Chapter 3
Variational Principles

In many problems, we need to find an optimal solution, one that maximises a benefit or minimises a cost for example. Various branches of science include formulations based on such principles:

- Fermat's principle of least time in optics
- Hamilton's principle of least action in mechanics
- "paths of least resistance" in electrostatics, hydrology, and other areas.

The criteria selecting an "optimal" solution in a physical system may not always be clear (especially in biological problems). However, if we are given a quantity to be minimised or maximised, the *calculus of variations* provides a natural methodology for reformulating the question in terms of a differential equation problem.

In this chapter we focus on problems that require the determination of a *function*, say $y_*(x)$, as the optimal solution to a given system. A classical example known as the *brachistochrone* problem (meaning "shortest time" in Greek) is motivated by the question: *What shape should a ramp take in order to deliver a mass (moving under the influence of gravity) to a specified final position in the least time?*; Fig. 3.1 shows schematics of some possible trial solutions.

Our approach will extend elementary methods from calculus for finding local optimal points to problems where solution functions, $y_*(x)$, are optimal relative to all small possible variations of that function.

3.1 Review and Generalisation from Calculus

Since our presentation will build on the basic formulations for finding maxima and minima from single- and multi-variable calculus, we briefly review that background as a means of introducing the terminology that we will employ.

Consider the problem of finding local extrema of a smooth function $y = f(x)$, called the *objective function*. If f has a local maximum at $x = x_*$ with value $y_* = f(x_*)$ then y_* must be greater or equal to all values of f achieved in a small neighbourhood

T. Witelski and M. Bowen, *Methods of Mathematical Modelling*,
Springer Undergraduate Mathematics Series, DOI 10.1007/978-3-319-23042-9_3

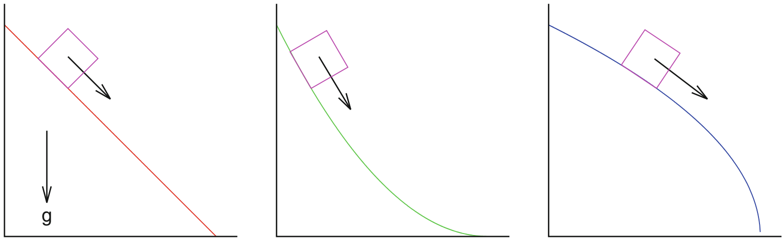

Fig. 3.1 Three trial solutions for the brachistochrone problem for the shape of a ramp to move a mass down from (0, 1) to (1, 0) under the influence of gravity as quickly as possible, $\tilde{y} = (1 - x)^{\beta}$ for $\beta = 1, 2, 1/2$

around x_*,

$$f(x_*) \geq f(x_* + \varepsilon) \qquad \text{for all } |\varepsilon| \to 0. \tag{3.1}$$

Taking a Taylor series expansion of f at x_* yields

$$f(x_* + \varepsilon) = f(x_*) + f'(x_*)\varepsilon + \tfrac{1}{2}f''(x_*)\varepsilon^2 + \cdots \qquad \text{as } \varepsilon \to 0. \tag{3.2}$$

If the slope $f'(x_*)$ were nonzero, then for $|\varepsilon| > 0$ either $f(x_* + \varepsilon)$ or $f(x_* - \varepsilon)$ would have a value greater than $f(x_*)$, violating our assumption that $f(x_*)$ is a maximum. Consequently, we must have $f'(x_*) = 0$ at a maximum. An analogous argument applies for local minima of $y = f(x)$, and so local extrema for smooth functions can only arise at *critical points*, where

$$f'(x_*) = 0.$$

Having eliminated the linear term from the Taylor series, the higher order terms in (3.2) must also respect the requirement (3.1). This leads to the standard condition on the second derivative for a local maximum ($f''(x_*) < 0$) (or minimum ($f''(x_*) > 0$)). If $f''(x_*) = 0$ at a critical point, then the next nonzero higher order terms need to be considered in order to determine if the critical point indeed yields a maximum or a minimum of f, or is neither (an inflection point).

Similarly, for a two-variable objective function, having a local maximum of f at (x_*, y_*) with value $z_* = f(x_*, y_*)$ implies that all nearby points satisfy

$$f(x_*, y_*) \geq f(x_* + \varepsilon_1, y_* + \varepsilon_2) \qquad \text{for all } |\varepsilon_1|, |\varepsilon_2| \to 0.$$

The corresponding multi-variable Taylor series for $\varepsilon_1, \varepsilon_2 \to 0$ yields

$$f(x_* + \varepsilon_1, y_* + \varepsilon_2) = f(x_*, y_*) + \underbrace{\partial_x f(x_*, y_*)\varepsilon_1 + \partial_y f(x_*, y_*)\varepsilon_2}_{\text{[first derivative terms]}} + \cdots .$$

As in the single variable case, the linear terms must vanish at a local maximum or minimum, and since this must hold for all choices of ε_1 and ε_2, we arrive at the generalisation of the previous critical point condition,

$$\partial_x f_* = 0, \qquad \partial_y f_* = 0. \tag{3.3}$$

Namely, the first partial derivatives of the objective function with respect to all of its independent variables must vanish, which can be more compactly expressed in terms of the gradient

$$\nabla f_* = \mathbf{0}. \tag{3.4}$$

This definition of critical points in terms of the gradient applies to functions of any number of independent variables. Being a critical point is a necessary condition for a local extrema, but it is not sufficient. Conditions on the second derivatives (represented by the *Hessian matrix* for functions of two or more variables) also extend to arbitrary numbers of variables for determining whether a critical point yields a maximum, minimum or inflection point of the objective function [24, 68]. Whether we are seeking a maximum or a minimum, all local extrema are determined by the critical point conditions and hence we focus on obtaining critical points of the more challenging class of problems that we will consider.

3.1.1 Functionals

In general, a *functional* is a mathematical expression that can be evaluated to give a scalar value corresponding to each function to which it is applied. Two simple examples are provided by the integrals that correspond to the area under a curve and the arclength of a curve,

$$A(y) = \int_0^1 y(x)\,dx, \qquad S(y) = \int_0^1 \sqrt{1 + y'(x)^2}\,dx,$$

for any given function $y(x)$ on $0 \le x \le 1$.

For the most part, we will only consider functionals that are definite integrals of a function and its derivatives,

$$J(y) = \int_a^b L(x, y(x), y'(x), y''(x), \ldots)\,dx. \tag{3.5}$$

Here the integrand function $L(x, y, \ldots)$ is called the *Lagrangian* (named after Joseph Lagrange (1736–1813) who reformulated classical mechanics based on a variational principle—see Sect. 3.3).

For the above geometric examples of area and arclength, the dependence of the functional on the solution is straightforward, while for the brachistochrone problem from Fig. 3.1, the connection is a little less direct. The time of travel to be minimised can be expressed as

$$T(y) = \int_0^T dt = \int \frac{ds}{v} = \int_0^1 \frac{\sqrt{1+(y')^2}}{v(y)}\,dx, \tag{3.6}$$

where we have used the relationship between the speed and distance travelled, $v = ds/dt$, and a further relationship is also needed between the speed and position (see Exercise 3.8).

Now consider the minimisation of an *objective functional*; by analogy with the previous analysis, we see that for $J(y)$ to have a local minimum[1] for a particular function $y_*(x)$ the functional must satisfy

$$J(y_*(x)) \leq J(y_*(x) + \varepsilon h(x)),$$

for all $\varepsilon \to 0$ and for all admissible "perturbation functions" $h(x)$ of $y_*(x)$. For the moment, $h(x)$ is unspecified and serves to describe a neighbourhood of the optimal solution $y_*(x)$ within the space of smooth functions. While there are many technical issues underlying these concepts that are deserving of careful analysis, we defer detailed considerations to more advanced texts [71, 99].

We will describe a straightforward procedure (as an extension of the process of finding critical points of an objective function) for determining optimal solutions of a functional. In particular, we will have to explain what it means to take a derivative of a functional with respect to a function, and we will distinguish these from standard derivatives by using the terminology *variational derivatives*.

3.2 General Approach and Basic Examples

Given an objective functional, there is a systematic four-step approach which can be used to obtain smooth solutions that correspond to local extrema of the functional:

(i) Assume the existence of an optimal solution $y_*(x)$ and perform an expansion of the functional in the neighbourhood of $y_*(x)$ using

$$\tilde{y} = y_*(x) + \varepsilon h(x) \qquad \tilde{J} = J(\tilde{y}), \tag{3.7}$$

where the perturbation function $h(x)$ is independent of ε. Formally expanding $\tilde{J}$ as $\varepsilon \to 0$ like a Taylor series yields

$$\tilde{J} = J(y_* + \varepsilon h) = J(y_*) + \text{“}\left(\left.\frac{dJ}{dy}\right|_{y_*}\right)(\varepsilon h)\text{”} + O(\varepsilon^2), \tag{3.8}$$

[1]The behaviour at a local maximum of $J(y)$ follows similarly.

where we still have to precisely define the notion of a variational derivative (in this case the evaluation of the derivative of an integral with respect to a function). The $O(\varepsilon)$ term[2] in the Taylor series of J will be called the *first variation*, the $O(\varepsilon^2)$ term the *second variation* and so on.

(ii) Apply the critical point condition: the first variation of the functional must be zero at the solution $y_*(x)$ for all admissible perturbations $h(x)$,

$$\text{``}\left(\left.\frac{dJ}{dy}\right|_{y_*}\right)(\varepsilon h)\text{''} = 0 \tag{3.9}$$

We will explain the meaning of 'admissible' solutions and perturbations below.

(iii) Assuming the solution to be smooth, convert the critical point condition on the functional into a differential equation for the optimal[3] solution $y_*(x)$.

(iv) Solve the differential equation problem.

The details of this approach will be discussed and expanded out in following examples. This framework applies to many more complicated problems. However, an important disclaimer is that not every problem can be simplified to a differential equation using this framework, those that can be are called *variational problems*. A number of difficulties can arise to make a problem non-variational, such as the functional not having critical points, or it not being possible to reduce the first variation to a differential equation.

3.2.1 The Simple Shortest Curve Problem

We begin by considering the simple problem of finding the function $y = y(x)$ that gives the shortest path from the origin $(0, 0)$ to the given fixed point $(1, b)$, $b \in \mathbb{R}$, see Fig. 3.2. While we know the answer to be the straight line connecting the two points, it will be instructive to see how the calculus of variations framework constructs this result.

The shortest path is the one that minimises the total arclength and hence our objective functional is

$$J(y) = \int_0^1 \sqrt{1 + (y')^2}\, dx. \tag{3.10a}$$

[2]The $O(\varepsilon^n)$ order symbol will be defined precisely in Chap. 6, but for the current context we use this to refer to the $\varepsilon \to 0$ limit of the remainder of the terms in the series with coefficients ε^N for $N \geq n$.

[3]Showing that a solution given by a critical point is a local maximum or minimum involves evaluating the second variation at $y_*(x)$.

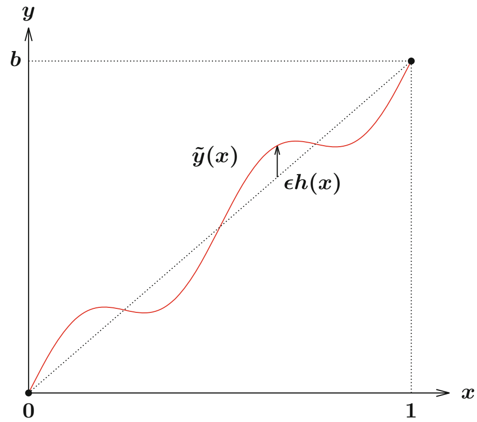

Fig. 3.2 An admissible trial solution $\tilde{y}(x)$ for problem (3.10a, 3.10b) and its perturbation from the optimal solution $y_*(x)$

The statement of the problem identifies boundary conditions that all solutions must satisfy,

$$y(0) = 0, \qquad y(1) = b. \tag{3.10b}$$

Analysis of the problem begins by identifying the properties of the family of perturbations $h(x)$ to the optimal solution $y_*(x)$ that define all admissible trial solutions (3.7), $\tilde{y}(x) = y_*(x) + \varepsilon h(x)$. Applying boundary conditions (3.10b) to the admissible solution $\tilde{y}$ yields that

$$\tilde{y}(0) = y_*(0) + \varepsilon h(0) = 0, \qquad \tilde{y}(1) = y_*(1) + \varepsilon h(1) = b.$$

Since the boundary conditions on the optimal solution are necessarily the same as those on the trial solutions ($y_*(0) = 0$ and $y_*(1) = b$), we obtain that the perturbation functions must satisfy

$$h(0) = 0, \qquad h(1) = 0. \tag{3.11}$$

Next, we substitute (3.7) into (3.10a) giving

$$\tilde{J} = \int_0^1 \sqrt{1 + \left(y_*' + \varepsilon h'\right)^2}\, dx = \int_0^1 \sqrt{1 + (y_*')^2 + 2\varepsilon h' y_*' + \varepsilon^2 (h')^2}\, dx.$$

For investigating the limit $\varepsilon \to 0$, it is convenient to factor the integrand as

$$= \int_0^1 \sqrt{\left(1 + (y_*')^2\right)\left(1 + \frac{2\varepsilon h' y_*'}{1 + (y_*')^2} + \frac{\varepsilon^2 (h')^2}{1 + (y_*')^2}\right)}\, dx$$

and then using the Taylor series expansion $\sqrt{1+z} = 1 + \frac{1}{2}z + O(z^2)$ as $z \to 0$ applied to the second factor allows us to write

$$\tilde{J} = \left(\int_0^1 \sqrt{1+(y_*')^2}\,dx\right) + \varepsilon\left(\int_0^1 \frac{h'y_*'}{\sqrt{1+(y_*')^2}}\,dx\right) + O(\varepsilon^2). \tag{3.12}$$

The first term in this expansion gives the expression for the functional at the optimal solution, $J_* = J(y_*)$, but since y_* is unknown, in order to evaluate this integral, we must examine further terms. The $O(\varepsilon)$ term provides the first variation (3.8) that will determine the critical points.

It is important to note that the form of the first variation in (3.8) was chosen to intentionally suggest that the perturbation function $h(x)$ should appear *linearly and undifferentiated*. We will always seek to put the first variation term into this form.

For (3.12), getting to this form can be accomplished by performing integration by parts to shift the derivative off of the h' factor,

$$\int_0^1 \frac{h'y_*'}{\sqrt{1+(y_*')^2}}\,dx = \left.\frac{y_*'h}{\sqrt{1+(y_*')^2}}\right|_{x=0}^{x=1} - \int_0^1 \frac{d}{dx}\left(\frac{y_*'}{\sqrt{1+(y_*')^2}}\right)h\,dx;$$

integration by parts will be a fundamental calculational tool in most of our problems. The boundary conditions on h given by (3.11) eliminate the contributions from the boundary terms, and leave the final form of the first variation of the functional as a single integral

$$\left.\frac{\delta J}{\delta y}\right|_{y_*} h = -\int_0^1 \frac{d}{dx}\left(\frac{y_*'}{\sqrt{1+(y_*')^2}}\right)h\,dx, \tag{3.13}$$

where we use the δ-notation to indicate that this is a variational (functional) derivative (also sometimes called a *Frechet derivative*).

The next step is to enforce the critical point condition (3.9) on (3.13) for all admissible perturbations. At this point, we need a basic, but important result from analysis called the *Fundamental Lemma of the Calculus of Variations* which states that

$$\text{If} \quad \int_a^b g(x)h(x)\,dx = 0 \quad \forall h(x) \quad \text{then} \quad \boxed{g(x) \equiv 0 \text{ on } a \le x \le b.} \tag{3.14}$$

In other words, if the integral of a product of functions is zero for all choices of one of the factors, then this is only possible if the other factor is identically zero on the whole interval of integration.[4,5]

Applying the fundamental lemma to the critical point condition for (3.13),

$$-\int_0^1 \frac{d}{dx}\left(\frac{y_*'}{\sqrt{1+(y_*')^2}}\right) h\, dx = 0, \tag{3.15}$$

for all possible h, we obtain the differential equation for $y_*(x)$

$$-\frac{d}{dx}\left(\frac{y_*'}{\sqrt{1+(y_*')^2}}\right) = 0. \tag{3.16}$$

We have consequently replaced an integral condition (the *weak form*, that applies over the whole domain) with an equivalent differential equation (the *strong form*) that applies pointwise on smooth solutions. Together with boundary conditions (3.10b), (3.16) provides us with a complete ODE boundary value problem defining the optimal solution $y_*(x)$.

In this problem it is straightforward to integrate the ODE once yielding a constant of integration C_1, then following some algebra,

$$\frac{y_*'}{\sqrt{1+(y_*')^2}} = C_1 \qquad \Rightarrow \qquad (y_*')^2 = \frac{C_1^2}{1-C_1^2} = C_*^2 \geq 0, \tag{3.17}$$

for some constant C_*. Then using the boundary conditions (3.10b) yields $y_*(x) = bx$, confirming that the shortest path is a straight line.

A second derivative test can be applied to verify that this solution is a local minimum of (3.10) (see Exercise 3.2), but here we can use an understanding of the structure of the problem to show that it is indeed the solution we seek.[6] In many cases, problem-specific intuition can be used to distinguish whether critical point solutions minimise or maximise the functional.

A more general version of this problem, allowing for solutions as parametric curves in the plane is given in Exercise 3.3.

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[4]In this simplified statement of this result, we are assuming that $g(x)$ is smooth and hence has no discontinuities.

[5]The du Bois Reymond lemma (2.11) is closely related to this lemma, choosing $h(x) = 1$ on arbitrary sub-intervals, and otherwise $h = 0$.

[6]For the area and arclength examples, the functionals become unbounded for large amplitude solutions and hence there are no local maxima.

3.2.2 The Classic Euler–Lagrange Problem

Consider an objective functional defined in terms of an integral on the interval $a \leq x \leq b$ with an integrand that is a function of the independent variable x, the solution $y(x)$ and its first derivative $y'(x)$

$$J(y) = \int_a^b L(x, y, y')\,dx, \tag{3.18a}$$

subject to prescribed boundary conditions on y given by

$$y(a) = c, \qquad y(b) = d. \tag{3.18b}$$

Parallelling the previous example, we find that admissible perturbation functions must satisfy the homogeneous boundary conditions

$$h(a) = 0, \qquad h(b) = 0. \tag{3.19}$$

The expanded form of $J(y_* + \varepsilon h)$ is given by

$$\tilde{J} = \int_a^b L(x, y_* + \varepsilon h, y'_* + \varepsilon h')\,dx$$

and by applying the multi-variable Taylor series, this can be written for $\varepsilon \to 0$ as

$$\begin{aligned}\tilde{J} &= \int_a^b \left[L(x, y_*, y'_*) + \varepsilon \left(\frac{\partial L}{\partial y} h + \frac{\partial L}{\partial y'} h' \right)\bigg|_{y=y_*} + O(\varepsilon^2) \right] dx \\ &= J_* + \varepsilon \left(\int_a^b \frac{\partial L}{\partial y} h\,dx + \int_a^b \frac{\partial L}{\partial y'} h'\,dx \right) + O(\varepsilon^2).\end{aligned}$$

In order to address the h' derivative factor in the third term, we apply integration by parts to obtain

$$\int_a^b \frac{\partial L}{\partial y'} h'\,dx = \frac{\partial L}{\partial y'} h \bigg|_{x=a}^{x=b} - \int_a^b \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) h\,dx.$$

Imposing the boundary conditions (3.19) on the perturbation function, we can then re-group the first variation as

$$\tilde{J} = J_* + \varepsilon \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \right] h\,dx + O(\varepsilon^2). \tag{3.20}$$

This was an important and necessary step because in order to apply the fundamental lemma (3.14) to the critical point condition, we must have the first variation expressed as a single integral having the perturbation $h(x)$ as a factor,

$$\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) \right] h\, dx = 0.$$

Since this holds for all admissible choices of $h(x)$, by (3.14) we obtain the differential equation on $a \le x \le b$,

$$\boxed{\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) = 0} \tag{3.21}$$

Subject to the boundary conditions (3.18b), (3.21) defines a differential equation problem that should have a unique solution for $y = y_*(x)$.

Equation (3.21) is known as the *Euler–Lagrange* equation for the functional (3.18a). In Sect. 3.3, we will see that this particular form is used extensively in problems for mechanical systems.

A very useful consequence of the general form of L in (3.18a) is that the Euler–Lagrange differential equation for any functional with $L = L(x, y, y')$ is given by substituting the specific L into (3.21). For example, the ODE (3.16) for the shortest distance problem (3.10a) is produced by (3.21) with $L = \sqrt{1 + (y')^2}$. Similarly, if our Lagrangian were $L(x, y, y') = f(x)y + g(x)y' + (y')^2$, then

$$\frac{\partial L}{\partial y} = f(x), \qquad \frac{\partial L}{\partial y'} = g(x) + 2y', \qquad \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) = g'(x) + 2y''$$

and (3.21) gives the Euler–Lagrange equation for $y_*(x)$ as

$$f(x) - g'(x) - 2\frac{d^2y}{dx^2} = 0.$$

3.3 The Variational Formation of Classical Mechanics

One of the most wide-spread applications of the calculus of variations is in deriving the equations of motion for mechanical systems from the critical points of a functional. The theory is based on a restatement of the previous Euler–Lagrange results—we relabel the variables to describe the motion of a mass:

$$x \to t \qquad : \text{independent variable (time)}$$
$$y(x) \to y(t) \qquad : \text{solution (position)}$$
$$y'(x) \to y'(t) \qquad : \text{derivative (velocity)}$$
$$L(x, y, y') \to L(t, y, y') \qquad : \text{still called the Lagrangian}$$
$$J = \int L\,dx \to I = \int L\,dt \qquad : \text{functional (action)}$$

The Euler–Lagrange equation (3.21) now takes the form

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial y'}\right) = 0, \tag{3.22}$$

and the key step in the application to mechanics is in identifying the Lagrangian function L as the difference between the kinetic energy T and potential energy V

$$L = T - V. \tag{3.23}$$

William Hamilton (1805–1865) used this form of Lagrangian based on the assumption that mechanical systems should satisfy what come to be called *Hamilton's principle of least action*, namely that the optimal (actual, observable) solution $y_*(t)$ will be the one that minimises the action integral, I.

A very simple example of a mechanical system is the vertical motion of a rigid object with mass m subject to gravity g. The kinetic energy of the point mass is $T = \frac{1}{2}m(y')^2$, while the potential energy due to gravity is given by $V(y) = mgy$. Substituting this form for $L(t, y, y')$ (3.23) into the Euler–Lagrange equation (3.22) yields

$$\frac{dV}{dy} - m\frac{d(y')}{dt} = 0 \quad \Rightarrow \quad m\frac{d^2y}{dt^2} = -\frac{dV}{dy} \quad \Rightarrow \quad m\frac{d^2y}{dt^2} = -mg,$$

exactly as would be obtained from Newton's second law, but without the need to explicitly work out the forces acting on the system. The difference between the Lagrangian and force-based approaches is not significant in this example, but for more complicated systems, the Euler–Lagrange approach can notably simplify the process of deriving the equations of motion.

Thorough treatments of the implications of this formulation are given in books on intermediate and advanced mechanics [40, 62, 67]; we limit ourselves here to the basic concepts that are most immediately useful for the formulation of models for mechanical systems.

3.3.1 Motion with Multiple Degrees of Freedom

One generalisation of the Euler–Lagrange equation of motion is the derivation of the equations of motion for a multi-variable systems. As a specific example, consider the motion of a projectile in two dimensions with position $\mathbf{x} = (x(t), y(t))$ subject to gravity acting in the y-direction. The action can be written in terms of the difference of the kinetic and potential energies,

$$I = \int \left[\frac{1}{2} m \left((x')^2 + (y')^2 \right) - mgy \right] dt.$$

The principle of least action then requires that $I(x, y)$ is minimised over all possible choices of the unknown functions $x(t)$ and $y(t)$. We assume that an optimal solution $(x_*(t), y_*(t))$ exists and express all admissible solutions in terms of independent perturbations of the unknowns

$$\tilde{x}(t) = x_*(t) + \varepsilon h_1(t), \qquad \tilde{y}(t) = y_*(t) + \varepsilon h_2(t).$$

Substituting into the action, $I(\tilde{x}, \tilde{y})$ and Taylor expanding in the limit $\varepsilon \to 0$,

$$\tilde{I} = I_* + \varepsilon \int \left[m(x_*' h_1' + y_*' h_2') - mgh_2 \right] dt + O(\varepsilon^2).$$

By application of integration by parts and eliminating the boundary terms, we obtain the $O(\varepsilon)$ term as

$$-\int (mx_*'')h_1 + (my_*'' + mg)h_2 \, dt.$$

Satisfying the critical point condition implies that this integral must be equal to zero for all choices of $h_1(t)$ and $h_2(t)$. One class of possibilities is given by $h_2 \equiv 0$ and $h_1(t)$ arbitrary, this yields the equation $mx_*'' = 0$. Another choice is $h_1 \equiv 0$ with $h_2(t)$ arbitrary, yielding $mg + my_*'' = 0$; the overall result comes from the intersection of these cases using the linear independence of the individual perturbations.

This can be understood as a generalisation of fundamental lemma (3.14) for the dot product of vector functions, $\mathbf{g}(x), \mathbf{h}(x) \in \mathbb{R}^n$,

$$\text{If} \quad \int_a^b \mathbf{g}(x) \cdot \mathbf{h}(x) \, dx = 0 \quad \forall \mathbf{h}(x) \quad \text{then} \quad \mathbf{g}(x) \equiv \mathbf{0} \text{ on } a \leq x \leq b. \tag{3.24}$$

In other words, if the value of the integral is zero for all choices of component perturbation functions in $\mathbf{h}$ then each component of $\mathbf{g}$ must be identically zero, $g_i(x) \equiv 0$ for $i = 1, 2, \ldots, n$.

Hence we can arrive at the (general) result that the critical point condition will yield separate Euler–Lagrange equations with respect to each variable with an independent perturbation. In this case

$$\forall h_1 : \quad x\text{-Euler–Lagrange:} \qquad \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) = 0, \tag{3.25a}$$

$$\forall h_2 : \quad y\text{-Euler–Lagrange:} \qquad \frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial y'}\right) = 0. \tag{3.25b}$$

Thus systems of differential equations for individual particles moving in two or three dimensions ($\mathbf{x} = (x, y, z)$), or sets of particles ($\mathbf{x}_1, \mathbf{x}_2, \ldots$), can be obtained from the Lagrangian for the complete system, $L(t, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_1', \mathbf{x}_2', \ldots)$, through the application of the Euler–Lagrange equation to each independent unknown function.

3.4 The Influence of Boundary Conditions

In the previous examples, we have seen the important roles played by the boundary conditions in:

(i) specifying boundary conditions for the Euler–Lagrange problem
(ii) determining boundary conditions on the perturbation functions
(iii) eliminating the boundary terms generated by integration by parts in the calculation of the first variation

The last point is perhaps the most important in terms of reducing the problem to a differential equation; if it is not possible to eliminate the boundary terms, then the fundamental lemma cannot be applied. Choices of boundary conditions that are compatible with this requirement are called *natural boundary conditions*.[7]

So far, we have seen how to work with Dirichlet boundary conditions (e.g. $y(a) = c$). We will now consider how other types of boundary conditions affect problems.

3.4.1 Problems with a Free Boundary

Consider a modification of the problem of determining the shortest path we considered in Sect. 3.2.1: find the function $y(x)$ passing through the origin that gives the shortest path to the vertical line $x = 1$.

Being a problem in terms of the minimum distance, the functional to be minimised is the same as in (3.10a),

[7]Given the lack of elementary approaches if the fundamental lemma can not be used, it is tempting to call them *necessary boundary conditions*.

$$J(y) = \int_0^1 \sqrt{1 + (y')^2}\, dx,$$

but the problem statement now provides only one definite boundary condition, $y(0) = 0$. There is no specific requirement on y at $x = 1$ and hence this is called a *free boundary*. In order to see how to deal with such a situation, we revisit (3.7) to consider the form of admissible solutions,

$$\tilde{y}(x) = y_*(x) + \varepsilon h(x).$$

At $x = 0$, we should have zero perturbation, $h(0) = 0$ since $\tilde{y}(0) = y_*(0) = 0$. At $x = 1$, $\tilde{y}(1)$ and $y_*(1)$ are arbitrary, and so we have no information about $h(1)$.

Proceeding as before, we expand $J(y_* + \varepsilon h)$ for $\varepsilon \to 0$ as

$$\tilde{J} = J_* + \varepsilon \left(\left. \frac{y_*' h}{\sqrt{1 + (y_*')^2}} \right|_{x=0}^{x=1} - \int_0^1 \frac{d}{dx} \left(\frac{y_*'}{\sqrt{1 + (y_*')^2}} \right) h\, dx \right) + O(\varepsilon^2).$$

We need the total contribution from the boundary terms to vanish, requiring

$$\frac{y_*'(1) h(1)}{\sqrt{1 + y_*'(1)^2}} - 0 = 0,$$

where the $x = 0$ boundary term vanishes due to $h(0) = 0$. Since we do not know anything about $h(1)$, in order to guarantee that the remaining term vanishes, we require that the optimal solution satisfies the natural boundary condition, $y_*'(1) = 0$.

It is important to note that irrespective of the form taken by the natural boundary conditions, they are only involved in eliminating the boundary terms in order to leave the integral in a form compatible with the fundamental lemma. Consequently, the Euler–Lagrange equation is again given by (3.21) and (in this case) leads to the same ODE as found for the prescribed end-point case (3.16). However, the new boundary conditions for the Euler–Lagrange problem, $y_*(0) = 0$ and $y_*'(1) = 0$, now select a different solution, $y_*(x) \equiv 0$ (corresponding to the line from the origin along the x-axis up to the point of intersection with the line $x = 1$).

3.4.2 Problems with a Variable Endpoint

Another version of the "find the shortest path" problem is to find $y(x)$ that gives the shortest path from the origin to a given curve $y = f(x)$ (see Fig. 3.3).

The boundary condition $y_*(0) = 0$ holds as before, but now the other end-point can be any point on the curve $y = f(x)$. Let us denote this unknown point as $x = b$ and then the boundary condition is simply

$$y(b) = f(b). \tag{3.26a}$$

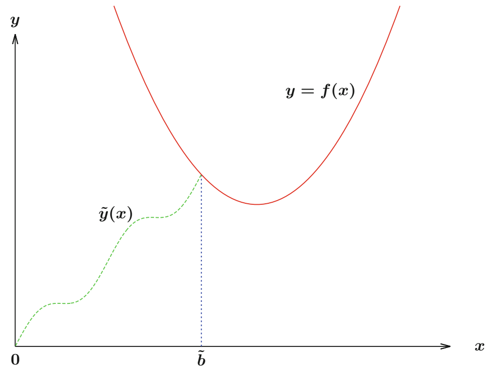

Fig. 3.3 A trial solution, $\tilde{y}(x)$ on $0 \le x \le \tilde{b}$, for the "shortest path to a given *curve*" variable endpoint problem (3.26)

The objective functional is (as before) the total arclength

$$J(y) = \int_0^b \sqrt{1 + (y')^2}\, dx. \tag{3.26b}$$

We begin the analysis as usual, by adding a perturbation to *each unknown*, $y(x)$ and b. As before, the admissible solutions take the form

$$\tilde{y}(x) = y_*(x) + \varepsilon h(x),$$

while the admissible endpoints can be written as

$$\tilde{b} = b_* + \varepsilon c,$$

where c is a perturbation constant. Consequently, we have

$$\tilde{J} = \int_0^{b_* + \varepsilon c} \sqrt{1 + (y'_* + \varepsilon h')^2}\, dx. \tag{3.27}$$

At this point in previous examples, we have Taylor expanded $\tilde{J}$ in the limit $\varepsilon \to 0$. A new complication present in this problem is that a perturbation appears in the limits of integration as well as in the integrand. One effective tool for dealing with this issue is Leibniz's rule for the derivative of an integral

$$\begin{aligned}\frac{d}{dz}&\left(\int_{a(z)}^{b(z)} g(x, y(x), z)\, dx\right)\\ &= \int_a^b \frac{\partial g}{\partial z}\, dx + \left[g(b, y(b), z)\frac{db}{dz} - g(a, y(a), z)\frac{da}{dz}\right].\end{aligned} \tag{3.28}$$

Applying Leibniz's rule to (3.27) with ε playing the role of z, we obtain the first variation in the form

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^{b_*} \frac{y_*' h'}{\sqrt{1+(y_*')^2}}\,dx + c\sqrt{1+y_*'(b_*)^2}.$$

After using integration by parts and imposing the critical point condition, this yields

$$\left(c\sqrt{1+y_*'(b_*)^2} + \frac{y_*'(b_*)h(b_*)}{\sqrt{1+y_*'(b_*)^2}}\right) - \int_0^{b_*} \frac{d}{dx}\left(\frac{y_*'}{\sqrt{1+(y_*')^2}}\right) h\,dx = 0.$$

If the boundary terms vanish, then applying the fundamental lemma to the integral will yield the expected Euler–Lagrange equation, but it is not yet clear which boundary conditions on y achieve this since c and $h(b_*)$ are unknown. However, we have not yet fully utilised (3.26a), namely that $\tilde{y}(\tilde{b}) = f(\tilde{b})$. Writing this out more explicitly, we have

$$y_*(b_* + \varepsilon c) + \varepsilon h(b_* + \varepsilon c) = f(b_* + \varepsilon c). \tag{3.29}$$

Expanding (3.29) in the limit $\varepsilon \to 0$, matching first terms in the respective expansions of the left- and right-hand sides, we recover $y_*(b_*) = f(b_*)$ for the optimal solution. Matching $O(\varepsilon)$ terms, we find that

$$y_*'(b_*)c + h(b_*) = f'(b_*)c$$

which can be solved for c

$$c = \frac{h(b_*)}{f'(b_*) - y_*'(b_*)}.$$

Substituting this result into the boundary terms yields

$$\left(\frac{y_*'(b_*)}{\sqrt{1+y_*'(b_*)^2}} + \frac{\sqrt{1+y_*'(b_*)^2}}{f'(b_*) - y_*'(b_*)}\right)h(b_*).$$

Since no constraints are known on the value of $h(b_*)$, in order to ensure that this term vanishes for all admissible perturbations, we require that the terms in parentheses sum to zero, which (after a little algebra) yields the natural boundary condition

$$y'(b_*) = -\frac{1}{f'(b_*)}. \tag{3.30}$$

Recalling that the Euler–Lagrange equation yields straight lines as solutions (3.17), this condition selects the minimum distance to the curve via the line that intersects the curve perpendicularly (recall the negative reciprocal slope relation from analytic geometry). While it may seem unusual to have three boundary conditions (on $y(0)$, $y(b)$, $y'(b)$) for a second order differential equation, it is not overdetermined since b_* is unknown and has to be determined as part of the solution.

3.5 Optimisation with Constraints

While the solutions of many systems are determined from optimising some property of the system, in some cases, the choice of the optimal solution is constrained further by some implicit structure in the problem—such as a finite amount of building materials limiting the maximum size of a structure, or finite time limiting how thoroughly a task can be performed. These are examples of *constrained optimisation problems*—the specific constraints are crucial in selecting the relevant solution. In simple problems, the constraints can be substituted directly into the objective function in order to obtain a reduced problem describing all achievable solutions (see Exercise 3.5, for example), but for problems where this cannot be done, the method of *Lagrange multipliers* can be employed instead. This approach will be used to investigate several classes of calculus of variations problems in the remaining sections of this chapter,

(i) Isoperimetric problems: optimising a functional subject to a condition on an integral of the solution.
(ii) Holonomic systems: optimising a functional subject to a geometric condition applied pointwise on the solution.
(iii) Optimal control: optimising a functional subject to a differential equation applied pointwise on the solution.

We begin with a brief review of the method of Lagrange multipliers applied to problems from multivariable calculus.

3.5.1 Review of Lagrange Multipliers

In constrained optimisation problems, feasible solutions lie within a subset of the space of all possible solutions. The method of *Lagrange multipliers* identifies the feasible solutions through the same process used in unconstrained optimisation problems—namely obtaining the possible solutions from solving equations determined by the critical point condition (cf. (3.4))

$$\frac{\partial(\text{objective})}{\partial(\text{variables})} = \mathbf{0}.$$

The Lagrange multiplier approach constructs an "augmented objective function" $\mathscr{L}$ incorporating the original objective function, call it $f(x, y)$, along with all constraints so that the critical points are still described by a gradient condition

$$\nabla \mathscr{L} = \mathbf{0} \qquad \Leftrightarrow \qquad \text{all critical point solutions.} \tag{3.31}$$

Consider the fundamental problem of maximising a function of two variables subject to a single constraint:

Find (x_*, y_*) yielding $\max f(x, y)$ from among the points (x, y) on the curve $g(x, y) = 0$,

where $g(x, y) = 0$ is the implicit equation of a given curve.

Suppose that the parametric equations describing the $g = 0$ curve are known, $x = x(t)$, $y = y(t)$, such that

$$g(x(t), y(t)) = 0 \qquad \forall t.$$

Differentiating this equation using the chain rule yields

$$0 = \frac{d}{dt}\big[g(x(t), y(t))\big] = \nabla g \cdot \frac{d\mathbf{x}}{dt} = 0. \tag{3.32}$$

If the parameter t is taken to represent time, the geometric interpretation of this equation is that the "velocity" vector $d\mathbf{x}/dt$ (which is tangent to the curve $\mathbf{x}(t)$) is perpendicular to the level-curve $g = 0$ since the gradient ∇g is orthogonal to contours of constant function-value.

Substituting $x(t), y(t)$ into $f(x, y)$ yields a function of a single variable

$$F(t) = f(x(t), y(t))$$

that is parametrised along the curve $g = 0$. Finding the critical points of $F(t)$ with respect to t gives the critical points of f on g,

$$\frac{dF}{dt} = 0 \qquad \Longrightarrow \qquad \frac{dF}{dt} = \nabla f \cdot \frac{d\mathbf{x}}{dt} = 0. \tag{3.33}$$

Hence, the chain rule shows that at a critical point, ∇f is also orthogonal to the velocity vector. In two dimensions, this forces the two gradients to be a scalar multiple of each other and then we obtain

$$\nabla f = \lambda \nabla g$$

where λ (called the *Lagrange multiplier*) is another unknown that needs to be determined. In component form, the resulting set of equations of three variables in the three unknowns is

$$f_x(x_*, y_*) = \lambda g_x(x_*, y_*), \quad f_y(x_*, y_*) = \lambda g_y(x_*, y_*), \quad g(x_*, y_*) = 0.$$

A typical solution strategy for solving the equations is to first solve the $\nabla f = \lambda\nabla g$ equations to obtain $x(\lambda), y(\lambda)$, and then substitute these into $g(x(\lambda), y(\lambda)) = 0$ to obtain a final equation for the values of λ.

This approach generalises in a straightforward manner to higher dimensions and to multiple constraints ($\mathscr{L} \equiv f - \lambda g - \mu h - \cdots$) [24, 68]. We note that had the parametric equations for $x(t), y(t)$ been available, then solutions could have been obtained directly from (3.33). The Lagrange approach does not actually use these parametric equations, just the assumption of their existence, to reformulate the problem into a convenient form.

The above discussion motivates the introduction of the augmented (constrained) Lagrange objective function satisfying (3.31),

$$\mathscr{L}(x, y, \lambda) \equiv f(x, y) - \lambda g(x, y). \tag{3.34}$$

This is just a convenient form that reproduces the set of equations above as the critical point equations for $\mathscr{L}$ with respect to its three variables,

$$\nabla\mathscr{L} = \mathbf{0} \qquad \leftrightarrow \qquad \{\, \mathscr{L}_x = 0, \qquad \mathscr{L}_y = 0, \qquad \mathscr{L}_\lambda = 0 \,\}. \tag{3.35}$$

3.6 Integral Constraints: Isoperimetric Problems

Consider problems where the solution $y_*(x)$ satisfies

$$\max_y \left(J \equiv \int_a^b L(x, y, y')\, dx \right) \qquad \text{subject to} \qquad G \equiv \int_a^b g(x, y, y')\, dx = 0. \tag{3.36}$$

These are sometimes called *isoperimetric problems* with the name referring back to a classic geometric problem of maximising the area enclosed by a curve constrained to have a fixed perimeter (see Exercise 3.20).

Maximising J subject to the constraint $G = 0$ can be expressed in terms of an augmented Lagrange functional of the form

$$I(y, \lambda) = J - \lambda G, \tag{3.37}$$

where λ is a constant. This can also be restated in terms of an augmented Lagrangian function

$$I = \int_a^b \mathscr{L}\, dx \qquad \text{where} \qquad \mathscr{L}(x, y, y', \lambda) = L(x, y, y') - \lambda g(x, y, y'). \tag{3.38}$$

Once this functional has been identified, we follow the standard variational process described in Sect. 3.2, starting with the introduction of perturbations to all of the unknowns

$$\tilde{y}(x) = y_*(x) + \varepsilon h(x), \qquad \tilde{\lambda} = \lambda_* + \varepsilon\gamma.$$

Expanding $\tilde{I}$ as a Taylor series for $\varepsilon \to 0$ and enforcing the critical point condition on the $O(\varepsilon)$ term yields

$$\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \lambda_* \left\{\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right\}\right] h\,dx + \gamma \int_a^b g(x, y_*, y_*')\,dx = 0.$$

Requiring this to hold with respect to (i) all possible γ perturbations recovers the geometric constraint

$$\int_a^b g(x, y_*, y_*')\,dx = 0, \tag{3.39a}$$

and (ii) all $h(x)$ perturbations using the fundamental lemma yields the Euler–Lagrange equation

$$\frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial y'}\right) = 0. \tag{3.39b}$$

This again shows that the form of the Euler–Lagrange equation (3.21) generalises to a wide array of problems.

As an example, consider the problem of finding the function $y(x) > 0$ on $0 < x < 1$ satisfying boundary conditions $y(0) = 0$ and $y(1) = 0$, with a given arclength, $S = \int_0^1 \sqrt{1 + (y')^2}\,dx$ that maximises the area under the curve, $A = \int_0^1 y\,dx$.

We can write the augmented Lagrangian $\mathscr{L} = A - \lambda P$ to maximise the enclosed area subject to the constraint of a fixed perimeter, $P = S + 1$ (which includes the contribution from the lower boundary, $y \equiv 0$). Applying (3.39a, 3.39b), we obtain

$$1 + \lambda_* \frac{d}{dx}\left(\frac{y_*'}{\sqrt{1 + (y_*')^2}}\right) = 0, \qquad \int_0^1 \left(\sqrt{1 + (y_*')^2} - S\right) dx = 0.$$

Solving the ODE and imposing the boundary conditions, yields

$$y_*(x) = \sqrt{\lambda_*^2 - (x - \tfrac{1}{2})^2} - \sqrt{\lambda_*^2 - \tfrac{1}{4}}, \tag{3.40}$$

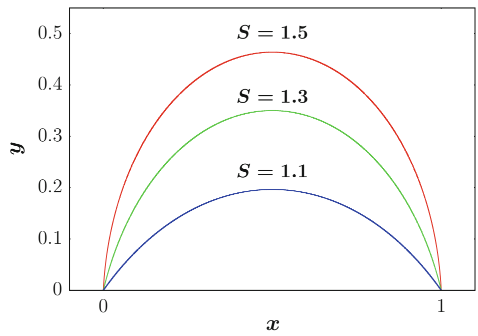

Fig. 3.4 The solution $y_*(x)$ (3.40) maximising the area under the curve on $0 \le x \le 1$ for given arclength S

which can be integrated to yield the resulting area and arclength for a given value of the Lagrange multiplier λ_*,

$$a(\lambda_*) \equiv \int_0^1 y_*(x)\,dx = \lambda_*^2 \arcsin\left(\frac{1}{2\lambda_*}\right) - \sqrt{\lambda_*^2 - \tfrac{1}{4}},$$

$$s(\lambda_*) \equiv \int_0^1 \sqrt{1 + y_*'(x)^2}\,dx = 2\lambda_* \arcsin\left(\frac{1}{2\lambda_*}\right).$$

To complete the solution of the constrained problem, the arclength constraint, $s(\lambda_*) = S$, must be applied to determine a value for λ_* in terms of S.

From (3.40) we see that acceptable (real-valued) solutions must have $\lambda_* \ge \frac{1}{2}$, which limits S in the problem statement to be in the range $1 \le S \le \pi/2$. Noting that (3.40) describes arcs of circles going through the fixed endpoints, with centres given by $(\frac{1}{2}, -\sqrt{\lambda_*^2 - \frac{1}{4}})$, with λ_* being the radius of the circle, see Fig. 3.4. The fact that (3.40) is not valid for $S > \pi/2$ suggests that the assumed form of the solutions (in this case, graphs of functions $y = y(x)$) may be too restrictive and different representations of the problem could be helpful, see Exercise 3.20.

3.7 Geometric Constraints: Holonomic Problems

Consider the problem of finding parametric equations for a curve $(x_*(t), y_*(t)$ on $0 \le t \le T)$ that minimises the integral

$$I = \int_0^T L(t, x, y, x', y')\,dt, \tag{3.41a}$$

subject to the constraint that for each value of t,

$$g(x(t), y(t), t) = 0. \tag{3.41b}$$

Problems of this form are called *holonomic problems* in the context of dynamics of mechanical systems. The constraint (3.41b) imposes a condition that the motion in the system must satisfy at each instant of time, as typically imposed by structural *geometric constraints* (as in the case of a roller coaster car moving on its track).

It can be shown that the appropriate generalisation of (3.34) for holonomic problems is the augmented Lagrangian

$$\mathscr{L}(x, y, \lambda) = L(t, x, y, x', y') - \lambda(t) g(x, y, t) \quad \Longrightarrow \quad \mathscr{I} = \int_0^T \mathscr{L}\, dt. \tag{3.42}$$

Here, a notable difference from (3.38) for isoperimetric problems is that the Lagrange multiplier is a function of t, rather than a constant, and cannot be factored out of the integral as we did in (3.37). Applying independent perturbations to each unknown,

$$\tilde{x}(t) = x_*(t) + \varepsilon h_1(t), \quad \tilde{y}(t) = y_*(t) + \varepsilon h_2(t), \quad \tilde{\lambda}(t) = \lambda_*(t) + \varepsilon \gamma(t),$$

and expanding $\mathscr{I}$ as $\varepsilon \to 0$, we obtain

$$\begin{aligned} \delta \mathscr{I}_* = \int_0^T \Bigg\{ & \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial x'} \right) \right] h_1 + \left[\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial y'} \right) \right] h_2 \\ & -\lambda_* \left(\frac{\partial g}{\partial x} h_1 + \frac{\partial g}{\partial y} h_2 \right) - g\gamma \Bigg\} dt, \end{aligned} \tag{3.43}$$

where the terms in the first line give the contributions from the original objective function and the terms in the second line represent the influence of the constraint. Noting that all perturbations are independently allowable, in enforcing the critical point condition, $\delta \mathscr{I}_* = 0$ (through linear independence, as in (3.24)) we conclude that the coefficient of each perturbation must vanish

$$\begin{aligned} &\forall h_1: \quad x\text{Euler–Lagrange:} && \frac{\partial \mathscr{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial x'} \right) = 0, \\ &\forall h_2: \quad y\text{Euler–Lagrange:} && \frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial y'} \right) = 0, \\ &\forall \gamma: \quad \text{Geometric constraint:} && g(x_*(t), y_*(t), t) = 0, \end{aligned}$$

where we have combined the L, g terms from (3.43) to write the Euler–Lagrange equations in a more compact form using (3.42). In fact, this system can be condensed further to be expressed as

$$u \text{ Euler–Lagrange:} \qquad \frac{\partial \mathscr{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial u'}\right) = 0 \qquad u = \{x, y, \lambda\},$$

where u denotes each unknown function of t in the system. In other contexts, systems containing both geometric constraints and differential equations are often also called *differential algebraic equations* (DAE).

3.8 Differential Equation Constraints: Optimal Control

For holonomic problems, we considered optimising a functional subject to constraints on the possible solution applied pointwise over the entire domain. We now extend the analysis to systems where the constraint is given by a differential equation. In such cases, we want to minimise

$$J = \int_0^T L(t, x(t), u(t))\, dt, \tag{3.44a}$$

subject to the constraint that at each t on $0 \le t \le T$,

$$\frac{dx}{dt} = f(t, x(t), u(t)), \tag{3.44b}$$

along with given conditions on $x(t)$

$$x(0) = x_0 \qquad x(T) = x_1. \tag{3.44c}$$

Problems of this form arise in *optimal control theory* which seeks to determine a *control function* $u(t)$ that allows the *state function* $x(t)$ to achieve a desired *target value* (or "final state") x_1 while also minimising the "cost" of the overall process in imposing the control on the evolution of the state. The cost will be represented by the integral (3.44a). The ending time T will also be considered as a degree of freedom and used to help reduce the cost and make the target achievable, as some systems might be able to run cheaply, if perhaps slowly.

Equation (3.44b) is called a *state equation*; it describes how the system evolves based on a dynamical model relevant to the problem (e.g. mechanics or chemical kinetics). In the absence of external forcing, described here by the control function $u(t)$, ($u \equiv 0$), then the natural (*uncontrolled*) dynamics starting from the initial condition (3.44c)$_1$ are given by the solution of

$$\frac{d\hat{x}}{dt} = f(\hat{x}, 0, t) \qquad \hat{x}(0) = x_0. \tag{3.45}$$

Under typical assumptions on the rate function f, this initial value problem has a well-defined solution $\hat{x}(t)$. However, this solution might not equal the target value x_1 at any value of $T \geq 0$, or might reach it only after an undesirably long time. Both of these situations would make the uncontrolled solution $\hat{x}$ unacceptable, and hence we need to consider modifying the evolution of the state from (3.45) by forcing the system with some control function $u(t)$ yielding (3.44b).

Appropriate control functions may allow the state to reach the target, but may also involve excessive input of energy, or could still take too long. Hence, the "optimality" in optimal control theory refers to minimising the cost involved in imposing the control. Two basic types of cost functionals that can be used, depending on whether system speed or minimisation of the energy of the control function is the priority, are given respectively by

$$J_{\text{speed}} = \int_0^T 1\, dt = T, \qquad J_{\text{energy}} = \int_0^T u^2\, dt. \tag{3.46}$$

Examples of systems that can be described in this form include the input control of chemicals in reaction systems to maintain a steady output, the design of controlled-release timed drug delivery, and car power-steering systems. In addition to uses in "engineered" systems, such models can also be applied to describe how biological systems adapt to their environments.

System (3.44) is a classic optimal control theory problem—its solution involves determining the evolution of the state $x(t)$, the control $u(t)$, and the optimal stopping time T. The imposition of the state equation at each time suggests forming an augmented Lagrangian with a time-dependent Lagrange multiplier, as in (3.42),

$$\mathscr{L}(x, x', u, \lambda) = L(t, x, u) - \lambda(t)\left(\frac{dx}{dt} - f(x, u, t)\right) \tag{3.47}$$

and corresponding augmented functional

$$I = \int_0^T \mathscr{L}(x, x', u, \lambda)\, dt. \tag{3.48}$$

Since the stopping time is also an unknown, we will also draw on the analysis carried out for the variable endpoint problem in Sect. 3.4.2.

Applying perturbations to all of the unknowns

$$\tilde{x} = x_* + \varepsilon h(t) \qquad \tilde{u} = u_* + \varepsilon v(t) \qquad \tilde{\lambda} = \lambda_* + \varepsilon\gamma(t) \qquad \tilde{T} = T_* + \varepsilon S,$$

the cost functional is then

$$\tilde{I} = \int_0^{T_*+\varepsilon S} \mathscr{L}(x_* + \varepsilon h, x_*' + \varepsilon h', u_* + \varepsilon v, \lambda_* + \varepsilon\gamma)\, dt. \tag{3.49}$$

Expanding $\tilde{I}$ as a Taylor series for $\varepsilon \to 0$ gives

$$\tilde{I} = I_* + \varepsilon \left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} + O(\varepsilon^2),$$

where, using Leibniz' rule (3.28), we obtain the first variation in the form

$$\int_0^{T_*} \left[\frac{\partial \mathscr{L}}{\partial x} h + \frac{\partial \mathscr{L}}{\partial x'} h' + \frac{\partial \mathscr{L}}{\partial u} v + \frac{\partial \mathscr{L}}{\partial \lambda} \gamma \right] dt + \left.\mathscr{L}_*\right|_{t=T_*} S.$$

Applying integration by parts to the h' term, the first variation becomes

$$= \int_0^{T_*} \left[\left\{ \frac{\partial \mathscr{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial x'} \right) \right\} h + \frac{\partial \mathscr{L}}{\partial u} v + \frac{\partial \mathscr{L}}{\partial \lambda} \gamma \right] dt + \left.\left(\mathscr{L} S + \frac{\partial \mathscr{L}}{\partial x'} h \right)\right|_{t=T_*}, \tag{3.50}$$

where we have used the initial condition $(3.44c)_1$ to obtain that the perturbation satisfies $h(0) = 0$. Recalling the form of the condition at the variable endpoint (3.26a), we write the target/final value condition, $\tilde{x}(\tilde{T}) = x_1$, as

$$x_*(T_* + \varepsilon S) + \varepsilon h(T_* + \varepsilon S) = x_1. \tag{3.51}$$

Expanding this equation in the limit $\varepsilon \to 0$, we recover $x_*(T_*) = x_1$ by matching leading terms, and at $O(\varepsilon)$ the perturbation function satisfies

$$h(T_*) = -x_*'(T_*) S. \tag{3.52}$$

Substituting this into the first variation (3.50), the boundary term reduces to

$$\left.\left(\mathscr{L} - x' \frac{\partial \mathscr{L}}{\partial x'} \right)\right|_{t=T_*} S.$$

This combination of terms is called the *Legendre transform* of the Lagrangian with respect to the state variable. It will be convenient to define a new function, the *Hamiltonian*,[8] from this combination,

$$\mathscr{H} \equiv \mathscr{L} - x' \frac{\partial \mathscr{L}}{\partial x'}. \tag{3.53}$$

[8]Note that in a different context, there is another definition of the Hamiltonian having $H = -\mathscr{H}$. Despite the difference in sign conventions, both are called Hamiltonians, see Exercise 3.7.

Since $\mathscr{L} = L - \lambda(x' - f)$, we find that the Hamiltonian is given by

$$\mathscr{H} = L + \lambda f, \tag{3.54}$$

and hence the boundary term becomes $\mathscr{H}|_{t=T_*}S$. Imposing the critical point condition $\delta I_* = 0$ for all independent perturbations ($\forall h$, $\forall v$, $\forall \gamma$, $\forall S$), we obtain the four equations:

$$\frac{\partial \mathscr{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial x'}\right) = 0, \qquad \frac{\partial \mathscr{L}}{\partial u} = 0, \qquad \frac{\partial \mathscr{L}}{\partial \lambda} = 0, \qquad \mathscr{H}(T_*) = 0. \tag{3.55}$$

The last equation provides a natural final condition on the Hamiltonian. Making use of the fact that it can be shown that if $\mathscr{L}$ does not explicitly (non-autonomously) depend on t, then the Hamiltonian is constant (see Exercise 3.7), the condition at T_* determines that $\mathscr{H}(t) \equiv 0$ for $0 \le t \le T_*$.

The third equation in (3.55) returns the state equation (3.44b). Similarly, the first equation yields an ordinary differential equation for the evolution of the Lagrange multiplier,

$$\left(\frac{\partial L}{\partial x} + \lambda_* \frac{\partial f}{\partial x}\right) + \frac{d\lambda_*}{dt} = 0, \tag{3.56}$$

which is called the *co-state equation*, with $\lambda_*(t)$ (responsible for imposing the state-equation constraint) being called the *co-state* in control theory.

Finally, the perturbation with respect to the control, $(3.55)_2$ gives a geometric constraint on the control at each time t on $0 \le t \le T_*$,

$$\frac{\partial L}{\partial u} + \lambda_* \frac{\partial f}{\partial u} = 0. \tag{3.57}$$

We apply the above derivation to a simple example of an optimal control problem. Consider the initial value problem for $x(t)$,

$$\frac{dx}{dt} = -3x + u \qquad x(0) = 2.$$

We immediately see that the uncontrolled system would have the exponentially decaying solution, $\hat{x}(t) = 2e^{-3t}$. As an illustrative problem where the influence of control is crucial, let the final target state for $T > 0$ be

$$x(T) = 5,$$

which is unattainable with the uncontrolled solution. Consider minimising a control-based cost functional, like J_{energy} from (3.46),

$$\min J = \frac{1}{2}\int_0^T u^2\,dt.$$

We can directly identify the Lagrangian function from J as $L = \frac{1}{2}u^2$ and the rate function from the state equation as $f = -3x + u$. Hence the augmented Lagrangian is

$$\mathscr{L} = \tfrac{1}{2}u^2 - \lambda(x' + 3x - u),$$

and the Hamiltonian is

$$\mathscr{H} = \tfrac{1}{2}u^2 + \lambda(u - 3x).$$

From the Euler–Lagrange equations (3.55), we then find

$$-3\lambda + \frac{d\lambda}{dt} = 0, \qquad u + \lambda = 0, \qquad \frac{dx}{dt} + 3x - u = 0.$$

Starting with the algebraic relation, we can eliminate the control in terms of the co-state, $u = -\lambda$. The co-state equation for λ can then be re-expressed for u as

$$\frac{du}{dt} = 3u \qquad \Longrightarrow \qquad u(t) = Ae^{3t},$$

where A is a constant of integration. With this expression for $u(t)$, the state equation becomes

$$\frac{dx}{dt} = -3x + Ae^{3t} \qquad \Rightarrow \qquad x(t) = Be^{-3t} + \tfrac{1}{6}Ae^{3t},$$

where B is a second constant of integration. Substituting x, u and λ into the Hamiltonian yields

$$\mathscr{H} = 3AB = 0,$$

so that either A or B is zero. The choice $A = 0$ returns the uncontrolled state, and so we conclude that $B = 0$ and hence $x(t) = 2e^{3t}$. Illustrating the dramatic influence of the control, the optimal controlled solution is exponentially growing, whereas the uncontrolled solution is exponentially decaying. The optimal stopping time can then be obtained directly from the optimal solution for this simple problem as $T_* = \frac{1}{3}\ln(5/2)$.

3.9 Further Directions

There is an extensive literature on both the theory and the applications of the calculus of variations. Some good texts with additional introductory material and other advanced topics include [64, 66, 71, 84, 104]. Further texts also provide more rigorous analysis [99]. Presentations of the applications of the calculus of variations in mechanics [40, 62, 67] and other areas of applied physics, including electrostatics and quantum mechanics, [104] leading to ODE and PDE problems can also be found in many classic engineering and physics textbooks. There are also numerous books that present constrained optimisation and optimal control problems as extensions of the basic method of the calculus of variations [59, 62, 66, 71, 84, 99].

While the introductory presentation in this chapter as focused on formulating problems leading to ODEs, some of the exercises will show that PDEs can be derived similarly from multiple integrals. More challenging problems arise when assumptions on the smoothness of solutions are removed, then optimal solutions may be composed of multiple piecewise-smooth sections that must satisfy appropriate connection conditions at transition points.

3.10 Exercises

3.1 Consider the functional for $y(x)$ on $0 \leq x \leq 1$,

$$J = \int_0^1 \tfrac{1}{2}(y')^2 + \frac{k}{x}yy' + x^2 y\, dx,$$

where k is a constant

(a) Determine the ODE for $y_*(x)$ by taking the variation of J.
(b) Show that the ODE can be also obtained directly from the Euler–Lagrange equation (3.21).
(c) For $k = 0$, determine $y_*(x)$ satisfying the boundary conditions $y(0) = 1$, $y(1) = 1$.
(d) For $k = 2$, determine $y_*(x)$ satisfying the boundary conditions $y'(0) = 1$, $y(1) = 1$. Show there is no solution that satisfies $y(0) = 1$.

3.2 (*The second derivative test*) As in single- and multi-variable calculus, whether a critical point is a local maximum or minimum can be determined from the next term in the local expansion of the objective function. Recall the problem of minimising the arclength of the curve $y = y(x)$, (3.10a),

$$J = \int_0^1 \sqrt{1 + (y')^2}\, dx, \qquad y(0) = 0, \qquad y(1) = b.$$

Setting $y(x) = y_*(x) + \varepsilon h(x)$ and expanding J to $O(\varepsilon)$ yielded the Euler–Lagrange equation.

(a) Continue the expansion of J to the $O(\varepsilon^2)$ term to obtain the *second variation*, $\delta^2 J$.
(b) Show that for this problem the solution $y_*(x) = bx$ is indeed a local minimiser by showing that the second variation is positive for all $h(x) \not\equiv 0$.

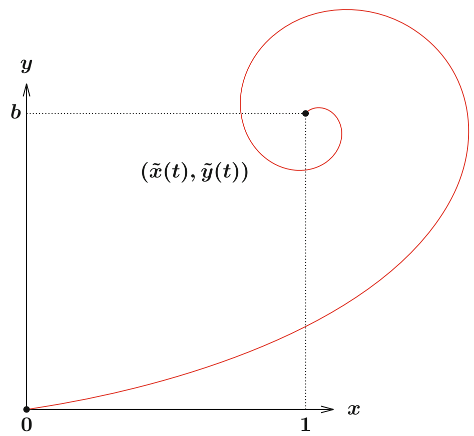

Fig. 3.5 Exercise 3.3: An admissible trial solution in the form of a parametric curve

3.3 Consider finding the shortest path between two points from among all possible smooth parametric curves in the xy plane $\mathbf{x}(t) = (x(t), y(t))$ for $0 \le t \le T$. Let the curve start at $(x, y) = (0, 0)$ at $t = 0$ and end at $(x, y) = (1, b)$ at $t = T$ for T fixed (see Fig. 3.5). The functional for the arclength of the curve is

$$J(x, y) = \int_0^T \sqrt{(x')^2 + (y')^2}\, dt.$$

Show that the Euler–Lagrange equations for x, y give the parametric equations of the straight line found as the solution of (3.10a, 3.10b).

3.4 For problems where the distance between two points is not given by the Euclidean distance, *geodesics* are curves giving the shortest path between two points. Consider the problem of determining a geodesic between two points on a surface $z = F(x, y)$.

(a) For a parametric curve $(x(t), y(t), z(t))$ that minimises the arclength,

$$J(x, y) = \int \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt,$$

observe that the resulting Euler–Lagrange equations for $(x(t), y(t))$ are rather complicated, even if the surface is the simple paraboloid, $z = k(x^2 + y^2)$.

(b) Show that even if we restrict the solutions to be functions, with $y = y(x)$, the Euler–Lagrange equation for $y(x)$ on the paraboloid is still a challenging nonlinear ODE for $k \neq 0$.

(c) Consider the problem of finding the geodesic from $(x, y) = (-1, 0)$ to $(1, 0)$ on the paraboloid. Determine the arclengths of the following trial solutions for $k \ge 0$ (from geometry or by calculating the integral):

(i) the path along the x-axis,
(ii) the semicircular path connecting the points.
What can you infer about the geodesic from the limits $k \to 0$ and $k \to \infty$ of (i, ii)?

3.5 The Euler–Lagrange equations for constrained motion can be obtained by starting with the Lagrangian for general unconstrained motion and then substituting-in the parametric equations describing the geometric constraint (a curve or surface) before taking variations.

(a) Consider the action integral for two-dimensional motion of a particle subject to gravity

$$I = \int \tfrac{1}{2}m\left[x'(t)^2 + y'(t)^2\right] - mgy(t)\,dt.$$

Consider a particle constrained to be on a circle, $x^2+y^2 = \ell^2$. Derive the equation of a pendulum by first plugging the parametric equations $x(t) = \ell\sin\theta(t)$, $y(t) = -\ell\cos\theta(t)$ into I and then applying the principle of least action with respect to $\theta(t)$.

(b) Repeat (a) for the motion of a particle constrained to the curve $y = f(x)$.

(c) Consider the action integral for three-dimensional motion of a particle subject to gravity

$$I = \int \tfrac{1}{2}m\left[x'(t)^2 + y'(t)^2 + z'(t)^2\right] - mgz(t)\,dt$$

Derive the equations of motion for a particle moving on the surface of a cone, $x^2 + y^2 = z^2$, using the parametric equations $x(t) = r(t)\cos\theta(t)$, $y(t) = r(t)\sin\theta(t)$ and $z(t) = r(t)$.

3.6 Consider a pendulum whose suspension point is vertically oscillated. Let the length of the pendulum be ℓ and its mass m. The acceleration due to gravity, g, is acting downward in the $-y$ direction. If the suspension point is oscillating according to $\bar{y}(t) = -\sigma\sin(\omega t)$, then the position of the pendulum's mass is

$$x(t) = \ell\sin\theta(t), \qquad y(t) = -\sigma\sin(\omega t) - \ell\cos\theta(t),$$

where θ is measured from the $-y$ axis.

Write the Lagrangian for this system in terms of θ, θ' and obtain the Euler–Lagrange equation for this problem, called the *parametrically driven pendulum*.

3.7 Define the Hamiltonian, H, in terms of the Lagrangian $L(t, y, y')$, through the Legendre transform as

$$H \equiv -L + y'\frac{\partial L}{\partial y'}. \tag{3.58}$$

(Note the opposite sign convention relative to (3.53).)

(a) If L is independent of time (i.e. $L = L(y, y')$) show that the Hamiltonian is constant. (Hint: show that $dH/dt = 0$)
Further, show that for this case, the second-order Euler–Lagrange ODE (3.21) is equivalent to the first order ODE,

$$-L(y, y') + y' \frac{\partial L}{\partial y'} = C, \tag{3.59}$$

where C is a constant ($=H$); this reduction of the Euler–Lagrange equation to a first order equation is called the *Beltrami identity*, and is a special case of *Noether's theorem* [62].

(b) If the potential energy V does not depend on y', determine the most general form for the kinetic energy $T(t, y, y')$ so that the Hamiltonian is the total energy, $H = T + V$.

3.8 (*The brachistochrone*) Recall the problem of finding the curve of least-time descent under gravity from $(x, y) = (0, 1) \to (1, 0)$, starting from rest (Fig. 3.1 and Eq. (3.6)). We now complete the problem as follows:

(a) Assuming there is no friction, the total mechanical energy, $E = \frac{1}{2}mv^2 + mgy$, (where $v^2 = x'(t)^2 + y'(t)^2$) remains constant, set by its initial value. Use this to determine $v(y)$ and complete the expression of the functional in (3.6).
(b) Apply (3.21) to obtain the Euler–Lagrange second order ODE for $y_*(x)$.
(c) Noting that the integrand in (3.6) does not explicitly depend on x, use the Beltrami identity (3.59) to obtain a simpler first order ODE for $y_*(x)$. What is the value of the constant C in this ODE?
(d) Show that the solution can be expressed in parametric form by the equations of a cycloid with $k > 0$ (see Fig. 3.6),

$$x(\theta) = k(\theta - \sin\theta), \qquad y(\theta) = 1 - k(1 - \cos\theta). \tag{3.60}$$

What is the pair of equations that must be solved numerically to determine the value of k?

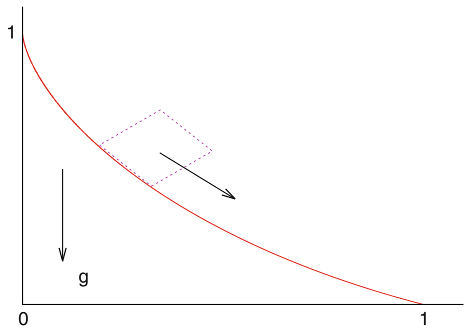

Fig. 3.6 Exercise 3.8: The solution of the brachistochrone problem, given by the cycloid (3.60)

3.9 Consider the functional for $y(x)$ on $1 \le x \le 2$

$$J(y) = \frac{1}{2}\int_1^2 \left[2y^2 + x^2\left(\frac{dy}{dx}\right)^2\right] dx.$$

Find the solution $y_*(x)$ that minimises J and satisfies the boundary condition $y(2) = 17$. What is the natural boundary condition on y_* at $x = 1$?

3.10 Obtain the curve $y = y_*(x)$ on $0 \le x \le b_*$ that starts at the origin, ends on the curve $y = 1 + (x - 1)^2$, and minimises the functional

$$J(y) = \frac{1}{2}\int_0^b (y')^2\, dx.$$

3.11 (*Higher-order Euler–Lagrange equations*) Derive the Euler–Lagrange equation for the functional

$$J(y) = \int_a^b L(x, y(x), y''(x))\, dx$$

Describe the kinds of natural boundary conditions that are needed.

3.12 Consider the functional for $y(x)$ on $0 \le x \le 1$,

$$J = \int_0^1 \left[\frac{dy}{dx}\frac{d^3y}{dx^3} - 240xy\right] dx$$

where $y(x)$ satisfies the boundary conditions

$$y'(0) = 0, \qquad y''(1) = 0, \qquad y'''(0) = 0, \qquad y(1) = 5.$$

(a) Write the expression for the first variation, δJ.
(b) Show that the critical point condition can be reduced to an ODE boundary value problem. Justify how each of the six boundary terms in δJ are eliminated. Solve the ODE problem for $y_*(x)$.

3.13 Consider the functional for $y(x)$ on $0 \le x \le 1$,

$$J = \int_0^1 \left[(y')^2 + (1 - 2x)\left(\int_0^x y^2(t)\, dt\right)\right] dx.$$

(a) Write the expression for the first variation, δJ.
(b) Write the four possible boundary conditions on $y(x)$ (two at each boundary) under which, the critical point condition, $\delta J = 0$, can be reduced to an ODE for $y(x)$.

3.14 Consider the functional for $y(x)$ on $0 \leq x \leq 1$,

$$J(y) = \int_0^1 \left[y \left(\frac{d^2y}{dx^2} \right)^2 + y^2 \right] dx + y'(0)y'(1)$$

with $y(x)$ satisfying the boundary conditions

$$y(0) = 2, \qquad y(1) = 5.$$

Determine the ODE boundary value problem for solutions that minimise or maximise J.

3.15 *Fermat's principle of least time* states that a beam of light will take a path that minimises its time of travel. The index of refraction, n, of a material gives the ratio of the speed of light in vacuum to the slowed speed of light in the material, $n = c/v \geq 1$.

Consider a layer of glass on $0 \leq x \leq 1$ whose index of refraction varies with position, $n = n(x)$. Let $y(x)$ describe the path of a beam of light entering the layer at $x = 0$ at a 45° angle, $y'(0) = 1$ (see Fig. 3.7 (left)).

(a) Making use of functional (3.6), derive the second-order ODE problem for $y(x)$ and show it can be reduced to a first-order equation.
(b) If $n(x)$ is piecewise constant, n_1 for $x < \frac{1}{2}$ and n_2 for $x > \frac{1}{2}$, show that part (a) reduces to *Snell's law* [91] (see Fig. 3.7 (right)).
(c) What is the form of the Euler–Lagrange equation if $n = n(x, y)$?

3.16 (*The beam equation*) Hamilton's principle of least action can be applied to derive the time-dependent partial differential equation for the transverse deflections $y = u(x, t)$ of a flexible solid rod or beam. The problem for a one-dimensional beam is characterised by the following properties:

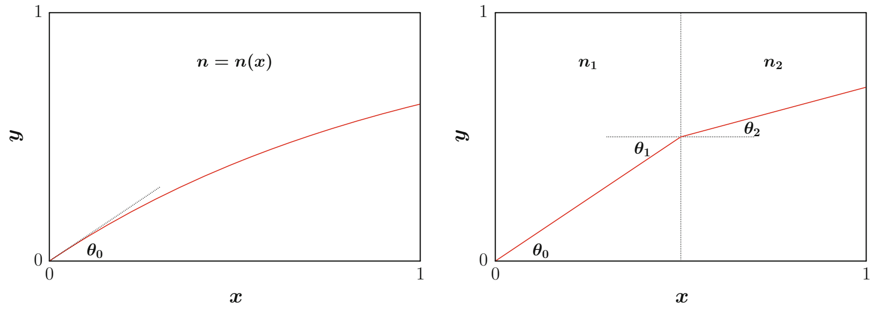

Fig. 3.7 Exercise 3.15: (*Left*) diffraction of the path of a ray of light due the change of the index of refraction with position, $n = n(x)$, (*Right*) The special case when n is piecewise constant

- The length of the beam is ℓ (domain: $0 \le x \le \ell$) with constant mass density ρ (mass per unit length), constant bending stiffness E (like a spring constant) and constant moment of inertia I.
- The overall total kinetic and potential energies of the beam are given by integrals over the length of the beam of the corresponding energy density functions, $e(u)$, $\mathscr{E} = \int_0^\ell e(u)\,dx$.
- The kinetic energy density is $\frac{1}{2}\rho(\partial_t u)^2$.
- The potential energy density due to bending (i.e. transverse deflection or buckling) is $\frac{1}{2}EI(\partial_{xx}u)^2$.

Follow these steps to derive the beam equation:

(a) Write the integrals for the total kinetic T and potential V energies of the beam.
(b) Write the action integral $J(u)$. (Hint: J is a double integral)
(c) Apply the principle of least action to derive the beam equation for $u(x,t)$ and state the choices of natural boundary conditions for u at $x = 0$ and $x = \ell$. Two boundary conditions (out of 2 sets of 2 options for natural boundary conditions) are needed at each boundary.

3.17 For a given positive function $k(x, y) > 0$, consider the integral for the function $u(x, y)$ on a finite two-dimensional domain D,

$$J(u) = \iint_D \tfrac{1}{2}k(x,y)\left(u_x^2 + u_y^2\right)\,dy\,dx.$$

Derive the partial differential equation for the solution $u(x, y)$ and two appropriate forms of natural boundary conditions on $u(x, y)$ that produce minima of $J(u)$.
If k is a constant, show that the PDE reduces to Laplace's equation.
Hint: Recall the vector version of the derivative product (for the divergence of a scalar times a vector), $\nabla \cdot (f\mathbf{g}) = (\nabla f) \cdot \mathbf{g} + f(\nabla \cdot \mathbf{g})$, and then make use of the divergence theorem.

3.18 How does the problem of minimising the arclength of a non-negative function $y(x) \ge 0$ on $0 \le x \le 1$ that encloses a fixed given area $A = \int_0^1 y\,dx$ relate to the isoperimetric example given in Sect. 3.6?

3.19 Find the solution $y(x)$ on $0 \le x \le \pi$ that minimises

$$J(x) = \int_0^\pi 1 + (y')^2\,dx \quad \text{subject to the constraint} \quad \int_0^\pi y^2\,dx = 80$$

and boundary conditions $y(0) = 0$ and $y(\pi) = 1$.

(a) Write the augmented Lagrangian $\mathscr{L}$ for this problem. Determine the Euler–Lagrange problem and solve for $y(x, \lambda)$ subject to the boundary conditions, where λ is the Lagrange multiplier.

(b) Evaluate the integral, $I(\lambda) = \int y^2\, dx$, as a function of λ. Plot this function and estimate the values of λ that yield solutions. If the integral constraint is required to equal one instead of 80, what happens to the solutions? What is the minimum value of I for which solutions exist?

3.20 (*The classic isoperimetric problem*) This problem will lead you through deriving that the circle is the closed curve of fixed perimeter P that encloses the maximum area:
Let $x(t)$, $y(t)$ be the parametric equations for a closed curve with $0 \le t \le 1$ that goes through the origin:

$$x(0) = x(1) = 0, \qquad y(0) = y(1) = 0.$$

(a) Use Green's theorem to show that the area enclosed by the curve is given by

$$\frac{1}{2}\int_0^1 \left[x(t)y'(t) - y(t)x'(t)\right]\, dt.$$

Show that the augmented objective function

$$\mathscr{L}(x, y, \lambda) = \tfrac{1}{2}(xy' - yx') - \lambda\left[\sqrt{(x')^2 + (y')^2} - P\right],$$

with $J = \int_0^1 \mathscr{L}\, dt$, defines the problem of maximising the area for a closed curve with perimeter P.
(b) Obtain the Euler–Lagrange equations for $x(t)$ and $y(t)$.
(c) Integrate each once with respect to t and show that they can be combined to yield the equation for a circle with the radius given in terms of $|\lambda|$.
(d) Determine λ so that the perimeter constraint is satisfied. Determine the possible positions for the centre of the circle.

3.21 (*Sturm-Liouville eigenvalue problems*) For given functions $p(x) > 0$ and $q(x)$, show that the functional for $y(x)$

$$I = \int_0^1 p(x)(y')^2 - q(x)y^2\, dx$$

yields the Euler–Lagrange equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0.$$

For homogeneous boundary conditions on $y(x)$, the trivial solution $y(x) \equiv 0$, is a critical point for all p, q. Show that seeking nontrivial critical points of I satisfying the normalisation condition,

$$\int_0^1 y^2\sigma(x)\, dx = 1,$$

for a given weight function $\sigma(x) > 0$, yields the Sturm-Liouville equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda\sigma(x))y = 0,$$

where λ are eigenvalues and $y(x)$ are the corresponding eigenfunctions.

3.22 (*The pendulum revisited*) Consider again, the action integral for motion of a point mass in two dimensions subject to gravity (recall Exercise 3.5(a)),

$$I = \int \tfrac{1}{2}m\left[x'(t)^2 + y'(t)^2\right] - mgy(t)\,dt.$$

Suppose that the mass is constrained to move on the circle, $x^2 + y^2 = \ell^2$. Consider this as a problem with a holonomic constraint. Write the Euler–Lagrange equations for $x(t)$, $y(t)$. Note that the Lagrange multiplier appearing in these equations can be interpreted as a force required to keep the mass on the circle. Show that the equations can be reduced to the pendulum equation for $\theta(t)$.

3.23 Consider the problem of finding a locally optimal solution of the functional

$$J = \int_1^2 \left[6y^2 + x^2\left(\frac{dy}{dx}\right)^2 + x^7\right] dx$$

subject to the conditions that

$$y(2) = y(1) + 3, \qquad \int_1^2 24xy\,dx = 5.$$

(a) Write the augmented functional for the constrained optimisation problem.
(b) Determine the natural boundary condition that allows the critical point condition to be reduced to an ODE problem.
(c) Write the general solution of the ODE (homogeneous and particular terms) and the system of equations to determine the three constants in your solution.

3.24 Find the locally optimum solution of the functional

$$J = \int_1^2 \left[\frac{3y^2}{x^5} - \frac{(y')^2}{x^3}\right] dx$$

subject to the conditions that

$$y(1) = 4, \qquad y(2) = -10, \qquad \int_1^2 y\,dx = -3.$$

3.25 The *Pontryagin Maximum Principle* (PMP)[9] is a classic result from optimal control theory. For the classic optimal control problem (3.44) the PMP states that the equations for the optimal solution can be very concisely given in terms of the Hamiltonian, $\mathscr{H} = L + \lambda f$,

$$\frac{dx}{dt} = \frac{\partial \mathscr{H}}{\partial \lambda}, \qquad \frac{d\lambda}{dt} = -\frac{\partial \mathscr{H}}{\partial x}, \qquad \frac{\partial \mathscr{H}}{\partial u} = 0,$$

Subject to $\mathscr{H}(T_*) = 0$. Show that these equations reproduce (3.55) and these equations are consistent with the Hamiltonian being a constant for all times. (Hint: Apply the chain rule to evaluate $d\mathscr{H}/dt$)

3.26 Determine the solution $x(t)$ and the control function $u(t)$ that satisfy the state equation

$$\frac{dx}{dt} = 3x + u \qquad 0 \le t \le T$$

with initial and final conditions

$$x(0) = 2, \qquad x(T) = 1,$$

while minimising the cost functional

$$J = \int_0^T \left(4x^2 + 3xu + u^2\right)\, dt.$$

(a) Use the Pontryagin principle from Exercise 3.25 for the case where the final time is the optimal stopping time $T = T_*$.
(b) If instead the final time is specified as $T = 1/4$, what is the optimal solution? What is the value of the Hamiltonian?

3.27 (*The brachistochrone revisited*) Recall Exercise 3.5(b), where we determined that the equation of motion for the horizontal position $x(t)$ of a particle sliding down a ramp $y = f(x)$ under the influence of gravity is

$$\frac{d}{dt}\left([1 + f'(x)^2]\frac{dx}{dt}\right) = -gf'(x) + f'(x)f''(x)\left(\frac{dx}{dt}\right)^2.$$

Use this result applied to functions satisfying the boundary conditions $f(0) = 1$ and $f(1) = 0$ to show that the brachistochrone problem can be expressed as an optimal control problem on f subject to minimising travel time T for a particle satisfying $x(0) = 0$ and $x(T) = 1$. Write a Lagrangian analogous to (3.47) and the corresponding functional to determine the Euler–Lagrange problem.

[9] Sometimes called the Pontryagin *minimum* principle, depending on the choice of sign convention used for H versus $\mathscr{H}$, recall page 71.