *perturbation methods* can often be employed in order to generate accurate approximations to the solution.

Perturbation methods provide a systematic approach to constructing approximate solutions to equations such as

$$F(x, \varepsilon) = 0, \qquad \mathbf{G}(x, y, \varepsilon) = \mathbf{0}, \qquad \frac{dx}{dt} = H(x, \varepsilon), \quad \dots, \qquad (6.1)$$

in the limit of a vanishing small *perturbation parameter*, $\varepsilon \to 0$. This is accomplished through the introduction of *asymptotic expansions*, whereby the original problem is decomposed into an ordered sequence of simpler sub-problems. The solutions of the sub-problems are then recombined to form an approximate solution to the full original problem. This may sound very similar to superposition principles that are often used to construct solutions of linear ODE and PDE, but a notable difference here is that the original equation (algebraic, ODE or PDE) may be nonlinear.

6.1 Asymptotic Analysis: Concepts and Notation

Asymptotic analysis provides the mathematical framework that justifies perturbation methods. The term *asymptotic* implies a *limit* process and hence every asymptotic result must be given in terms of a stated parameter approaching a limiting value; for example: $e^{-x} \to 1$ as $x \to 0$ or $e^{-x} \to 0$ as $x \to +\infty$.

Two functions $f(\varepsilon)$ and $g(\varepsilon)$ are said to be *asymptotically equivalent* in the limit $\varepsilon \to 0$, if

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 1. \tag{6.2}$$

This relationship is often written more compactly as "$f \sim g$ as $\varepsilon \to 0$". It is important to note that this definition does not simply just mean that f and g have the same limiting values; for functions that approach infinity or vanish in the limit, equivalence states that they have the same limiting rate of growth or decay. For example, in the limit $\varepsilon \to 0$

$$\frac{1}{\varepsilon} + 1 \sim \frac{1}{\varepsilon}, \qquad \tan\left(\frac{\pi}{2} - \varepsilon\right) \sim \frac{1}{\varepsilon}, \qquad \sin(\varepsilon) \sim \varepsilon, \qquad \frac{1}{1 - \varepsilon x} \sim 1 + \varepsilon x,$$

but $\sin(\varepsilon) \nsim \varepsilon^2$ despite each vanishing in the limit. We therefore see that asymptotic equivalence does <u>not</u> imply equality of the functions, but it is a necessary condition for them to have the same limiting behaviour. As a consequence, asymptotic equivalence is not a unique relationship (it effectively defines *equivalence classes* of functions). For example, $\cos(\sqrt{\varepsilon}) \sim (1 - \varepsilon/2) \sim e^{\varepsilon} \sim (1 + \varepsilon)$ as $\varepsilon \to 0$.

A related description of asymptotic behaviour is given by the *order relation*, "$f = O(g)$ as $\varepsilon \to 0$" (read as "f is big-Oh of g") defined by

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = A, \tag{6.3}$$

where A is finite. The statement $f = O(g)$, identifies the function $f(\varepsilon)$ as having comparable growth or decay as $g(\varepsilon)$ as $\varepsilon \to 0$. For example, $\sin(\varepsilon) = O(4\varepsilon)$ as $\varepsilon \to 0$, and $N! = O(N^{N+1/2}e^{-N})$ as $N \to \infty$. The latter result is known as Stirling's formula [13], a central result for estimating large combinatorial values in computing and probability applications. The notation $O(1)$ is often used to describe expressions having finite limiting values and separates quantities that are singular ($f \to \infty$ as in $f = O(\varepsilon^{-1})$) from those that vanish ($f \to 0$ as in $f = O(\varepsilon^2)$) as $\varepsilon \to 0$.

There is also a "little oh" relation, "$f = o(g)$" as $\varepsilon \to 0$ (also written as "$f \ll g$") describing f being *asymptotically smaller than* g if

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0. \tag{6.4}$$

The $o(\cdot)$ notation is used to indicate weak effects due to smaller ("*higher order*") terms that can be neglected in comparison with other larger terms. Since, for example, $\varepsilon^2 = o(\varepsilon)$ we may then write $2\varepsilon + 3\varepsilon^2 \sim 2\varepsilon$. The $o(\cdot)$ notation is also often used

to state that the difference between two results is smaller than a specified reference quantity, as in

$$e^\varepsilon - \left(1 + \varepsilon + \tfrac{1}{2}\varepsilon^2\right) = o(\varepsilon^2) \quad \Longrightarrow \quad e^\varepsilon = 1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 + o(\varepsilon^2). \qquad (6.5)$$

We note that our descriptions of these asymptotic relations in terms of limits of ratios are actually convenient simplifications of more rigorous definitions. In particular, $f = O(g)$ if $|f| \le A|g|$ for all sufficiently small ε [13, 72]. The simplifications are sufficient for most cases, but can fail if the limit is not defined. For example, it is true that $\sin(x) = O(1)$ as $x \to \infty$ since $|\sin(x)|$ is bounded by one, even though is does not have a limit. Asymptotic equivalence is more generally defined along the lines of (6.5), in terms of the difference between the functions being small, $f \sim g$ if $f - g = o(g)$.

6.2 Asymptotic Expansions

Having established the order notation, we can now describe the structure of basic asymptotic expansions. Consider a function $x(t, \varepsilon)$ that can be expanded in terms of a "separation of variables"-type series

$$x(t, \varepsilon) = \delta_0(\varepsilon)x_0(t) + \delta_1(\varepsilon)x_1(t) + \delta_2(\varepsilon)x_2(t) + \cdots, \qquad (6.6)$$

where it is assumed that all $x_n = O(1)$ and the *gauge functions* $\{\delta_n(\varepsilon)\}$ are asymptotically ordered in size as $\varepsilon \to 0$ so that

$$\delta_0(\varepsilon) \gg \delta_1(\varepsilon) \gg \delta_2(\varepsilon) \gg \cdots. \qquad (6.7)$$

Equations (6.6) and (6.7) define the basic form of an *asymptotic expansion (AE)* and, in particular, (6.6) separates the dependence on the asymptotic parameter ε in each term from the coefficients x_n (which may, in general, also be functions of other independent variables and parameters in the model). The first term in the expansion (6.6) is usually referred to as the *leading order term*, $x \sim \delta_0 x_0$.

There are subtle differences in the way one might write the basic content of the first few terms of an asymptotic expansion based on our previous discussion of asymptotic ordering. Consider

$$\begin{aligned} x &\sim \delta_0 x_0 + \delta_1 x_1 + \delta_2 x_2, \\ x &= \delta_0 x_0 + \delta_1 x_1 + O(\delta_2), \\ x &= \delta_0 x_0 + \delta_1 x_1 + o(\delta_1). \end{aligned} \qquad (6.8)$$

The first expression gives the first three terms explicitly, the second gives two terms and an estimate on the asymptotic order of the remainder of the expansion, and the third expression does not predict δ_2 but just states that the omitted contributions are all smaller than the last given $O(\delta_1)$ term which thus effectively provides only the information contained in the two term estimate $x \sim \delta_0 x_0 + \delta_1 x_1$.

A simple example of an asymptotic expansion is provided by the Taylor series of a smooth function $x(t)$ expanded in a neighbourhood of a point t_*

$$x(t) = x(t_*) + \varepsilon x'(t_*) + \tfrac{1}{2!}\varepsilon^2 x''(t_*) + \tfrac{1}{3!}\varepsilon^3 x'''(t_*) + \cdots , \qquad (6.9)$$

where $\varepsilon = t - t_*$ (cf. (6.6)); in this case, the small parameter is ε, the separation between the fixed point t_* and the variable t. We know from calculus that in the limit $\varepsilon \to 0$, an accurate estimate of the value of $x(t)$ in a neighbourhood of t_* can be obtained from a limited number of terms in the expansion.

For some functions, the asymptotic expansion can be handled by symbolic algebra programs (such as `Maple` or `Mathematica`). In `Maple`, for example, to calculate the first six terms in the expansion of $x(t) = e^{\tan t}$ for $t \to 0$ (here t is the small parameter), the command is simply

```
> series(exp(tan(t)),t=0);
```

$$1 + t + \frac{1}{2}t^2 + \frac{1}{2}t^3 + \frac{3}{8}t^4 + \frac{37}{120}t^5 + O(t^6).$$

`Maple` can also generate asymptotic expansions in other limits, such as for $x(t) = 1/(1 + 4t)$ as $t \to \infty$

```
> asympt(1/(1+4*t),t);
```

$$\frac{1}{4}t^{-1} - \frac{1}{16}t^{-2} + \frac{1}{64}t^{-3} - \frac{1}{256}t^{-4} + \frac{1}{1024}t^{-5} + O\left(t^{-6}\right),$$

where the gauge functions are now inverse powers of t; this result can also be derived as a Taylor series expansion with respect to the variable $t = 1/\varepsilon$ in the limit $\varepsilon \to 0$.

We note that asymptotic expansions can take more complex forms than (6.6). For example, the function $x(t, \varepsilon) = \exp(\sin(\varepsilon\sqrt{t})/\varepsilon^2)$, has the asymptotic expansion

$$x \sim e^{\sqrt{t}/\varepsilon}\left(1 - \frac{1}{6}\varepsilon t^{3/2} + \frac{1}{72}\varepsilon^2 t^3 + \cdots\right) \quad \text{as } \varepsilon \to 0, \qquad (6.10)$$

which is clearly not of the form (6.6). Our investigations of solutions to equations like (6.1) will be further complicated by the fact that the form of the solution is not known beforehand and so we will need to find the gauge functions as well as the coefficients.

6.2.1 Divergence of Asymptotic Expansions

Example (6.10) indicates that asymptotic expansions exist even in cases where convergent Taylor series do not. In fact, summed over all terms, asymptotic expansions

can diverge, but, as we will shall show, this does not prevent partial sums like (6.8) from providing accurate results in the vanishing limit of the perturbation parameter.

In order to illustrate the above points, we consider an example of a function defined in terms of an integral, namely

$$I(\varepsilon) = \int_0^\infty \frac{e^{-t}}{1 + \varepsilon t} \, dt; \tag{6.11}$$

such a definition is rather common and includes Fourier and Laplace integrals, as well as Mellin, Hankel and many other integral transformations, and a number of special functions, such as the Gamma, Beta, Airy and error functions [11, 13, 29, 56].

The Taylor series (6.9) of the integrand in the limit $\varepsilon \to 0$ leads to the asymptotic expansion

$$I(\varepsilon) \sim 1 - \varepsilon + 2\varepsilon^2 - 6\varepsilon^3 + \cdots = \sum_{n=0}^\infty (-1)^n n! \varepsilon^n. \tag{6.12}$$

The standard ratio test for power series requires $(n+1)\varepsilon < 1$, so we find that the series has a zero radius of convergence (i.e. it diverges for all $|\varepsilon| > 0$ at sufficiently large values of n). Furthermore, convergence for ε in an interval around zero, $-\delta < \varepsilon < \delta$, should not have been expected since for $\varepsilon < 0$ the singularity at $t = -1/\varepsilon$ is not integrable. Despite these issues, Fig. 6.1 shows that (6.12) truncated at a finite number of terms provides a good estimate of the value of the integral as $\varepsilon \to 0^+$.

To understand this behaviour, consider (6.11) with $\varepsilon = \frac{1}{10}$; the integral can be numerically evaluated to give $I(0.1) \approx 0.915633$. For (6.12), the terms in the expansion remain asymptotically ordered while $(n+1)\varepsilon < 1$, namely for $n < 10$. Recalling that the first term neglected in the asymptotic expansion gives an estimate of the error (cf. (6.8)), we see that truncating the expansion up to and including ten terms will give a decreasing magnitude for the error, but going beyond ten will lead
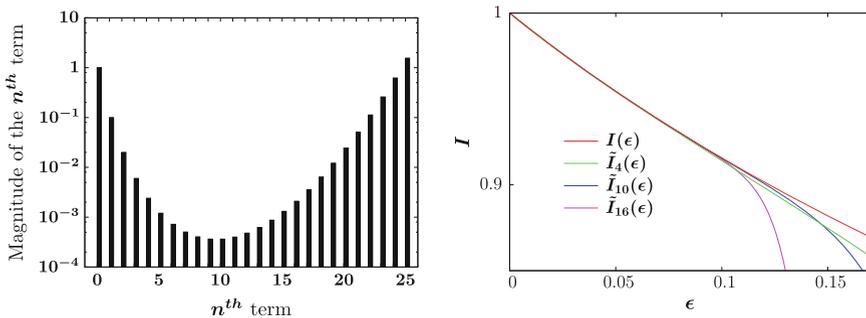

Fig. 6.1 (*Left*) The magnitude of term a_n in (6.12) for $\varepsilon = 0.1$. For $n < 10$, the terms decay in size, but thereafter start to grow. (*Right*) $I(\varepsilon)$ from (6.11) compared with the partial sums, $\tilde{I}_N(\varepsilon)$, of series (6.12) showing the error growing for $\varepsilon \geq 0.1$ for above the optimal truncation

to the error increasing in size (see Fig. 6.1). This is in stark contrast to convergent series, where retaining more terms always reduces the error.

To summarise, asymptotic expansions are always accurate in the limit $\varepsilon \to 0$, but for finite ε, there will be an *optimal truncation $n = 0, 1, \ldots, N(\varepsilon)$*, that minimises the error. In most cases, however, even just the first few terms from the asymptotic expansion can yield an excellent approximation to the true solution, as illustrated by $\tilde{I}_4(\varepsilon)$ in Fig. 6.1. Complications stemming from attempting to interchange limits ($\varepsilon \to 0$ and $n \to \infty$) will appear in several situations and often signal subtle changes of behaviour in the asymptotic expansion.

6.3 The Calculation of Asymptotic Expansions

Rather than calculating asymptotic expansions of given functions, we are usually more interested in constructing an asymptotic expansion of solutions to problems of forms like (6.1).

We begin by calculating asymptotic expansions of solutions to algebraic equations, for which the coefficients $\{x_n\}_{n=0,1,2,\ldots}$ in (6.6) are necessarily constants. It will be useful to classify solutions (and the corresponding asymptotic expansions) in the limit $\varepsilon \to 0$ into the following types:

- *Regular solutions*, which have finite limits: $\lim_{\varepsilon \to 0} x = x_0$ with the leading order gauge function in (6.6) being $\delta_0 \equiv 1$.
- *Vanishing solutions*, which are regular solutions with a zero limit, i.e. $\lim_{\varepsilon \to 0} x = 0$ with the leading order gauge function satisfying $\delta_0(\varepsilon) \ll 1$.
- *Singular solutions*, which have divergent limits: $\lim_{\varepsilon \to 0} |x| = \infty$, corresponding to a finite x_0 with a singular $\delta_0(\varepsilon) \gg 1$.

We now introduce two approaches for constructing solutions in the form of asymptotic expansions known as the *expansion method* and the *iteration method*. We will illustrate both methods applied to a simple example for which the solutions can be found explicitly.

Consider the quadratic equation

$$x^2 - x + \tfrac{1}{4}\varepsilon = 0 \qquad \text{as } \varepsilon \to 0, \tag{6.13}$$

its exact solutions being

$$x_{A,B} = \frac{1 \pm \sqrt{1 - \varepsilon}}{2}. \tag{6.14}$$

Employing the *generalised binomial expansion* for $f(z) = (1 + z)^r$, which can be derived by application of Taylor series,

$$(1 + z)^r \sim 1 + rz + \frac{r(r-1)}{2}z^2 + \frac{r(r-1)(r-2)}{3!}z^3 + \cdots \qquad \text{as } z \to 0, \qquad (6.15)$$

with $r = \frac{1}{2}$, $z = -\varepsilon$, allows us to write the two solutions as

$$\begin{aligned} x_A &= 1 - \tfrac{1}{4}\varepsilon - \tfrac{1}{16}\varepsilon^2 + \cdots = O(1), \\ x_B &= 0 + \tfrac{1}{4}\varepsilon + \tfrac{1}{16}\varepsilon^2 + \cdots = O(\varepsilon). \end{aligned} \qquad (6.16)$$

We note that taking $\varepsilon \ll 1$ directly in (6.13) gives $x^2 - x \approx 0$ yielding approximate solutions $x \approx 0, 1$. In other words, the balance of the first two terms in (6.13) is important in locating the roots, with the third term only slightly adjusting their values (cf. (6.16)). This is the hallmark of a *regular perturbation problem*, for which the limit $\varepsilon \to 0$ only yields regular (i.e. non-singular) solutions with well-defined limits for $\varepsilon \to 0$.

We will begin our analysis with the more concise 'expansion method', which benefits from an a priori assumption on the form for the asymptotic expansion being sought.

6.3.1 The Expansion Method

In order to solve (6.13) without explicit reference to the exact solutions, we assume the solutions are regular and have $\delta_n = \varepsilon^n$ for $n = 0, 1, 2, \ldots$ (i.e. the sequence of gauge functions $\{\delta_n(\varepsilon)\}$ is already known), so that

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots. \qquad (6.17)$$

This is a commonly occurring asymptotic expansion for regular solutions, which can be thought of as being a Taylor series expansion of the solution with respect to the parameter ε around $\varepsilon = 0$.

Substituting (6.17) into Eq. (6.13) yields

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 - (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + \frac{1}{4}\varepsilon = 0,$$

and ordering terms in powers of ε yields

$$\left(x_0^2 - x_0\right) + \varepsilon\left(2x_1 x_0 - x_1 + \frac{1}{4}\right) + \varepsilon^2(x_1^2 + 2x_0 x_2 - x_2) + \cdots = 0 + \varepsilon 0 + \varepsilon^2 0 + \cdots.$$

Assuming that the coefficients are $O(1)$, requiring both sides of the equation to balance leads to the system of equations

$$
\begin{array}{llll}
O(\varepsilon^0) : x_0^2 - x_0 = 0 & \Rightarrow & x_0 = 1 & \text{or } x_0 = 0, \\
O(\varepsilon^1) : 2x_1x_0 - x_1 + \frac{1}{4} = 0 & \Rightarrow & x_1 = -1/4 \mid & x_1 = 1/4, \\
O(\varepsilon^2) : x_1^2 + 2x_0x_2 - x_2 = 0 & \Rightarrow & x_2 = -1/16 \mid & x_2 = 1/16,
\end{array}
\tag{6.18}
$$

and so on for higher orders. Note that only the leading order equation is nonlinear; subsequent corrections depend on which of the leading order solutions is considered. Comparing the coefficients resulting from (6.18) with (6.16), we observe that we have obtained the asymptotic expansions for both solutions of (6.13).

6.3.2 The Iteration Method

In contrast to the expansion method, we no longer assume a form for the entire asymptotic expansion, but instead only look (initially) at the leading order term $x \sim x_0\delta_0(\varepsilon)$, with both x_0 and $\delta_0(\varepsilon)$ to be determined. There are two fundamental assumptions made in relation to the leading order term:

 (i) For every nontrivial solution (i.e. not all $x_n = 0$), the leading order term must be nontrivial: $x_0 \neq 0$ and $\delta_0 \neq 0$.[1]
(ii) The leading order coefficient is finite, $x_0 = O(1)$ as $\varepsilon \to 0$ (i.e. we are looking for regular solutions; we tackle singular solutions later).

Substituting the leading order term into (6.13) yields

$$
\underbrace{x_0^2\delta_0^2}_{(1)} - \underbrace{x_0\delta_0}_{(2)} + \underbrace{\tfrac{1}{4}\varepsilon}_{(3)} = 0.
\tag{6.19}
$$

In order for this equation to hold as $\varepsilon \to 0$, at least two of the terms must balance in asymptotic scales, with the remaining terms being *sub-dominant* (i.e. asymptotically smaller than the retained terms as $\varepsilon \to 0$). The smaller terms can then be neglected in determining the leading order solution. The set of dominant terms that balance to yield the leading order solutions are called the *dominant terms* and this argument is referred to as the *principle of dominant balance*.

Ignoring all $O(1)$ coefficients for the time-being, consider the three possibilities for balancing the asymptotic gauges of the potential dominant terms in (6.19):

$$
\begin{array}{llll}
\text{(a) Terms (1, 2): } \delta_0^2 = \delta_0 & \Rightarrow & \delta_0 = 1 \\
\text{(b) Terms (1, 3): } \delta_0^2 = \varepsilon & \Rightarrow & \delta_0 = \sqrt{\varepsilon} \\
\text{(c) Terms (2, 3): } \delta_0 = \varepsilon & \Rightarrow & \delta_0 = \varepsilon
\end{array}
\tag{6.20}
$$

---

[1]Trivial solutions ($x \equiv 0$) are exact solutions only if they satisfy the full problem for all ε.

In case (a), $\delta_0 = 0$ is excluded by the first fundamental assumption (that δ_0 should be non-zero). It is also clear that for this problem, all three terms cannot be of the same asymptotic order.

For each of the proposed balances, we must now attempt to verify the principle of dominant balance. Namely, it is *essential* to confirm that all sub-dominant terms which have been omitted from the dominant balance are indeed smaller than the dominant terms. When this occurs, the balance is called a *distinguished limit*.

For the three possible balances in (6.20) we see that

(a) Terms $(1, 2) : \delta_0^2 = \delta_0 = O(1)$ $\qquad \gg \qquad$ Term $(3) : epsilon = O(\varepsilon)$
(b) Terms $(1, 3) : \delta_0^2 = \varepsilon = O(\varepsilon)$ $\qquad \not\gg \qquad$ Term $(2) : \delta_0 = O(\sqrt{\varepsilon})$
(c) Terms $(2, 3) : \delta_0 = \varepsilon = O(\varepsilon)$ $\qquad \gg \qquad$ Term $(1) : \delta_0^2 = O(\varepsilon^2)$

In case (b), we see that the second term is not sub-dominant to the first and third terms, and so is not a valid balance. This leaves cases (a) and (c) as valid dominant balances leading to distinguished limits.

The coefficients in each case can now be obtained:

(a) $\underline{\delta_0 = 1}$: Eq. (6.19) becomes $x_0^2 - x_0 + \frac{1}{4}\varepsilon = 0$. Thus, as $\varepsilon \to 0$, the leading order dominant balance determines the leading order coefficient from

$$x_0^2 - x_0 = 0 \qquad \Rightarrow \qquad x_0 = 1,$$

where we reject the root $x_0 = 0$ in line with assumption (i) above. We see that $x_0 = 1$ corresponds to the first term in x_A from (6.16).

(c) $\underline{\delta_0 = \varepsilon}$: Eq. (6.19) takes the form $\varepsilon x_0^2 - x_0 + \frac{1}{4} = 0$ (where we have divided through by the common factor ε). The rescaled leading order equation yields $x_0 = 1/4$, which together with $\delta_0 = \varepsilon$ determines the first term in $x_B \sim \frac{1}{4}\varepsilon$ from (6.16).

Further terms in the expansion of each solution can be obtained by repeating the process: determining the distinguished limits for the gauge functions $\delta_i(\varepsilon)$ and the values for the coefficients x_i using

$$x_A \sim 1 + \delta_{1A} x_{1A} \qquad \text{and} \quad x_B \sim \frac{1}{4}\varepsilon + \delta_{1B} x_{1B},$$
$$x_A \sim 1 + \delta_{1A} x_{1A} + \delta_{2A} x_{2A} \quad \text{and} \quad x_B \sim \frac{1}{4}\varepsilon + \delta_{1B} x_{1B} + \delta_{2A} x_{2A},$$

and so on, substituting into (6.13) at each step and requiring that the gauge functions be asymptotically ordered, i.e. $1 \gg \delta_{1A} \gg \delta_{2A}$ and $\varepsilon \gg \delta_{1B} \gg \delta_{2B}$. In this manner, we can obtain the expansions of x_A and x_B from (6.16).

Obviously, as more terms are retained in the expansion, the determination of the dominant balance requires an increasing amount of work and this can be a substantial drawback of the iterative approach. Consequently, the expansion method is often favoured in obtaining a rapid result. However, while both roots for this example could be obtained using the expansion (6.17), we will encounter many situations for which the principle of dominant balance will be particularly useful.

6.3.3 Further Examples

Suppose we wish to find the roots of the transcendental equation

$$x^2 - 2x + \varepsilon \sin x = 0 \quad \text{as} \quad \varepsilon \to 0. \tag{6.21}$$

This equation does not admit explicit expressions for all roots, but it can be seen that $x = 0$ is a solution for any ε. Setting $\varepsilon = 0$ in (6.21) gives the leading order equation

$$x_0^2 - 2x_0 = 0 \quad \Rightarrow \quad x_0 = 0 \text{ or } x_0 = 2. \tag{6.22}$$

We will focus on the expansion for the nontrivial root, see Fig. 6.2.

While we may again proceed by hand, we take this opportunity to demonstrate how symbolic algebra software can be employed to perform some of the computationally intensive calculations. Using Maple, we write
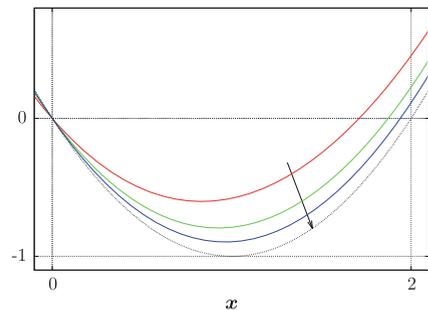
```
>   eps:=epsilon;
```
$$eps := \varepsilon$$
```
>   eq:=x^2-2*x+eps*sin(x);
```
$$eq := x^2 - 2x + \varepsilon \sin(x)$$
```
>   x:=x0+eps*x1+eps^2*x2+eps^3*x3;
```
$$x := x0 + \varepsilon\, x1 + \varepsilon^2 x2 + \varepsilon^3 x3$$
```
>   eqser:=series(eq,eps=0,3);
```
$$eqser := x0^2 - 2x0 + (2x0\,x1 - 2x1 + \sin(x0))\,\varepsilon$$
$$+ \left(2x0\,x2 + x1^2 - 2x2 + \cos(x0)\,x1\right)\varepsilon^2 + O\left(\varepsilon^3\right)$$
```
>   order0:=coeff(eqser,eps,0);
```
$$order0 := x0^2 - 2x0$$
```
>   order1:=coeff(eqser,eps,1);
```
$$order1 := 2x0\,x1 - 2x1 + \sin(x0)$$
```
>   order2:=coeff(eqser,eps,2);
```
$$order2 := 2x0\,x2 + x1^2 - 2x2 + \cos(x0)\,x1$$

Fig. 6.2 Convergence of the function $f(x, \varepsilon) = x^2 - 2x + \varepsilon \sin x$ for $\varepsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ to the $\varepsilon = 0$ limit

The `series()` command performs a Taylor series expansion on the equation. The `order0` equation is simply our leading order problem (6.22). Separating out the coefficient of $O(\varepsilon^1)$ and substituting the value $x_0 = 2$ into `order1` yields the value for $x_1 = -\frac{1}{2}\sin 2$. Continuing on, the `order2` equation gives $x_2 = \frac{1}{4}\sin 2 - \frac{1}{8}\sin^2 2$. The expansion up to $O(\varepsilon^3)$ for the nontrivial root is therefore

$$x = 2 - \frac{1}{2}(\sin 2)\varepsilon + \left(\frac{1}{4}\sin 2 - \frac{1}{8}\sin^2 2\right)\varepsilon^2 + O(\varepsilon^3).$$

To recap, when the expansion method works, it is a very straightforward approach to carry out, either by hand or with the aid of computer software. The limitation of the method is that it will fail to construct solutions that are not of the form assumed for the asymptotic expansion. The trial solution need not be (6.17) (experience can provide good guesses), but the choice of the gauge functions δ_n must be known before this method can be used.

Our next example introduces a problem for which the expansion method using the asymptotic expansion (6.17) fails, and consequently illustrates the strengths of the iteration method. Consider the $\varepsilon \to 0$ limit of the equation

$$(x-1)^2 - 9\varepsilon = 0. \tag{6.23}$$

Substituting the expansion (6.17) into Eq. (6.23) leads to the system of equations

$$
\begin{array}{llll}
O(\varepsilon^0): & (x_0 - 1)^2 = 0 & \Rightarrow & x_0 = 1 \text{ or } x_0 = 1, \\
O(\varepsilon^1): & 2x_1(x_0 - 1) - 9 = 0 & \Rightarrow & -9 = 0 \quad (\times?) \\
O(\varepsilon^2): & x_1^2 + 2(x_0 - 1)x_2 = 0 \\
\vdots
\end{array}
$$

Substituting the leading order double root into the $O(\varepsilon)$ equation has yielded $-9 = 0$, a clear contradiction. This indicates that our choice for the expansion of x was incorrect, namely that the solutions of (6.23) are not of the assumed form (6.17). In fact, as is trivial to verify, the exact solutions are $x = 1 \pm 3\sqrt{\varepsilon}$.

Applying the iteration method to (6.23) with $x \sim x_0\delta_0(\varepsilon) + x_1\delta_1(\varepsilon)$, we would find directly that the first two gauge functions are given by $\delta_0 = 1$ and $\delta_1 = \sqrt{\varepsilon}$ and the coefficients are $x_0 = 1, 1$ and $x_1 = \pm 3$; looking for higher-order terms would yield coefficients $x_i = 0$ for $i \geq 2$.

Problems having degenerate (repeated) leading order roots should *always* be treated with caution regarding their expansions. There are also many problems requiring non-algebraic gauge functions ($\delta(\varepsilon) \neq \varepsilon^\alpha$); such problems frequently arise from transcendental equations (see [47, 49]).

6.4 A Regular Expansion for a Solution of an ODE Problem

The expansion and iteration methods also extend to solving ODE and PDE problems, the only major difference being that at each order we will have to determine a function rather than just a constant coefficient.

We now illustrate the application of the expansion method to find the solution of an ordinary differential equation using the projectile problem (4.7a) introduced in Chap. 4,

$$\frac{d^2 x}{dt^2} = -\frac{1}{(1 + \varepsilon x)^2}, \qquad x(0) = 1, \qquad x'(0) = \alpha, \tag{6.24}$$

in the limit $\varepsilon \to 0$. We begin by substituting the asymptotic expansion $x(t) \sim x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$ into the ODE and initial conditions. The expansion for the left-hand side of the ODE is simply

$$\frac{d^2 x}{dt^2} = x_0''(t) + \varepsilon x_1''(t) + \varepsilon^2 x_2''(t) + \cdots$$

and using the binomial expansion (6.15) with $r = -2$, $z = \varepsilon x$, the right-hand side becomes

$$-\frac{1}{(1 + \varepsilon x)^2} = -1 + 2\varepsilon x - 3\varepsilon^2 x^2 + 4\varepsilon^3 x^3 + \cdots \tag{6.25}$$

In fact, this needs to be further expanded using the asymptotic expansion for $x(t)$ to yield

$$= -1 + 2\varepsilon(x_0 + \varepsilon x_1 + \cdots) - 3\varepsilon^2(x_0 + \varepsilon x_1 + \cdots)^2 + \cdots$$
$$= -1 + 2\varepsilon x_0 + \varepsilon^2(2x_1 - 3x_0^2) + \varepsilon^3(2x_2 - 6x_0 x_1 + \cdots) + O(\varepsilon^4).$$

The initial conditions provide initial conditions on the coefficient functions $x_n(t)$ of the asymptotic expansion through a comparison at each order in ε

$$x(0) = 1 \quad \implies \quad x_0(0) + \varepsilon x_1(0) + \varepsilon^2 x_2(0) + \cdots = 1 + \varepsilon 0 + \varepsilon^2 0 + \cdots,$$

$$x'(0) = \alpha \quad \implies \quad x_0'(0) + \varepsilon x_1'(0) + \varepsilon^2 x_2'(0) + \cdots = \alpha + \varepsilon 0 + \varepsilon^2 0 + \cdots.$$

We can now separate the original ODE problem (6.24) into sub-problems for each $x_n(t)$ at $O(\varepsilon^n)$

$$
\begin{array}{llll}
O(\varepsilon^0): & x_0'' = -1 & x_0(0) = 1 & x_0'(0) = \alpha, \\
O(\varepsilon^1): & x_1'' = 2x_0 & x_1(0) = 0 & x_1'(0) = 0, \\
O(\varepsilon^2): & x_2'' = 2x_1 - 3x_0^2 & x_2(0) = 0 & x_2'(0) = 0,
\end{array}
$$

and so on. The solution to the higher order problems depends on the solutions from the lower order ones and so we must solve the sub-problems in sequence. Solving the

$O(\varepsilon^0)$ problem yields the leading order solution, $x_0(t) = -\frac{1}{2}t^2 + \alpha t + 1$. Substituting x_0 into the $O(\varepsilon^1)$ equation we obtain,

$$x_1'' = -t^2 + 2\alpha t + 2 \qquad \Longrightarrow \qquad x_1(t) = -\frac{1}{12}t^4 + \frac{\alpha}{3}t^3 + t^2$$

and results at higher orders follow analogously. Reassembling the asymptotic expansion for the solution gives

$$x(t) = \left(-\frac{1}{2}t^2 + \alpha t + 1\right) + \varepsilon\left(-\frac{1}{12}t^4 + \frac{\alpha}{3}t^3 + t^2\right) + O(\varepsilon^2). \qquad (6.26)$$

Physically, the flight of the projectile ends when the ground is reached ($0 \leq t \leq t_*$ for $x(t_*) = 0$), but mathematically, there is nothing stopping us from considering the behaviour predicted by (6.26) for longer times. Note that the asymptotic ordering of the expansion breaks down with $O(x_0) = O(\varepsilon x_1) = O(1/\varepsilon)$ when $t = O(1/\sqrt{\varepsilon})$ and the construction of the solution would also break down at this point, since the assumption that $|\varepsilon x_1| \ll |x_0|$ implicit in (6.25) would be violated. This is a common occurrence for asymptotic expansions and we will also see examples of systems in Chaps. 7 and 8 where such difficulties arise. Such a breakdown indicates a transition in scaling regimes and the resolution of the problem involves identification of the appropriate new scaling. In the next section, we will consider this issue further in the context of determining solutions to singular perturbation problems.

6.5 Singular Perturbation Problems

Problems having one or more solutions that exhibit singular (divergent) behaviour in the limit $\varepsilon \to 0$ are called *singular perturbation problems*. The singular solutions are not obtainable from asymptotic expansions appropriate to regular solutions, such as (6.17); attempts to use such asymptotic expansions will return only a subset of the solutions of the full problem, or at worst, no solutions at all (see Exercise 6.3).

This behaviour is inherent to all classes of singular perturbation problems:

- For algebraic equations, a singularly perturbed Nth degree polynomial will only have M regular solutions, with $M < N$ if the leading order equation reduces to an Mth degree polynomial for x_0 due to terms of the form $\varepsilon^p x^K$ ($p > 0$) vanishing for $M < K \leq N$.
- For differential equations, if the limit $\varepsilon \to 0$ causes the highest order derivative to vanish (e.g. $\varepsilon^p d^N x/dt^N$), the leading order solution of the remaining lower order problem will typically not have enough degrees of freedom to satisfy all of the initial or boundary conditions imposed on the original problem.

- Likewise, in other classes of singular problems, the $\varepsilon = 0$ leading order problem may be dramatically reduced from the full problem for $\varepsilon > 0$, for example: a singular PDE reducing to an ODE, a singular ODE reducing to an algebraic equation, or a system of equations reducing to a single equation.

The non-regular solutions are not truly lost by such reductions, and can be recovered through appropriate rescalings of the original problem. Which solutions are obtained depends on the scaling of the problem and the form of the asymptotic expansion assumed for the solution.

As an example, consider the $\varepsilon \to 0$ limit of the (immediately solvable) algebraic equation

$$\varepsilon x^2 - 2x + 1 = 0. \tag{6.27}$$

Substituting $\varepsilon = 0$ into (6.27) gives

$$-2x + 1 = 0 \quad \Rightarrow \quad x = \tfrac{1}{2}. \tag{6.28}$$

This leading order equation therefore only yields one of the two roots expected from the second-degree equation (6.27). The exact solutions of (6.27) can be written as

$$x = \frac{1 \pm \sqrt{1 - \varepsilon}}{\varepsilon} \sim \frac{1 \pm \left(1 - \tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + O(\varepsilon^3)\right)}{\varepsilon}, \tag{6.29}$$

from which we see that

$$x_A \sim \frac{2}{\varepsilon} - \frac{1}{2} - \frac{\varepsilon}{8}, \qquad x_B \sim \frac{1}{2} + \frac{\varepsilon}{8} + \frac{\varepsilon^2}{16}. \tag{6.30}$$

Equation (6.28) produced only the leading order term of the regular solution x_B. The x_A solution diverges as $\varepsilon \to 0$ and cannot be expressed in terms of the regular expansion (6.17).

Substituting the leading term from each of these solutions back into (6.27) helps identify the cause of the discrepancy

$$x_A: \quad \varepsilon \left(\tfrac{2}{\varepsilon}\right)^2 - 2\left(\tfrac{2}{\varepsilon}\right) + 1 = 0, \qquad x_B: \quad \varepsilon \left(\tfrac{1}{2}\right)^2 - 2\left(\tfrac{1}{2}\right) + 1 = 0.$$

The solutions are determined by different dominant balances in Eq. (6.27). For x_A, the first two terms balance at $O(1/\varepsilon)$ while the third term is sub-dominant at $O(1)$. For x_B, the second and third terms balance at $O(1)$ with the first term being sub-dominant at $O(\varepsilon)$ (this being the regular solution that was obtained from (6.28)).

6.5.1 Rescaling to Obtain Singular Solutions

We now outline a systematic procedure for how the previously introduced methods for regular perturbation problems can be extended to handle singular solutions:

(1) Substitute $x = \delta_0(\varepsilon)X$ into the given problem.
(2) Choose $\delta_0(\varepsilon)$ so as to produce a consistent dominant balance; verify that all neglected terms are indeed sub-dominant. Different $\delta_0(\varepsilon)$'s lead to different dominant balances, and considering all of the possible choices will yield all of the regular and singular solutions.
(3) Factor out any common ε-scalings to yield a regular perturbation problem in X. Then solve this problem using either the iteration or expansion methods to generate regular solutions corresponding to this distinguished limit.
(4) Rescale X by δ_0 to obtain x in final form.

We will illustrate this methodology on the example from the previous section. Substituting $x = \delta_0 x_0$ into (6.27) yields

$$\underbrace{\varepsilon\delta_0^2 X^2}_{(1)} - \underbrace{2\delta_0 X}_{(2)} + \underbrace{1}_{(3)} = 0. \tag{6.31}$$

We compare the orders of magnitude of the terms (cf. Sect. 6.3.2). There are three possible balances:

$$
\begin{array}{lll}
\text{(a) Terms (2, 3): } \delta_0 = 1 & \Rightarrow & \delta_0 = 1 \\
\text{(b) Terms (1, 3): } \varepsilon\delta_0^2 = 1 & \Rightarrow & \delta_0 = 1/\sqrt{\varepsilon} \\
\text{(c) Terms (1, 2): } \varepsilon\delta_0^2 = \delta_0 & \Rightarrow & \delta_0 = 1/\varepsilon
\end{array}
$$

In case (a), the omitted first term is sub-dominant, $O(\varepsilon)$, and so this is a consistent balance; this balance yields the regular solution, x_B. In case (b), the dominant balance is at $O(1)$, while the omitted second term is $O(1/\sqrt{\varepsilon}) \gg 1$ in the limit $\varepsilon \to 0$ and is not sub-dominant. Hence this balance is inconsistent and must be rejected. Finally, in case (c) the balancing terms are $O(1/\varepsilon)$, while the third term is $O(1) \ll 1/\varepsilon$, therefore yielding a second consistent dominant balance; this case yields the singular solution.

In order to investigate the singular solution, we write $x = X/\varepsilon$ and substitute into (6.27) to arrive at the (rescaled) regular problem

$$X^2 - 2X + \varepsilon = 0, \tag{6.32}$$

where we have multiplied through by ε. Expanding $X(\varepsilon)$ as a regular expansion (here $X \sim X_0 + \varepsilon X_1 + \varepsilon^2 X_2$ will work) and seeking the leading order term yields

$$X_0^2 - 2X_0 = 0, \tag{6.33}$$

which has solutions $X_0 = 0$ and $X_0 = 2$. We ignore the root $X_0 = 0$ because it is not a nontrivial leading order term.[2] The other root yields the singular solution $x_A \sim 2/\varepsilon$.

A particular difference between the iteration method described in Sect. 6.3.2 and the current methodology is worth noting. Both seek to determine the leading order gauge function δ_0, and the set of coefficients x_n versus X_n will be identical. The iteration method seeks to identify one successive term in the asymptotic expansion at each iteration. In contrast, the rescaling approach re-casts the entire problem into a new (regular perturbation) form for $X(\varepsilon) = O(1)$ (as in (6.32)); whether to then use the expansion method or iteration method to solve for X is left as a separate decision. In terms of the asymptotic expansion of the solution, we have

$$
\begin{aligned}
x &\sim \delta_0 X(\varepsilon) \\
&\sim \delta_0 \left(X_0 + \tilde{\delta}_1 X_1 + \tilde{\delta}_2 X_2 + \tilde{\delta}_3 X_3 + \cdots \right) \\
&\sim \delta_0 x_0 + \delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3 + \cdots ,
\end{aligned}
\tag{6.34}
$$

so that the leading order gauge function δ_0 scales through the $\tilde{\delta}$ gauge functions in the expansion of the rescaled solution.

Singular problems for ODE and PDE introduce additional complexities and are of particular interest as they frequently arise in applications. The following two chapters consider these scenarios in detail.

6.6 Further Directions

There are many variations of the methods used in this chapter. Reference [47] employs an iterative procedure, where the original equation must be written in a form that is compatible with the contraction mapping theorem and allows for greater analysis of the convergence of the asymptotic expansion. The method described in [49] is somewhat more similar to that presented here; there, the gauge functions are assumed to be of the form $\delta_n = \varepsilon^{\alpha_n}$ where α_n need not be an integer. Finally, we note that many further approaches for constructing asymptotic expansions for integrals and differential equations build directly on the perturbation methods for algebraic equations described in this chapter [11, 13, 29, 47, 72, 92].

---

[2]It is a "ghost" of the x_B regular solution, which is $O(\varepsilon)$ in terms of X (and violates requirement (i) in Sect. 6.3.2).

6.7 Exercises

6.1 Use Taylor series expansions for $\sin y$ and e^y for $y \to 0$ and the basic property $\exp(\sum_k a_k) = \prod_k e^{a_k}$ to derive (6.10) for $\varepsilon \to 0$ from $x(t, \varepsilon) = \exp(\sin(\varepsilon\sqrt{t})/\varepsilon^2)$ when $t = O(1)$, but $x \sim 1 + \sqrt{\varepsilon T} + \frac{1}{2}\varepsilon T$ when $t = \varepsilon^3 T$ with $T = O(1)$.

6.2 Consider the limit of $\varepsilon \to 0$ for the equation

$$(x - 3)^3 = 24\varepsilon x^2.$$

Solve by iteration[3] to determine $x \sim \delta_0(\varepsilon)x_0 + \delta_1(\varepsilon)x_1 + \delta_2(\varepsilon)x_2$.

6.3 Consider the algebraic equation for $\varepsilon \to 0$

$$\varepsilon^6 x^3 - 5\varepsilon^3 x^2 - 20\varepsilon x + 60 = 0.$$

Show that there are no regular solutions and determine the leading order nontrivial term in the expansion of each of the three solutions.

6.4 Consider the projectile problem for $\varepsilon \to 0$

$$\frac{d^2x}{dt^2} = -\frac{1}{(1 + \varepsilon x)^2}, \qquad x(0) = 1, \qquad x'(0) = 3\varepsilon.$$

(a) Let $t^{\max}$ be the time when the projectile reaches its maximum height. How many terms in the expansion of $x(t)$ will you need to determine $t^{\max} = t_0 + \varepsilon t_1 + O(\varepsilon^2)$?

(b) Show that while (6.26) with $\alpha = 3\varepsilon$ could have been used to determine the solution in part (a), this could not be used with the initial condition $x'(0) = 4/\varepsilon$. Assume a singular form $x(t) = X(t)/\varepsilon \sim X_0(t)/\varepsilon + X_1(t)$ to find this solution.

6.5 Consider the problem

$$\frac{dv}{dt} + \varepsilon v^2 + t = 0, \qquad v(0) = 0, \qquad \varepsilon \to 0.$$

(a) Find the first three terms in the expansion of the solution, $v(t) \sim v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t)$.

(b) Determine the range of times, $0 \le t < O(\varepsilon^\alpha)$, for which the terms in the expansion retain asymptotic ordering, i.e. $v_0 \gg \varepsilon v_1 \gg \varepsilon^2 v_2 \gg \cdots$.

---

[3] Use of a computer algebra program (`Maple` or `Mathematica`) is recommended for solving many of the more algebraically intensive problems.

6.6 Consider the system of equations

$$\varepsilon x - y = 1$$
$$\varepsilon^2 x + y = 4$$

in the limit $\varepsilon \to 0$.

Explain why setting $\varepsilon = 0$ does not lead to an acceptable leading order solution. Determine the first two terms in the expansions of $x(\varepsilon)$, $y(\varepsilon)$.

6.7 Use the iteration method with $x \sim \delta_0(\varepsilon)x_0 + \delta_1(\varepsilon)x_1$ to find the singular real solution of the equation
$$2x^2 e^{-5x} = 8\varepsilon, \qquad \varepsilon \to 0.$$

Note: This equation has three real roots; the two vanishing roots are not of interest here (they are $x \sim \pm 2\varepsilon^{1/2} + 10\varepsilon$). The one you should locate has $\delta_0(\varepsilon) \to \infty$ (i.e. it is large and positive).
Hint: Take the logarithm of both sides of the equation before beginning iteration.

6.8 Re-consider the solutions of Eq. (6.21), but now in the limit of $\varepsilon \to \infty$: let $\tilde{\varepsilon} = 1/\varepsilon$ and find the real-valued solutions of

$$\sin x + \tilde{\varepsilon}(x^2 - 2x) = 0, \qquad \tilde{\varepsilon} \to 0.$$

Assume $x \sim x_0 + \tilde{\varepsilon}x_1 + \tilde{\varepsilon}^2 x_2$. Note that for $\tilde{\varepsilon} = 0$ the leading order problem has infinitely many solutions, $x_0 = n\pi$ for any integer n.

(a) Determine x_1, x_2.
(b) Plot $F(x, \tilde{\varepsilon}) = \sin x + \tilde{\varepsilon}(x^2 - 2x)$ for $\tilde{\varepsilon} = \frac{1}{100}$. How many zeroes does it have?
(c) Parts (a, b) show that there is a conflict, as $\varepsilon \to 0$ there is a finite number of solutions, namely there is a maximum value for n, call it $N(\tilde{\varepsilon})$, at each value of $\tilde{\varepsilon}$. Requiring that the expansion for x remain asymptotically ordered ($x_0 \gg \varepsilon x_1 \gg \dots$) suggests one estimate for N, but show that maintaining ordering in the expansion of the equation, given x_0 and x_1 from (a), yields the correct $N(\tilde{\varepsilon})$.

6.9 In the 1930s, Carleman showed that in an appropriate asymptotic limit a diffusion model could be derived from a system of reactive wave equations. Consider the system of PDEs for $p(x, t), q(x, t)$ with $\varepsilon > 0$:

$$\varepsilon^2 \frac{\partial p}{\partial t} + \varepsilon \frac{\partial p}{\partial x} = q - p, \qquad \varepsilon^2 \frac{\partial q}{\partial t} - \varepsilon \frac{\partial q}{\partial x} = p - q.$$

(a) If the solutions are uniform in space (i.e. $p = p(t), q = q(t)$), then p, q satisfy a reversible transformation reaction $P \rightleftharpoons Q$ (see Sect. 1.2). What is the conserved quantity for this reaction system? What is the reaction rate?
(b) If the reactions (the right-hand side terms) are eliminated, find the travelling waves for the reduced equations for $p(x, t)$ and $q(x, t)$. What are the speeds of these waves?

(c) Define $u = p + q$ and $v = (p - q)/\varepsilon$. Combine the full equations for p and q to obtain two equations for u and v. In the limit $\varepsilon \to 0$, use regular perturbation expansions $u(x, t) \sim u_0 + \varepsilon u_1 + \cdots$ and $v(x, t) \sim v_0 + \varepsilon v_1 + \cdots$ to find the leading order equations for u and v, and then derive a PDE for $u_0(x, t)$ alone.

6.10 The nondimensional form of the shallow water equations is

$$\frac{\partial h}{\partial t} + h\frac{\partial u}{\partial x} + u\frac{\partial h}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\text{Fr}^2}\frac{\partial h}{\partial x} = 0,$$

where $u(x, t)$ is the fluid speed, $h(x, t)$ is the height of the fluid, and Fr is a dimensionless parameter. A uniform, steady solution is given by $h \equiv 1$ and $u \equiv 1$, which can represent a river having uniform speed and depth.

Consider waves generated due to some small disturbance on this steady state. Let

$$h = 1 + \varepsilon\eta(x, t), \qquad u = 1 + \varepsilon v(x, t).$$

(a) For $\varepsilon \to 0$, determine the $O(\varepsilon)$ linearised wave equations for η, v.
(b) Apply the approach of Sect. 2.4 to the linearised system to determine the critical value of the Froude number at which waves from the disturbance change from spreading in both up- and down-stream directions to having all ripples being swept downstream (called *sub-* and *super-critical behaviours*).